Dirichlet eigenvalues via Γ-convergence and optimal transportation

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Set-up and the continuous problem

- $D \subset \mathbb{R}^d (d \geq 2)$ is an open, bounded, connected domain with Lipschitz boundary
- $\nu$ is a Borel probability measure with a continuous density $\rho$ where $0 < m \leq \rho \leq M$ pointwise for some constants $m, M$
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- $\nu$ is a Borel probability measure with a continuous density $\rho$ where $0 < m \leq \rho \leq M$ pointwise for some constants $m, M$.

Find the Neumann eigenvalues of the weighted Laplacian

$$\mathcal{L}u = \frac{-1}{\rho} \text{div}(\rho^2 \nabla u),$$

i.e., solve

$$\mathcal{L}u = \lambda u$$
$$\frac{\partial u}{\partial n} = 0$$

for $\lambda \in \mathbb{R}, u \in H^1(D)$. 
Sample points \( \{x_n\}_{n \in \mathbb{N}} \) from \((D, \nu, \rho)\) iid

Choose a similarity function \( \eta: \mathbb{R}^d \to [0, \infty] \) radial, nonincreasing, \( \eta(0) > 0 \) and continuous at 0, and \( \sigma_{\eta} = \int_{\mathbb{R}^d} \eta(h) |h|^2 dh < \infty \). (Example: \( \eta = \chi_{B_1(0)} \))

Make similarity functions for each \( n \in \mathbb{N} \):

\[
\eta_{\varepsilon}(x) := \frac{1}{\varepsilon} \eta_{\varepsilon}(x)
\]

where \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) is a sequence of positive numbers \( \varepsilon_n \to 0 \) that don't decrease "too quickly."
Discrete problem

- Sample points \(\{x_n\}_{n \in \mathbb{N}}\) from \((D, \nu, \rho)\) iid
- For each \(n \in \mathbb{N}\), form the (weighted, undirected) graph \(G_n\) on \(\{x_1, \ldots, x_n\}\) with edge weights \(W_{ij}^{(n)}\). How?
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Discrete problem (continued)

- Set the edge weights: \( W_{ij}^{(n)} = \eta_{\varepsilon_n} (x_i - x_j) \)

Now form the graph Laplacian \( L^{(n)} = D^{(n)} - W^{(n)} \) of \( G_n \) and find its eigenvalues:

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0 = \lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)}
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As \( n \to \infty \) does the \( k \)th eigenvalue of \( L^{(n)} \) converge to the \( k \)th eigenvalue of \( \mathcal{L} \)?
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As \( n \to \infty \) does the \( k \)th eigenvalue of \( L^{(n)} \) converge to the \( k \)th eigenvalue of \( L \)? Yes!
Results of García Trillos-Slepčev

Theorem (García Trillos-Slepčev 2015)

Choose \( \varepsilon_n \to 0 \) to satisfy

\[
\lim_{n \to \infty} \frac{\left(\log n\right)^{3/4}}{n^{1/2}} \frac{1}{\varepsilon_n} = 0 \quad \text{if } n = 2
\]

\[
\lim_{n \to \infty} \frac{\left(\log n\right)^{1/d}}{n^{1/d}} \frac{1}{\varepsilon_n} = 0 \quad \text{if } n \geq 3.
\]

In the set-up above, for all \( k \in \mathbb{N} \),

\[
\lim_{n \to \infty} \frac{2\lambda_k^{(n)}}{n \varepsilon_n^2} = \sigma_\eta \lambda_k
\]

and \( \text{Proj}_{k}^{(n)}(v_n) \to \text{Proj}_k(v) \) if \( v_n \to v \) for functions \( v_n \) on \( G_n \) and \( v \in L^2(D, \nu) \).
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and $\text{Proj}^{(n)}_k(v_n) \to \text{Proj}_k(v)$ if $v_n \to v$ for functions $v_n$ on $G_n$ and $v \in L^2(D, \nu)$.

