1. Show that for any group $G$, the quotient group $G/Z(G)$ is never a nontrivial cyclic group. Here, $Z(G)$ is the center of the group $G$.

2. Let $F$ be a field, and show that the matrix group

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in F, \ ac \neq 0 \right\}$$

is a solvable group.

3. Let $F$ be any field and $G$ be a finite multiplicative subgroup of $F^\times$. Prove that if $|G| > 1$, then $\sum_{g \in G} g = 0$ in $F$.

4. Let $A$ be a commutative ring. Assume that every element $a$ of $A$ is either invertible or nilpotent (i.e., $a^n = 0$ for some $n$ depending on $a$). Show that $A$ has a unique maximal ideal.

5. Let $R$ be a ring with $1$ and $M$ an $R$-module. An element $x \in M$ is called torsion if there exists $r \in R$ and $r \neq 0$ such that $rx = 0$. Let $M_t$ be the set of all torsion elements in $M$. Show that if $R$ is an integral domain then $M_t$ is a submodule of $M$ and $M/M_t$ is a torsion-free $R$-module for any $R$-module $M$. Give an example of a commutative ring $R$ and an $R$-module $M$ such that $M_t$ is not a submodule.
6. Let $A$ be a commutative ring and $M$ be a finitely generated $A$-module. One form of Nakayama Lemma says that if $M = N + IM$, where $N \subseteq M$ is an $A$-submodule of $M$, and where $I$ is an ideal of $A$ contained in every maximal ideal of $A$, then $M = N$.

Now assume that $A$ is a commutative local ring (i.e., $A$ has a unique maximal ideal $m$), and assume that $f : E \to F$ is a homomorphism of $A$-modules. Therefore, $f(mE) \subseteq mF$ and so $f$ induces a homomorphism $\bar{f} : E/mE \to F/mF$. Use Nakayama’s Lemma to show that if $F$ is finitely generated as an $A$-module, then $f$ is surjective if and only if $\bar{f}$ is surjective.

7. Let $k$ be a field and let $A$ be an $k$-algebra. A $k$-linear transformation $D : A \to A$ is a called a $k$-derivation if

$$D(xy) = D(x)y + xD(y), \quad \text{for all } x, y \in A.$$ 

Show that if $D_1$ and $D_2$ are $k$-derivations on $A$, then the composition $D_1 \circ D_2$ need not be a $k$-derivation, but that $D_1 \circ D_2 - D_2 \circ D_1$ is always a $k$-derivation on $A$.

8. Let $F \supseteq k$ be a finite extension of degree $n$ and $f(x) \in k[x]$ be an irreducible polynomial of degree $m$. If $m$ and $n$ are relatively prime, then $f(x)$, as a polynomial over $F$, is still irreducible.

9. Let $k$ be a finite field of $p^r$ elements. If $f(x)$ is an irreducible polynomial in $k[x]$, show that the field $F = k[x]/k[x]f(x)$ contains all roots of $f(x)$ and that the Galois group $\text{Gal}(F/k)$ permutes the set of roots of $f(x)$ transitively.
10. Let $T : V \to V$ be a linear transformation on the $n$-dimensional complex vector space $V$. Give $V$ the usual $\mathbb{C}[x]$-module structure. Suppose that $V$ is isomorphic as a $\mathbb{C}[x]$-module to

$$\mathbb{C}[x]/\mathbb{C}[x]f_1(x) \oplus \mathbb{C}[x]/\mathbb{C}[x]f_2(x) \oplus \mathbb{C}[x]/\mathbb{C}[x]f_3(x) \oplus \mathbb{C}[x]/\mathbb{C}[x]f_4(x),$$

where

$$f_1(x) = (x - 2)^6(x - 3)^7(x - 4)^3,$$

$$f_2(x) = (x - 2)^7(x - 3)^9(x - 4)^3,$$

$$f_3(x) = (x - 2)^6(x - 3)^7(x - 4)^3,$$

$$f_4(x) = (x - 2)^5(x - 3)^5(x - 4)^2.$$

Now do the following:

(a) Compute $n$.

(b) List the characteristic polynomial and the minimal polynomial of $T$.

(c) List the invariant factors of $T$.

(d) List the elementary divisors of $T$.

(e) Write down the Jordan canonical matrix of $T$. 