Green’s function asymptotics near the internal edges of spectra of periodic elliptic operators. Spectral gap interior.

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Prairie Analysis Seminar 2015, Kansas State University, Manhattan, Kansas.

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Outline

1. Decay of the resolvents for $-\Delta$
2. Periodic PDOs and dispersion relations
3. Asymptotics of Green's functions of periodic elliptic operators
Laplacian $-\Delta$ on $\mathbb{R}^d$

$R_\lambda = (-\Delta - \lambda)^{-1}$, for $\lambda (< 0)$ in $\rho(-\Delta)$
Resolvents of Laplace operators

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By Fourier transform, Green’s function $G_\lambda$ of $-\Delta$ at $\lambda$ — kernel of $R_\lambda$:

$$G_\lambda(x, y) = \frac{1}{(2\pi)^d} \int \frac{e^{i(x-y)\xi}}{|\xi|^2 - \lambda} d\xi$$
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Q: Investigate the asymptotics of $G_\lambda(x, y)$ as $|x - y| \to \infty$. 

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Decay of resolvents for Laplacian

If $d = 3$, $G_\lambda(x, y) = \frac{1}{4\pi |x-y|} e^{-|\lambda|^{1/2}}$

If $d \geq 4$, $G_\lambda(x, y) \sim e^{-|\lambda|^{1/2}} |x-y|^{(d-2)/2} |\lambda|^{(d-3)/4} |x-y|^{(d-1)/2}$

Exponential decay of the Green's function as $|x-y| \to \infty$ related to distance between $\lambda$ and spectrum $\sigma(-\Delta)$.

Additional power decay
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Periodic PDOs and dispersion relations

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Periodic PDOs and dispersion relations

Asymptotics of Green’s functions of periodic elliptic operators

Periodic PDO

$L = -\Delta + V(x)$

where $\forall n \in \mathbb{Z}^d$, $V(x + n) = V(x)$

The results also hold for self-adjoint elliptic operator in $\mathbb{R}^d$

$L(x, D) = D^* A(x) D + V(x)$, $D = \{-i \partial_k\}_{d k=1}$

where $A: \mathbb{R}^d \to \mathbb{R}^d \times d$ and $V: \mathbb{R}^d \to \mathbb{R}$ smooth and periodic w.r.t $\mathbb{Z}^d$.
Dimension $d > 1$.
Periodic Schrödinger operator in $\mathbb{R}^d$:

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Dispersion relation

$T := \mathbb{R}^d / \mathbb{Z}^d, \quad T^* := \mathbb{R}^d / (2\pi \mathbb{Z})^d$

$L(k) := -\Delta - 2i k \cdot \nabla + k^2 + V(x)$ on $T$, for $k \in \mathbb{C}^d$

$L(k)$ is bounded below and for $k \in \mathbb{R}^d$

$\sigma(L(k)) = \{ \lambda_i(k) | \lambda_1(k) \leq \lambda_2(k) \leq ... \leq \lambda_n(k),... \rightarrow \infty \}$

Eigenvalues $\lambda_i(k)$ - continuous and piecewise analytic

Operator $L$ decomposes into direct sum of multiplication operators by $\lambda_i(k)$
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The multi-valued mapping 
\( k \mapsto \sigma(L(k)) \) is called dispersion relation. Its graph is:

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B_L = \{ (k, \lambda) \in \mathbb{C}^{d+1} | \exists u \neq 0 : L(k)u = \lambda u \}
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$k$ – quasimomentum
Dispersion relation of $-\Delta$

$B_{-\Delta}$ union of $(2\pi \mathbb{Z})^d$-shifted of the single paraboloid $k^2 = \lambda$
Dispersion, bands, and gaps

Floquet-Bloch theory: $\sigma(L) = \bigcup_{k \in \mathbb{R}^d} \sigma(L(k)) = \bigcup_{j} I_j$. Bands can overlap when $d > 1$, but may leave spectral gaps between them.
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Green's function asymptotics near the internal edges of spectra of periodic elliptic operators.
The problem

Spectral gap $(\beta, \alpha)$ and $\lambda \in (\beta, \alpha)$.

$\lambda$ is close to spectral edge $\beta$ or $\alpha$.

Schwartz kernel of $(L - \lambda)^{-1}$ in $L^2(\mathbb{R}^d)$ – Green's function.

Goal: study asymptotics of $G_{\lambda}(x, y)$ when $|x - y| \to \infty$. 

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A “generic” spectral edge behavior of $L$

WLOG – spectral edge of interest $\lambda = 0$, i.e., the minimum of a dispersion curve $\lambda_j$ for some $j \in \mathbb{N}$. 

Assume $\exists$ quasimomentum $k_0$ s.t. the following ‘generic’ conditions hold:

A1 $\lambda_j(k_0) = 0$

A2 $\min_{k, i \neq j} |\lambda_i(k)| > 0$

A3 $k_0$ is the only minimum of $\lambda_j$ (modulo $(2\pi \mathbb{Z})^d$)

A4 The Hessian matrix of $\lambda_j$ at $k_0$ is positive definite.

