Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems that you would like to be graded by circling the problem numbers on the problem sheet. Note: All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let $G$ be a finite group, let $p$ be a prime, and let $S$ be a Sylow $p$-subgroup of $G$. Prove that if $N \triangleleft G$ is a normal subgroup of $G$, then $S \cap N$ is a Sylow $p$-subgroup of $N$.

2. Let $G$ be a finite group of odd order and let $x \in G$ be a nonidentity element. Prove that $x$ and $x^{-1}$ are not conjugate in $G$.

3. Let $G$ be a group acting transitively on the finite set $X$. Let $x \in X$ and denote by $G_x = \{g \in G| gx = x\}$ the isotropy subgroup of $x$. Let $H \triangleleft G$ be a normal subgroup of $G$; note that $HG_x$ is a subgroup of $G$ and that $H$ acts on $X$. Prove that the number of distinct orbits of the action of $H$ on $X$ equals the index $[G:HG_x]$.

4. Let $I = (2,x) = \{2f(x) + xg(x)| f(x), g(x) \in \mathbb{Z}[x]\}$ be the ideal in the ring $R = \mathbb{Z}[x]$ of polynomials in the indeterminate $x$ with integer coefficients. Prove that $I$ is not a free $R$-module.

5. Let $R$ be a Euclidean domain with respect to the function $d: R - \{0\} \rightarrow \mathbb{Z}^+ (= \{1, 2, \ldots \})$. Assume that $d$ satisfies

(a) $d(ab) = d(a)d(b)$, for all $a, b \in R - \{0\}$,
(b) $d(a + b) \leq \max\{d(a), d(b)\}$, for all $a, b, a + b \in R - \{0\}$.

Prove that either $R$ is a field, or that there exists a field $\mathbb{F} \subseteq R$ such that $R \cong \mathbb{F}[x]$, the ring of polynomials in the indeterminate $x$ with coefficients in $\mathbb{F}$. [Hint: Let $\mathbb{F} = \{a \in R| d(a) = 1\}$.]

6. Let $A, B: V \rightarrow V$ be linear transformations on the finite dimensional vector space $V$ over the complex numbers $\mathbb{C}$. Prove that if
$AB = BA$ then there exists a nonzero vector $0 \neq v \in V$ that is simultaneously an eigenvector for both $A$ and $B$.

7. Let $T : V \to V$ be a linear transformation of the finite dimensional vector space $V$ over the field $\mathbb{F}$. Define the usual $\mathbb{F}[x]$-module structure on $V$ by setting $f(x) \cdot v = f(T)(v)$, $f(x) \in \mathbb{F}[x]$, $v \in V$. Prove that $V$ is a cyclic $\mathbb{F}[x]$-module if and only if the characteristic polynomial of $T$ equals the minimal polynomial of $T$.

8. Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of $q$ ($= p^r$) elements, where $p$ is prime, and let $\mathbb{K} = \mathbb{F}_{q^4} \supseteq \mathbb{F}$. Say that elements $\alpha, \beta \in \mathbb{K}$ are equivalent if they have the same minimal polynomial over $\mathbb{F}$. Clearly this is an equivalence relation on $\mathbb{K}$. Compute the number of equivalence classes in $\mathbb{K}$ as a function of $q$. (Hint: consider $\mathbb{F} \subseteq \mathbb{F}_{q^2} \subseteq \mathbb{F}_{q^4} = \mathbb{K}$.)

9. Let $\mathbb{F} \subseteq \mathbb{E} \subseteq \mathbb{K}$ be fields, where the extension degrees are finite, let $\alpha \in \mathbb{K}$, and let $f(x)$ be the minimal polynomial of $\alpha$ over $\mathbb{F}$. Assume that $[\mathbb{E} : \mathbb{F}]$ and $\deg f(x)$ are relatively prime. Prove that $f(x)$ is also the minimal polynomial of $\alpha$ over $\mathbb{E}$.

10. Let $n \geq 3$ be an integer and let $f(x) = x^n - 2 \in \mathbb{Q}[x]$. Prove that the Galois group of $f(x)$ is nonabelian but solvable.