

Algebra Qualifying Exam
January 21, 2003

Instructions: You are given six problems to work in a two-hour period.

Note: All rings in this exam are associative and with 1 and all integral domains are commutative.

(1) Let $G$ be a finite group and $H$ be a subgroup of $G$ such that the index of $H$ in $G$ is the smallest prime factor of the order of $G$. Show that $H$ has to be normal.

(2) Let $D$ be an integral domain containing a field $F$ as a subring. Suppose every element of $D$ is algebraic over $F$. Prove that $D$ is a field. Then show that any finite integral domain $D$ contains a finite field $F$ as a subring and therefore is a field.


(4) Let $R$ be a ring with 1 and $M$ be a left $R$-module. If $N$ and $P$ are two $R$-submodules of $M$, define the set

$$(N : P) = \{ r \in R \mid rP \subseteq N \}.$$ 

Show that $(N : P)$ is a two-sided ideal of $R$.

(5) Let $\mathbb{F}_q$ be a finite field with $q$ elements. Then

(a) Compute that number of distinct invertible $2 \times 2$-matrices with entries in $\mathbb{F}_q$.

(b) For any $n$-dimensional vector space $V$-over $\mathbb{F}_q$, compute the number of 2-dimensional subspaces of $V$.

(6) Let $K$ be a field of characteristic $\neq 2$, let $f(x) \in K[x]$ be an irreducible separable polynomial, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the roots of $f(x)$ in a splitting field $E$ over $K$. Set

$$\delta = \prod_{1 \leq j < i \leq n} (\alpha_i - \alpha_j) \in E,$$

If $G = \text{Gal}(E/K)$ and if $H = \text{Gal}(E/F(\delta))$, show that $[G : H] \leq 2$. (Hint: show that $\delta^2 \in K$.)

(7) Let $V$ be an $n$-dimensional vector space over an algebraically closed field $k$ and $T : V \rightarrow V$ be a linear transformation. Show that there are two linear transformations $S : V \rightarrow V$ and $N : V \rightarrow V$ such that (a) $S$ is diagonalizable, (b) $N$ is nilpotent, (c) $SN = NS$, and (d) $T = S + N$.

(8) Let $T_1$ and $T_2$ be two diagonalizable linear transformations of a finite dimensional vector space $V$ over a field $F$. Show that if $T_1T_2 = T_2T_1$, then both $T_1T_2$ and $T_1 + T_2$ are diagonalizable.