A5 All components of $k_0$ integer multiples of $\pi$. 

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Notations in the main theorem

Define $E(\beta) := \lambda_j(k_0 + i\beta)$. $E$ – real-valued, strictly concave, Hessian at 0 is negative-definite. For $\lambda \approx 0$, $\Gamma_\lambda := \{ \beta | E(\beta) = \lambda \}$ is strictly convex, compact. For $s \in S_{d-1}$, $\beta_s$ – unique point on $\Gamma_\lambda$ s.t. $-\nabla E(\beta_s)|\nabla E(\beta_s)| = s$.
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$\phi_z(x)$ – the corresponding $\mathbb{Z}^d$-periodic eigenfunctions of $L(z)$
The main result

Theorem (M.K, P. Kuchment, and A. Raich 2014)

Suppose conditions A1-A5 are satisfied. For $\lambda < 0$ sufficiently close to 0 (depending on the dispersion branch $\lambda_j$ and the operator $L$), $G_\lambda$ of $L$ at $\lambda$ admits the asymptotics as $|x - y| \to \infty$:

$$G_\lambda(x, y) = \frac{e^{(x-y)(ik_0-\beta_s)}}{(2\pi|x - y|)^{(d-1)/2}} \frac{|\nabla E(\beta_s)|^{(d-3)/2}}{\det (-P_s\text{Hess}E(\beta_s)P_s)^{1/2}}$$

$$\times \frac{\phi_{k_0+i\beta_s}(x)\phi_{k_0-i\beta_s}(y)}{(\phi_{k_0+i\beta_s}, \phi_{k_0-i\beta_s})_{L^2(\mathbb{T})}} + e^{(y-x)\cdot \beta_s}r(x, y).$$

Here $s = (x - y)/|x - y|$ and $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ (independent of $s$) s.t. the remainder term $r$ satisfies $|r(x, y)| \leq C_\varepsilon |x - y|^{-d/2+\varepsilon}$ when $|x - y|$ is large enough.
Generalization on abelian coverings

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**Goal**: Extend the main result in Euclidean case to this setting, i.e. Green’s function asymptotics of a periodic elliptic operator $L$ on $X$. 
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WLOG, assume $G = \mathbb{Z}^d$. 
Generalization on abelian coverings (Cont.)

**Additive function on abelian covering \( X \)**

- \( \exists \) smooth \( h : X \to \mathbb{R}^d \) i.e. \( h(g \cdot x) = h(x) + g \),
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- $|h(x) - h(y)| \simeq d_X(x, y)$ when $x, y$ are sufficiently far.
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Floquet-Bloch theory, assumptions and notations

- $\sigma(L)$ has **band-gap** structure.
- Dispersion relations, band functions, gaps etc still make sense.
Additive function on abelian covering $X$

- $\exists$ smooth $h : X \to \mathbb{R}^d$ i.e. $h(g \cdot x) = h(x) + g$,
  $\forall g \in \mathbb{Z}^d, x \in X$.

- $|h(x) - h(y)| \simeq d_X(x, y)$ when $x, y$ are sufficiently far.

**Floquet-Bloch** theory, **assumptions** and **notations**

- $\sigma(L)$ has **band-gap** structure.

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- Assume the spectral edge is 0.
Generalization on abelian coverings (Cont.)

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Floquet-Bloch theory, assumptions and notations

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- Notations $E, \beta_s, P_s, \phi_z$: similar to flat case.
Under the conditions \( \textbf{A1-A5} \), for \( \lambda < 0, \lambda \approx 0 \), then \( G_\lambda \) of \( L \) at \( \lambda \) admits the asymptotics as \( d_X(x, y) \to \infty \):

\[
G_\lambda(x, y) = \frac{e^{(h(x)-h(y))(ik_0-\beta_s)}}{(2\pi|h(x)-h(y)|)^{(d-1)/2}} \frac{|\nabla E(\beta_s)|^{(d-3)/2}}{\det(-P_s \text{Hess}E(\beta_s)P_s)^{1/2}} \times \frac{\phi_{k_0+i\beta_s}(x)\phi_{k_0-i\beta_s}(y)}{(\phi_{k_0+i\beta_s}, \phi_{k_0-i\beta_s})_{L^2(M)}} + e^{(h(y)-h(x)).\beta_s} r(x, y).
\]

Here \( s = (h(x)-h(y))/|h(x)-h(y)| \) and \( \forall \varepsilon > 0, \exists C_\varepsilon > 0 \) (independent of \( s \)) s.t. the remainder term \( r \) satisfies \( |r(x, y)| \leq C_\varepsilon d_X(x, y)^{-d/2+\varepsilon} \) when \( d_X(x, y) \) is large enough.
Thank you for your attention today!