ALGEBRA QUALIFYING EXAM

Do at least eight problems with at least two from each of the four sections.

Group Theory

1. Let \( \phi: G \rightarrow H \) be a surjective homomorphism of groups, and let \( K = \ker \phi \). If \( H_1 \) is a subgroup of \( H \) show that there is a unique subgroup \( G_1 \) of \( G \) such that
   (i) \( K \leq G_1 \),
   (ii) \( \phi(G_1) = H_1 \).

2. Let \( G \) be a group of order 56. Show that either
   (i) a 2-Sylow subgroup is normal, or
   (ii) a 7-Sylow subgroup is normal.
   (Extra credit: Give examples of groups \( G_1, G_2 \) of order 56 such that a 7-Sylow subgroup of \( G_1 \) is not normal and a 2-Sylow of \( G_2 \) is not normal.)

3. Let \( P \) be a finite \( p \)-group (\( p \) is prime), and let \( H \) be a proper subgroup of \( P \). Show that \( \ker(\phi) \varsubsetneq H \).

4. Prove that no group can be written as the union of two proper subgroups. Give an example of a group which is a union of three proper subgroups.

5. Let \( A \) be an abelian group with generators \( a, b \) and relations \( 2a - b = 0 \), \( -a + 2b = 0 \). Compute the structure of \( A \).

6. Let \( G \) be the group with presentation \( \langle a, b | a^2 = b^3 \rangle \). Show that \( G \) is infinite. (Hint: This is not hard at all! Let \( G_0 \) be the subgroup of \( GL(2, \mathbb{Z}) = 2 \times 2 \) nonsingular matrices with integer entries, generated by \( a_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), \( b_0 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \). Show that \( a_0, b_0 \) satisfy the given relation, and that \( G_0 \) is infinite.)
Rings and Modules

1. Let \( \phi: R_1 \to R_2 \) be a homomorphism of rings.
   
   a) If \( I_2 \) is an ideal of \( R_2 \), show that \( \phi^{-1}(I_2) \) is an ideal of \( R_1 \).
   
   b) If \( I_1 \) is an ideal of \( R_1 \) show by example that \( \phi(I_1) \) need not be an ideal of \( R_2 \).

2. Prove that "Chinese Remainder Theorem": If \( n \) is a positive integer with \( n = ab \), \( a \) and \( b \) relatively prime, then there is an isomorphism of rings
   
   \[
   \frac{\mathbb{Z}}{(n)} \cong \frac{\mathbb{Z}}{(a)} \times \frac{\mathbb{Z}}{(b)}.
   \]

3. Let \( R \) be a ring and let \( M \) be a left \( R \)-module. Let \( \text{Ann}(M) = \{ r \in R | rM = 0 \} \) be the annihilator of \( M \).
   
   a) Show that \( \text{Ann}(M) \) is a 2-sided ideal of \( R \)
   
   b) If \( M \) is irreducible, and if \( R \) commutative, show that there is an isomorphism of \( R \)-modules
      
      \[
      \frac{R}{\text{Ann}(M)} \cong M
      \]

4. Let \( R \) be an integral domain such that every ideal of \( R \) is free. Prove that \( R \) is a principal ideal domain.

5. Let \( R \) be a ring and let \( M \) be a left \( R \)-module. Prove the so-called Noether isomorphism theorem: if \( M_1, M_2 \) are \( R \)-submodules of \( M \) then
   
   \[
   \frac{M_1 + M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2}.
   \]

   (Hint: Map \( M_1 \to \frac{M_1 + M_2}{M_2} \) in the more or less obvious way.

   Is the map surjective? What is the kernel?)
Linear Algebra

1. Let $F$ be a field, and let $V$ be a vector space over $F$.
   a) Define what it means for a subset $S \subseteq V$ to be a basis.
   b) Using Zorn's lemma, show that any vector space has a basis.

2. Let $\{v_1, \ldots, v_n\}$ be a basis for the vector space $V$ over $F$.
   If $w \in V$ satisfies $w \notin \langle v_2, \ldots, v_n \rangle$ (where $\langle \rangle$ means $F$-span), show that $\{w, v_2, \ldots, v_n\}$ is a basis.

3. Let $T: V \to V$ be a linear transformation such that $T^2 = T$.
   Prove that the subspaces $TV$ and $(I - T)V$ are $T$-invariant
   and that $V = TV \oplus (I - T)V$.

4. Give an example of a matrix $A$ with rational entries such that
   minimal polynomial $= (x + 1)^2(x^2 + 1)^2(x^4 + x^3 + x^2 + x + 1)$,
   characteristic polynomial $= (x + 1)^3(x^2 + 1)^3(x^4 + x^3 + x^2 + x + 1)$

5. Let $T_1, T_2 : V \to V$ be linear transformations, where $V$ is a
   finite dimensional vector space over an algebraically closed
   field. If $T_1 T_2 = T_2 T_1$, prove that there exists a vector
   $v \in V$ which is an eigenvector for both $T_1$ and $T_2$.

Fields and Galois Theory

1. Let $F \subseteq K$ be fields and let $a \in K$.
   a) State what it means for $a$ to be algebraic over $F$.
   b) Prove that $a$ is algebraic over $F$ if $F(a)$ is a
      finite dimension $F$-vector space.

2. Let $F$ be a finite field, and let $F^*$ be the non-zero elements
   of $F$, regarded as a multiplicative group. Show that $F^*$ is a
   cyclic group. (Hint: If $e =$ exponent of $F^*$, how many roots
   in $F$ are there to the polynomial $x^e - 1$?)

3. Let $\sqrt[3]{2}$ be a real cube root of 2, and let $\zeta$ be the complex
   number $\zeta = \exp(\frac{2\pi i}{3})$. Let $K_1 = \mathbb{Q}[\sqrt[3]{2}]$, $K_2 = \mathbb{Q}[\zeta]$,
   $K_3 = \mathbb{Q}[\sqrt[3]{2}, \zeta]$. Prove that $K_1$ is not normal over $\mathbb{Q}$ but
   that $K_2, K_3$ are normal over $\mathbb{Q}$.
Let $F \subset K$ be a separable normal extension of $F_1$ and let $G$ be the Galois group of the extension. Let $H$ be a subgroup of $G$ and let $L$ be the field of invariants of $H$, i.e., $L = \{ a \in K \mid ha = a \text{ for all } h \in H \}$. Without using the fundamental theorem of Galois theory, prove that $L$ is normal over $F$ if and only if $H$ is a normal subgroup of $G$. 