Algebra - Do any five problems

1. Show that the field with 27 elements has a cyclic multiplicative group.

2. Let $T$ be a linear transformation of an $n$-dimensional vector space into itself. If $T^k = 0$ for some $k$, show that a basis may be selected for $V$ such that if $A = (a_{ij})$ is the matrix of $T$ with respect to this basis (i.e., $T(v_i) = \sum a_{ij}v_j$), then $a_{ij} = 0$ for $j \geq i$. As a corollary, can you show that $T^n = 0$?

3. Let $D$ be a commutative ring with 1. For $a \in D$, show that $D/aD$ is an integral domain iff $a$ is prime (that is, if $ar = bc$, then $b = aq$ or $c = aq$ for ring elements $r, b, c$). Give one or two nontrivial conditions on $a, D$ (or both) which insure that $D/aD$ is a field.

4. Let $G$ be a subgroup of the symmetric group $S_n$. If $G$ contains an odd permutation, show that $G$ has even order and that exactly half of the elements of $G$ are odd permutations.

5. If $G$ is a cyclic group of order $n$, show that every subgroup of $G$ is cyclic. Also show that if $m|n$, then $G$ has a unique subgroup of order $m$.

6. Let $R$ be a ring such that for all $r \in R$, $mr = 0$ for some fixed, square-free integer $m$. Show that $R$ is the direct sum of ideals $R_i$ and for each $i$ there is a prime factor $p_i$ of $m$ such that for all $r \in R_i$, $p_i r = 0$.

Analysis - Do four of the following six exercises

1. Suppose $\{K_n\}_{n=1}^\infty$ is a collection of closed sets contained in the compact set $K$ and $K \supset K_1 \supset K_2 \supset \ldots$. Prove $\bigcap_{n=1}^\infty K_n$ is not empty.

2. Prove that if $f$ is a continuous real-valued function defined on a compact set of real numbers, then $f$ is uniformly continuous.

3. State and prove the chain rule for functions of one real variable.
5. Let $K$ be algebraic over $F$ where $F$ has characteristic 0. Prove that $K$ is separable over $F$.

6. Let $f(x)$ be a polynomial in $F[x]$ which has no multiple roots in any extension field of $F$. If $K$ is the splitting field of $f(x)$ over $F$, and $G = \text{Gal}(K/F)$ show that $f(x)$ is irreducible in $F[x]$ if and only if $G$ transitively permutes the roots of $f(x)$ in $K$.

7. Let $F = \mathbb{Z}/(2)$. Let $f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. Let $K$ be a splitting field of $f(x)$ over $F$. Compute $[K:F]$.

8. If $F \subseteq K$ with $K$ a finite field, prove $\text{Gal}(K/F)$ is abelian.