Do two problems from each section.

**Group Theory**

1. Let $G$ be a non-abelian group of order $pq$, where $p$ and $q$ are primes and $p < q$. Prove that $p|q-1$.

2. Prove that there does not exist a simple group of order 112.

3. Prove the Frattini lemma: if $G$ is a finite group, $K$ is a normal subgroup of $G$, and $P$ is a Sylow $p$-subgroup of $K$, then $G = N_G(P)K$.

4. Prove that any finite nilpotent group $G$ is the (internal) direct product of its Sylow subgroups.

(One possible approach: use the fact that $N_G(H) \neq H$ for any proper subgroup $H$ of $G$.)

**Linear Algebra**

1. Let $V$ be a finite-dimensional $F$-vector space and let $T \in \text{End}_F(V)$. Assume that the minimal polynomial of $T$ is of the form $f(x)g(x)$ where $f(x)$ and $g(x)$ are relatively prime polynomials in $F[x]$. Prove that $V = \ker f(T) \oplus \ker g(T)$ (internal direct sum).

2. Let $V$ be an $F$-vector space and let $W$ be a subspace of $V$. Let $\hat{V}$ and $\hat{W}$ denote the dual spaces of $V$ and $W$, respectively. Prove that

$$\hat{W} \cong \hat{V}/\text{Ann}(W)$$

where $\text{Ann}(W)$ is $\{f \in \hat{V} | W \subseteq \ker f\}$.

3. Let $V$ be a vector space of dimension 7 over the rationals, and let $T \in \text{End}_Q(V)$. Suppose that the characteristic polynomial of $T$ is $(x-1)^5(x-2)^2$ and that the minimal polynomial of $T$ is $(x-1)^4(x-2)$. List the possibilities for the Jordan canonical form of $T$, up to re-ordering of Jordan blocks.

4. Let $V$ be a finite-dimensional vector space over an algebraically closed field $F$, and let $S$ and $T$ be two commuting members of $\text{End}_F(V)$. Show that $S$ and $T$ have a common eigenvector (not necessarily for the same eigenvalue).

**Rings and Modules**

(In these problems, rings are assumed to have a multiplicative identity element “1”, and modules are assumed to be unital. That is, if $R$ is a ring and $M$ is a (left) $R$-module, then $1 \cdot x = x$ for all $x \in M$.)

1. By definition, and $R$-module is irreducible if it is non-zero and has no proper non-zero submodules. Let $M$ and $N$ be two irreducible $R$-modules. Prove that either $\text{Hom}_R(M,N) = 0$ or $M \cong N$. Show also that $\text{Hom}_R(M,M)$ is a division ring.

2. Show that if $F$ is an infinite field and $f \in F[x_1, \ldots, x_n] \neq 0$.

3. Let $R$ be an integral domain and let $\rho$ be a non-zero prime ideal of $R$. Show that $R_\rho$ has a unique maximal ideal (where $R_\rho$ denotes the “localization” of $R$ at $\rho$).

4. Show that a module over a ring $R$ is always a homomorphic image of a free $R$-module.

**Fields and Galois Theory**

1. Let $E$ be a splitting field for the polynomial $x^3 - 5$ over $\mathbb{Q}$. Find all of the subfields of $E$. 


2. Let $F_0$ be a field of order 4 (i.e., having precisely four elements). Let $t$ be transcendental over $F_0$, and put $F = F_0(t)$, the function field in one variable over $F_0$. Finally, put $E = F(u)$ where $u^3 = t$.

   (a) Show that $E/F$ is normal and separable.

   (b) Determine the Galois group of the extension $E/F$.

3. Let $K$ be an extension field of the rationals, of finite degree. Prove that $K$ contains only a finite number of roots of unity.

4. Let $E/F$ be an extension field and let $\alpha \in E$. Show that $\alpha$ is algebraic over $F$ if and only if $[F(\alpha) : F]$ is finite.