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A joint work with:
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Prairie Analysis Seminar
Kansas State University, Manhattan, KS
Main Themes

1. A Littlewood-Paley-Stein theory approach to Calderón-Zygmund theory on Hardy spaces,
   
   Linear – $T : H^p \rightarrow H^p$ and $S_\Lambda : H^p \rightarrow L^p$
   
   Bilinear – $T : H^{p_1} \times H^{p_2} \rightarrow H^p$ and $S_\Theta : H^{p_1} \times H^{p_2} \rightarrow L^p$

2. A notion of moments for non-convolution operators.

3. Applications to paraproduct operators mapping $H^p \rightarrow H^p$ and $H^{p_1} \times H^{p_2} \rightarrow H^p$. 
**CZO:** A continuous linear operator $T$ from $S$ into $S'$ is a CZO if

$$\langle Tf, g \rangle = \int_{\mathbb{R}^{2n}} K(x, y)f(y)dy \, dx$$

when $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, where

$$|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C|x - y|^{-(n + |\alpha| + |\beta|)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^n \text{ and } x \neq y.$$
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LPSO: Let $\lambda_k : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ for $k \in \mathbb{Z}$, and define

$$\Lambda_k f(x) = \int_{\mathbb{R}^n} \lambda_k(x, y) f(y) \, dy.$$

A collection $\{\Lambda_k\} \in LPSO$ if

$$|D_y^\alpha \lambda_k(x, y)| \lesssim 2^k(n+|\alpha|) (1 + 2^k |x - y|)^{-N} \quad \text{for all } \alpha \in \mathbb{N}_0^n.$$
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Define the square function $S_\Lambda$ by $S_\Lambda f(x)^2 = \sum_{k \in \mathbb{Z}} |\Lambda_k f(x)|^2$. 
Main Research Questions for Linear Operators

**CZO:** Under what conditions on $T \in CZO$ is $T$ bounded $H^p \rightarrow H^p$ for $0 < p \leq 1$?
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The space \( H^p \) is the collection of \( f \in S' \) such that \( \sup_{t>0} |\phi_t^* f| \in L^p \).

Also, for \( f \in S'/P \) \( ||f||_{H^p} \approx \left( \sum_{k \in \mathbb{Z}} |Q_k f|^2 \right)^{1/2} \).
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The space $H^p$ is the collection of $f \in S'$ such that $\sup_{t>0} |\varphi_t * f| \in L^p$. 

Note: $f \in H^p$ "implies" $\int_{\mathbb{R}^n} f(x) x^\alpha dx = 0$ for $|\alpha| \leq \left(\frac{1}{p} - 1\right) n$. 

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Hardy Space Estimates for Bilinear Operators

Linear Theory

T1 Theorems for $L^2$

CZO:

**Theorem (David-Journé)**

Let $T \in \text{CZO}$. Then $T$ is bounded on $L^2$ if and only if $T1, T^*1 \in \text{BMO}$ and $T \in \text{WBP}$. 
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LSPO:

**Theorem (Carleson, Jones, Semmes)**

Let $\{\Lambda_k\} \in LPSO$. Then $S_{\Lambda}$ is bounded on $L^2$ if and only if

$$d\mu(x,t) = \sum_{k \in \mathbb{Z}} |\Lambda_k(1)(x)|^2 dx \delta_{t=2^{-k}}$$

is a Carleson measure.
CZO: Assume $T \in CZO$ is bounded on $L^2$.

Theorem (Fefferman-Stein)

If $T$ is a convolution operator, then $T$ is bounded on $H^p$ for all $0 < p \leq 1$. 
**CZO:** Assume $T \in \text{CZO}$ is bounded on $L^2$.

**Theorem (Fefferman-Stein)**

*If $T$ is a convolution operator, then $T$ is bounded on $H^p$ for all $0 < p \leq 1$.***

**Theorem (Torres, Frazier-Han-Jawerth-Weiss, Frazier-Torres-Weiss)**

*If $T$ satisfies $T1 = 0$ and $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$, then $T$ is bounded on $H^p$ for all $\frac{n}{n+L+1} < p \leq 1$.***
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**Theorem (Alvarez-Milman)**

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Overview and General Approach

Given $T \in CZO$, our goal is to prove $\|Tf\|_{H^p} \leq C\|f\|_{H^p}$. 

For appropriate $\psi \in C^\infty_0$ with $Q_kf = \psi_k^*f$ and $\psi_k(x) = 2^{kn}\psi(2^kx)$,

$\|Tf\|_{H^p} \approx \left\| \frac{1}{L_2} \right\|_{L^p} = \|S\Lambda f\|_{L^p},$

where $\Lambda_kf(x) = Q_kTf(x) = \langle Tf, \psi_k^* \rangle$. 

