Algebra Qualifying Exam
Spring 1994

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of rational, real and complex numbers, respectively.

1. Let $G$ be a finite group and $p$ is a prime number. Define $G(p) = \{ g \in G | o(g) = p^n \text{ for some } n \}$.
   (a) Show that $G(p)$ is the union of all Sylow $p$-subgroups of $G$.
   (b) Show that $G(p)$ is a subgroup if and only if $G$ has a normal Sylow $p$-subgroup.

2. Show that if $G$ is a finite $p$-group, then for any divisor $d$ of $|G|$, $G$ has a normal subgroup of order $d$.

3. Prove or disprove the following statements:
   (a) An ideal $I$ of a commutative ring $R$ with 1 is maximal if and only if $R/I$ is a field.
   (b) An ideal $I$ of a ring $R$ with 1 is maximal if and only if $R/I$ is a division ring.

4. A commutative ring $R$ with 1 is called local if $R$ has only one maximal ideal $m$. Show that in this case, the maximal ideal $m$ is precisely the set of all non-units in $R$. Is it true in general that for any commutative ring the set of all non-units is an ideal?

5. Let $R$ be a ring with 1. An element $e \in R$ is called a central idempotent if $e^2 = e$ and $e$ is in the center of the ring $R$.
   (a) Give an example of a ring $R$ having a central idempotent different from 0 and 1.
   (b) Let $e \in R$ be a central idempotent show that for any unitary $R$-module $M$, both $eM$ and $(1 - e)M$ are $R$-submodules of $M$ and that $M = eM \oplus (1 - e)M$.

6. Let $V$ be an $n$-dimensional vector space over a field $F$ and $T: V \rightarrow V$ be a linear transformation. Set $P = \{ x \in V | Tx = x \}$ to be the subspace of $T$-fixed points and assume that $T(V) \subseteq P$. Calculate the characteristic polynomial and minimal polynomial of $T$ in terms of $n$ and $k = \dim \ker(T)$. Can $T$ be diagonalized?

7. For $V$ a vector space over the field $F$, let $V^*$ denote the dual space of $V$, that is, $V^*$ is the vector space $\text{Hom}_F(V,F)$ of all linear transformations $\lambda : V \rightarrow F$. If $V$ is $n$-dimensional with a basis $\mathcal{B} = \{ x_1, x_2, \ldots, x_n \}$, define elements $\lambda_1, \ldots, \lambda_n$ of $V^*$ by setting

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\lambda_i \left( \sum_{j=1}^{n} a_j x_j \right) = a_i,
$$

$1 \leq i \leq n, a_j \in F$, and put $\mathcal{B}^* = \{ \lambda_1, \ldots, \lambda_n \}$.
   (a) Show that $\mathcal{B}^*$ is a basis of $V^*$.
   (b) If $V$ is infinite dimensional with a basis $\{ e_1, e_2, \ldots, e_n, \ldots \}$ and if the $\lambda_i$'s are defined similarly as above for $i = 1, 2, \ldots$, prove or disprove the statement that $\{ \lambda_1, \lambda_2, \ldots \}$ is a basis for $V^*$.

8. Give an example of a normal field extension which is not Galois.

9. Prove that any finite extension of degree $n$ over a finite field is Galois. What is the Galois group?