Please work exactly two of the problems in each of the four sections below. Clearly indicate which problems you wish to have graded.

Groups

1. Let $G$ be a group of order 28. Suppose that $G$ has a normal Sylow 2-subgroup. Prove that $G$ is abelian.

2. Let $G$ be a group, $F$ a free group. Let $\phi : G \to F$ be a surjective homomorphism. Show that $G$ contains a subgroup isomorphic to $F$.

3. Let $P$ be a finite $p$-group, for some prime $p$. Let $A$ be a non-trivial normal subgroup of $P$. Show that $A \cap Z(P) \neq 1$; in particular, show that $Z(P) \neq 1$.

Rings and Modules

1. Let $R$ be a commutative ring with identity. Show that

\[ \{x \in R; \ x^n = 0 \text{ for some } 0 \leq n \in \mathbb{Z}\} = \bigcap \{P; P \text{ a prime ideal of } R\}. \]

2. Let $R$ be a ring with identity, $I$ a minimal left ideal of $R$. Show that either $I^2 = 0$ or that there exists $e \in R$ satisfying

\[ I = Re, \quad e^2 = e. \]

3. Let $R$ be a commutative ring with identity having a unique maximal ideal $I$. Let $M$ be a finitely generated left $R$-module. Suppose that $IM = M$. Show that $M = 0$.

[Recall that $IM = \{ \sum i_j m_j; \ i_j \in I, \ m_j \in M \}$.]
Field and Galois Theory

1. (a) What is the Galois group of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$? Explicitly describe the elements of this group. Explicitly determine the Galois correspondence.

   (b) If $a, b$ are non-zero rational numbers, show that $\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(a\sqrt{3} + b\sqrt{5})$.

2. Let $\mathbb{F}_p$ denote the field of $p$ elements. Let $f \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree $d$.

   (a) Show that $f$ has a root in a field of order $p^d$.

   (b) Show that $f$ divides $x^{p^d} - x$.

3. Let $F \subseteq K$ be a Galois extension. Let $f \in F[x]$ be a polynomial irreducible over $F$. Suppose that $f = f_1 \cdots f_t$, with $f_i \in K[x]$ and the $f_i$ irreducible over $K$ for all $i$. Show that the $f_i$ all have the same degree.

Linear Algebra

1. Describe completely, but without proof, how the structure theorem for modules over a PID can be used to provide canonical form theorems for a linear map on a finite dimensional vector space over a field.

2. Let $V$ be a finite dimensional vector space over the field of real numbers $\mathbb{R}$. Let $A : V \to V$ be a linear map. Suppose that $A^{1989} - I = 0$. Show that $V$ is the direct sum of $n$ $A$-invariant 2-dimensional subspaces.

3. Let $A$ be an $n \times n$ matrix over an algebraically closed field. Show that there exists an invertible $n \times n$ matrix $X$ with $A^T = X^{-1}AX$. 