So we reduce the $T: H^p \to H^p$ bounds to two problems:

1. For which $\{\Lambda_k\} \in LPSO$ is $S\Lambda$ bounded $H^p \to L^p$?
2. For which $T \in CZO$ does $\Lambda_kf = Q_kTf$ verify (1)?
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Moments for Non-Convolution Operators

**CZO:** Let $T \in CZO$ and $\alpha \in \mathbb{N}_0^n$. Define $[[T]]_\alpha$ by

$$[[T]]_\alpha(x) = \int_{\mathbb{R}^n} K(x, y)(x - y)^\alpha dy.$$
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- $[[T]]_{\alpha}(x) = \sum_{\beta \leq \alpha} c_{\alpha, \beta} x^{\alpha - \beta} T(y^{\beta})(x),$

  in particular $T \mathbf{1} = [[T]]_{0}$.
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$[[\Lambda_k]]_0(x) = \Lambda_k(1)(x)$. 

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Carleson Measures and Polynomial Growth $BMO_M$ Spaces

A measure $d\mu(x, t)$ on $\mathbb{R}^n \times (0, \infty)$ is a Carleson measure if

$$d\mu(Q \times (0, \ell(Q))) \lesssim |Q|$$

for all cubes $Q \subset \mathbb{R}^n$. 
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$F \in BMO_M$ if

$$d\mu_\psi(x, t) = \sum_{k \in \mathbb{Z}} 2^{2Mk} |Q_k F(x)|^2 dx \delta_{t=2^{-k}}$$

is a Carleson measure for an appropriate $\psi \in S$; recall $Q_k f = \psi_k * f$. 
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When $M = 0$, we have $BMO_0 = BMO$. 
Answer to question (1):

**Theorem (H.-Lu)**

Assume \( \{\Lambda_k\} \in \text{LPSO} \) and \( L \geq 0 \). If

\[
d\mu_\alpha(x, t) = \sum_{k \in \mathbb{Z}} |[[\Lambda_k]]_\alpha(x)|^2 \, dx \, \delta_{t = 2^{-k}}
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is a Carleson measure for all \( |\alpha| \leq L \), then \( S_\Lambda \) is bounded from \( H^p \) into \( L^p \) for all \( \frac{n}{n+L+1} < p \leq 1 \).
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Answer to question (2):

**Theorem (H.-Lu)**

Assume \( T \in CZO \) is bounded on \( L^2 \).

If \( T^*(x^\alpha) = 0 \) and \( [[T]]_\alpha \in BMO_{|\alpha|} \) for \( |\alpha| \leq L \), then \( T \) is bounded on \( H^p \) for all \( \frac{n}{n+L+1} < p \leq 1 \).
A few remarks on the answer to question (1), about boundedness of $S_\Lambda$ for $\{\Lambda_k\} \in LPSO$. 

This is done in two steps:

- If $\Lambda_k$ is $\alpha = 0$ for $|\alpha| \leq L$, then $S_\Lambda: H^p \rightarrow L^p$.

- Decompose $\Lambda_k f = \Lambda(\langle L \rangle)k f + \Pi_k f$; use the previous step to bound $\Lambda(\langle L \rangle)k f$ and Carleson estimates for the $\Pi_k$ term.

Obviously, $\Pi_k f = \Lambda_k f - \Lambda(\langle L \rangle)k f$, but how should $\Lambda(\langle L \rangle)k f$ be defined?

$\Lambda_0 k f(x) = \Lambda k f(x) - \left[\Lambda k \right]_0 (x) \mathcal{P}k f(x)$

$\Lambda_1 k f(x) = \Lambda_0 k f(x) - \sum_{|\alpha| = 1} \left[\Lambda_0 \right]_\alpha (x) \alpha! (\langle -1 \rangle)^{|\alpha|} 2^{-|\alpha|} - k |\alpha| \mathcal{P}k \mathcal{D} \alpha f(x)$

$\Lambda(L-1) k f(x) = \Lambda(L-1) k f(x) - \sum_{|\alpha| = L} \left[\Lambda(L-1) \right]_\alpha (x) \alpha! (\langle -1 \rangle)^{|\alpha|} 2^{-|\alpha|} - k |\alpha| \mathcal{P}k \mathcal{D} \alpha f(x)$
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$$\Lambda_k^{(0)} f(x) = \Lambda_k f(x) - [[\Lambda_k]]_0(x) P_k f(x)$$
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\vdots

\[
\Lambda_k^{(L)} f(x) = \Lambda_k^{(L-1)} f(x) - \sum_{|\alpha| = L} \frac{[[\Lambda_k^{(L-1)}]]_\alpha f(x)}{\alpha!} (-1)^{|\alpha|} 2^{-k|\alpha|} P_k D^\alpha f(x)
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A remark on the answer to question (2), about when $T \in CZO$ gives rise to $\{\Lambda_k\} = \{Q_k T\} \in LPSO$.

**Theorem (H.-Lu)**

Assume that $T \in CZO$ is bounded on $L^2$. If $T^*(x^\alpha) = 0$ for all $|\alpha| \leq L$, then $\{\Lambda_k\} = \{Q_k T\} \in LPSO$. 

