I. Groups

1. Let $A$ be a finite abelian group of order $n$. Assume that for each divisor $m$ of $n$ the equation $x^m = 1$ has $m$ solutions, where $1$ is the identity in $A$. Prove that $A$ is cyclic.

2. Let $A$ be the abelian group with presentation

   $A = \langle a_1, a_2 | 2a_1 - a_2 = 0, -2a_1 + 2a_2 = 0 \rangle$.

   Find the structure of $A$.

3. Let $G$ be a finite group and let $P$ be a 2-Sylow subgroup. Let $M \triangleleft P$ be a subgroup of index 2 in $P$. Assume that $x \in P - M$ is not conjugate in $G$ to any element of $M$. Then show that $x \notin G'$, the commutator subgroup of $G$. (Hint: Look at the permutation representation of $G$ induced on the cosets of $M$. What is the cycle structure of $x$?)

4. (You may assume the conclusion of exercise (3).) Let $G$ be a finite simple group with dihedral 2-Sylow subgroups. Prove that $G$ has a single class of involutions.

5. Let $G$ be a simple group of order 60. Prove that $G$ has exactly 5 2-Sylow subgroups. (Thus $G \cong A_5$, the alternating group on 5 symbols.)

II. Rings and Modules

1. Let $R$ be an integral domain such that every ideal is a free $R$-module. Prove that $R$ is a principal ideal domain.

2. Let $R$ be a principal ideal domain.

   (a) Prove that any non-zero prime ideal in $R$ is maximal.

   (b) Using (a), prove that the polynomial ring $\mathbb{Z}[x]$ is not a principal ideal domain.
3. Let $R$ be a ring and let $M$ be a left $R$-module. Prove that the following are equivalent:

(i) $M = \bigoplus_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ is a collection of irreducible $R$-submodules of $M$;

(ii) $M = \bigoplus_{\alpha \in J} M_\alpha$, where $\{M_\alpha\}_{\alpha \in J}$ is a collection of irreducible $R$-submodules of $M$.

4. Let $R$ be a ring with identity. Define the Jacobson radical $J(R)$ by setting

$$J(R) = \{ r \in R | rM = 0 \text{ for every irreducible left } R\text{-module } M \}.$$ 

(a) Prove that $J(R)$ is a 2-sided ideal of $R$.

(b) Prove that $J(R/J(R)) = 0$.

III. Linear Algebra

1. Let $T$ be a linear transformation on the finite dimensional $Q$-vector space $V$. If $T^2 + I = 0$ prove that $\dim_Q V$ is even.

2. Let $F$ be any field over which the polynomial $x^2 + x + 1$ is irreducible. Prove that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ is not similar to any upper triangular matrix, with entries in the field $F$.

3. Let $T : V \to V$ be a linear transformation on the finite dimensional $F$-vector space $V$. Let $W \subseteq V$ be a $T$ invariant subspace of $V$.

(a) Prove that there exists a unique linear transformation $\overline{T} : V/W \to V/W$ such that $\overline{T}(v + W) = T(v) + W$.

(b) Prove that $m_T(x) | m_{\overline{T}}(x)$.

4. Let $T : V \to V$ be a linear transformation on the finite dimensional $F$-vector space $V$. Let $m_T(x) = \sum_{i=0}^{n} a_i x^i$. Prove that $T$ is invertible if and only if $a_0 \neq 0$. 
IV. Fields

1. Let $\zeta$ be the complex number $\zeta = e^{2\pi i / n}$. Prove that $[\mathbb{Q}([\zeta]) : \mathbb{Q}] = 2$ if and only if $n = 3, 4$ or $6$.

2. Let $F$ be a finite field, and let $F^* = F - \{0\}$. Prove that with respect to multiplication, $F^*$ is a cyclic group. (You may use the result of exercise (1) of I.)

3. Prove that the Galois group of $x^4 - 5$ cannot be abelian. (Bear in mind that every subgroup of an abelian group is normal.)

4. Let $K$ be a splitting field over $\mathbb{C}[x]$ for the polynomial $y^3 - (x - 1)(x - 2)(x - 3)$. Prove that the genus of $K$ over $\mathbb{C}$ is 2. (Just kidding!)

5. Let $p$ be a prime and let $\zeta$ be the complex number $\zeta = e^{2\pi i / p}$. Prove that $\text{Gal}(\mathbb{Q}(\zeta) / \mathbb{Q}) \cong \mathbb{Z}_{p-1}$, a cyclic group of order $p - 1$. 