Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet.

Note: All rings on this exam are associative and have multiplicative identity 1. All all integral domains are assumed to be commutative.

1. Let $P$ be a $p$-Sylow subgroup of the finite group $G$. Prove that $N_G(N_G(P)) = N_G(P)$.

2. Let $G$ be a finite group and let $C$ be a conjugacy class of elements in $G$. If $|C| = \frac{1}{2}|G|$, show that every element of $C$ is an involution (i.e., an element of order 2).

3. Let $x$ be an element of $p$-power order in the finite group $G$, where $p$ is prime. Assume that $|\{g^{-1}xg | g \in G\}| = p$. Show that $x$ lies in a normal $p$-subgroup of $G$.

4. Prove, or give a counterexample to the assertions below:
   (a) $\mathbb{Z}[x]$ is a principal ideal domain.
   (b) If $I$ is a maximal ideal of $\mathbb{Z}$, then $I[x]$ is a maximal ideal of $\mathbb{Z}[x]$.

5. Consider the commutative ring $R = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$. Show that the element $1 + 2\sqrt{-5}$ is irreducible but not prime in $R$.

6. Let $R$ be a ring and let $M$ be an irreducible left $R$-module. If $K$ is the kernel of the action of $R$ on $M$ (i.e., $K = \ker(R \to \text{End}_\mathbb{Z}(M))$), prove that $R/K$ is semisimple, i.e., the Jacobson radical is trivial. (Hint: the problem itself is trivial.)
7. Let $F$ be a field and let

$$0 \to V_1 \to V_2 \to V_3 \to V_4 \to 0$$

be an exact sequence of finite dimensional vector spaces over $F$. Prove that

$$\dim V_1 - \dim V_2 + \dim V_3 - \dim V_4 = 0.$$ 

8. Let $F$ be a field and let $T : V \to V$ be a linear transformation on $V$. Assume that $T$ has elementary divisors $x - a, (x - a)^2, (x - a)^2, (x - b)^2, x - c, x - c$, where $a, b, c \in F$ are distinct elements of $F$.

(i) What is the dimension of $V$?

(ii) What is the minimal polynomial of $T$?

(iii) What are the invariant factors of $T$?

(iv) Compute the Jordan canonical form of $T$.

9. Let $F \subseteq K$ be fields such that the extension degree $[K : F] < \infty$. Prove that every element of $K$ is algebraic over $F$.

10. Let $G$ be a finite Hamiltonian group, i.e., one such that every subgroup of $G$ is normal. Now assume that $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial whose Galois group is isomorphic to $G$. Prove that $\deg f(x) = |G|$. 