What’s new? Values of $\varepsilon_n$ (close to optimal) and a framework for discrete-to-continuum passage without strong regularity assumptions.
Motivation

- Approximate continuous problem using the discrete problem
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- They use this result to show consistency of spectral clustering: (Discrete) spectral clustering on graphs which approximate $D$ converges to (continuum) spectral clustering on $D$. 
Ingredient 1: $\Gamma$-convergence of discrete Dirichlet energies

- Main idea of $\Gamma$-convergence: Weaken the direct methods of calculus of variations by looking at a sequence of approximating functionals and relax lower semi-continuity/coercitivity.
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- Let $F, F_n : X \rightarrow \mathbb{R}$ be functionals.
- liminf inequality: For all $x_n \rightarrow X$, $\lim \inf_n F_n(x_n) \geq F(x)$.
- limsup inequality: For all $x \in X$, there is $x_n \rightarrow x$ such that $\lim \sup_n F_n(x_n) \leq F(x)$. 
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- Compactness: If $\{u_n\}_{n \in \mathbb{N}}$ is such that $\sup_n F_n(x_n) < \infty$ then this sequence is precompact.
We have that the discrete Dirichlet energies $G_{n,\varepsilon_n}$ $\Gamma$-converge to the continuum Dirichlet energy $\sigma_{\eta} G$ where

$$G_{n,\varepsilon_n}(u) = \frac{1}{\varepsilon_n^2 n^2} \sum_{i,j} W_{i,j}^{(n)}(u(x_i) - u(x_j))^2$$

$$G(u) = \int_D |\nabla u|^2 \rho(x)^2 \, dx \text{ or } \infty \text{ if } u \notin H^1(D).$$
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How does $\Gamma$-convergence help here? The defining inequalities and existence of useful sequences $\nu_n \rightarrow \nu$. 
Ingredient 2: Metric coming from optimal transportation

Think of \( G_n \) as \( D \) with the empirical measure \( \nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \) together with the edge data \( W^{(n)} \). How should functions \( u_n \) supported on the graph \( G_n \) converge to a function \( u \) on \( D \)?
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$\nu_n \in L^2(D, \nu_n)$ converge to $\nu \in L^2(D, \nu)$ in the $TL^2$-sense if $\nu_n \rightharpoonup \nu$ and if there are ”nice” transportation maps $T_n : D \to D$ (with $(T_n)_*(\nu) = \nu_n$) such that $\nu_n \circ T_n \rightharpoonup \nu$ in $L^2(D, \nu)$. 
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Main idea: Extend $\nu_n$ to $D$ (instead of restricting $\nu$ to $G_n$), but in way that's faithful to both $(D, \nu, \rho)$ and $x_1, x_2, \ldots, x_n$. 

Ingredient 2: Metric coming from optimal transportation
Sketch of the proof (for the eigenvalues)

▶ Prove $\Gamma$-convergence of the Dirichlet energies by interpolating with other functionals $G_{\varepsilon_n}$ that also $\Gamma$-converge to $\sigma_\eta G$.

▶ View the eigenvalues using the Courant-Fischer maxmini principle:

$$
\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\|u\|_\nu = 1, u \in S} G(u)
$$

where $\Sigma_{k-1}$ is the set of $(k-1)$-dimensional subspaces of $L^2(D, \nu)$. 

▶ Use induction on $k$ and establish lower and upper bounds for each $k$-dimensional subspace of $L^2(D, \nu)$; conclude by applying Courant-Fischer.
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Extensions to Dirichlet eigenvalues

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**Continuous problem:** Minimize the $G$ over all functions $u \in H^1_0(U)$ of unit norm to get the first Dirichlet eigenvalue $\mu_1$ of $D$. (Use the Courant-Fischer maxmini principle to get the higher Dirichlet eigenvalues $\mu_k$.)
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**Discrete problem:** Minimize \( G_{\varepsilon_n,n} \) over all functions \( u \) on \( G_n \) with unit norm that vanish outside of \( \{x_1, x_2, \ldots, x_n\} \cap U \) to get the first Dirichlet eigenvalue \( \mu_1^{(n)} \) of \( G_n \). (Use the Courant-Fischer maxmini principle to get the higher Dirichlet eigenvalues \( \mu_k^{(n)} \).)
Extensions to Dirichlet eigenvalues (cont.)

- We conjecture that results analogous to these for the Neumann eigenvalues hold for the Dirichlet eigenvectors.

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The proof for the Neumann eigenvalues largely carries over when we modify the Courant-Fischer maximi principle to look only at functions in $H^1_0(U) \subset L^2(D, \nu)$. 

Our idea: For a good choice of $\{T_n\} \subset \mathbb{N}$, which distinguish $U$ and $D - U$ asymptotically, we can show directly that $v|_{\partial U} = 0$. 

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Questions? Comments? Please contact me!
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Thanks!