Since $\lambda_k(x, y) = T^*\psi x_k(y)$ is the kernel of $Q_k T$, this says $|D^\alpha y T^*\psi x_k(y)| \lesssim 2^k (n + |\alpha|) (1 + 2^k |x - y|)^{-N}$ for appropriate $\alpha$ and $N$. This provides pointwise estimates for $Q_k T f(x)$, which is very useful for analyzing the action of $T$ on distribution spaces (including Hardy spaces, but also for Trieble-Lizorkin and Besov spaces).
A remark on the answer to question (2), about when $T \in CZO$ gives rise to \( \{\Lambda_k\} = \{Q_k T\} \in \text{LPSO} \).

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Since $\lambda_k(x, y) = T^* \psi^x_k(y)$ is the kernel of $Q_k T$, this says

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|D^\alpha_y T^* \psi^x_k(y)| \lesssim 2^k(n+|\alpha|)(1 + 2^k|x - y|)^{-N}
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for appropriate $\alpha$ and $N$. 
A remark on the answer to question (2), about when $T \in CZO$ gives rise to $\{\Lambda_k\} = \{Q_kT\} \in LPSO$.

**Theorem (H.-Lu)**

*Assume that $T \in CZO$ is bounded on $L^2$. If $T^* (x^\alpha) = 0$ for all $|\alpha| \leq L$, then $\{\Lambda_k\} = \{Q_kT\} \in LPSO$.*

Since $\lambda_k(x, y) = T^* \psi^x_k(y)$ is the kernel of $Q_kT$, this says

$$|D_y^\alpha T^* \psi^x_k(y)| \lesssim 2^{k(n+|\alpha|)} (1 + 2^k |x - y|)^{-N}$$

for appropriate $\alpha$ and $N$.

This provides pointwise estimates for $Q_kTf(x)$, which is very useful for analyzing the action of $T$ on distribution spaces (including Hardy spaces, but also for Triebel-Lizorkin and Besov spaces).
Application to the Bony Paraproducts

Given $b \in BMO$, define the Bony paraproduct

$$\Pi_b f(x) = \sum_{j \in \mathbb{Z}} Q_j(\tilde{Q}_j b \cdot P_j f)(x),$$

where $P_j$ is an approximation to identity and $Q_j$ has vanishing moments to order $M$. These can be selected so that $\Pi_b 1 = b$. It easily follows that $\Pi_b^*(x^\alpha) = 0$ for $|\alpha| \leq M$. 
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$$2^{|\alpha|} Q_k [[\Pi_b]]_\alpha(x) \approx C_\alpha \sum_{j \in \mathbb{Z}} 2^{|\alpha|(k-j)} \int_{\mathbb{R}^n} \tilde{Q}_j b(u) Q_j \psi^\alpha_k(u) du$$
Application to the Bony Paraproducts

Given $b \in BMO$, define the Bony paraproduct

$$\Pi_b f(x) = \sum_{j \in \mathbb{Z}} Q_j (\tilde{Q}_j b \cdot P_j f)(x),$$

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$$2^{|\alpha|k} Q_k [[\Pi_b]] \alpha(x) \approx C_\alpha \sum_{j \in \mathbb{Z}} 2^{|\alpha|(k-j)} \int_{\mathbb{R}^n} \tilde{Q}_j b(u) Q_j \psi_k^x(u) du$$

$$= C_\alpha \sum_{j \in \mathbb{Z}} 2^{|\alpha|(k-j)} Q_k Q_j \tilde{Q}_j b(x)$$

How do we address $2^{|\alpha|(k-j)} Q_k Q_j \tilde{Q}_j b(x)$?
\[ 2^{\alpha |(k-j)|} \mathcal{F} \left[ Q_k Q_j \tilde{Q} f \right] (\xi) = 2^{\alpha |(k-j)|} \hat{\psi}(2^{-k} \xi) \hat{\phi}(2^{-j} \xi) \hat{f}(\xi) \]
\[
2^{|\alpha|(k-j)} \mathcal{F} \left[ Q_k Q_j \tilde{Q} f \right] (\xi) = 2^{|\alpha|(k-j)} \hat{\psi}(2^{-k} \xi) \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)
\]
\[
= (2^{-k} |\xi|)^{-|\alpha|} \hat{\psi}(2^{-k} \xi) (2^{-j} |\xi|)^{|\alpha|} \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)
\]
2^{|\alpha|} (k-j) \mathcal{F} \left[ Q_k Q_j \tilde{Q} f \right] (\xi) = 2^{|\alpha|} (k-j) \hat{\psi}(2^{-k} \xi) \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)
= (2^{-k} |\xi|)^{-|\alpha|} \hat{\psi}(2^{-k} \xi) (2^{-j} |\xi|)^{|\alpha|} \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)
= \mathcal{W}(\alpha)(2^{-k} \xi) \mathcal{V}(\alpha)(2^{-j} \xi) \hat{f}(\xi)
\[
2^{\alpha |(k-j)|} \mathcal{F} \left[ Q_k Q_j \hat{Q}_j f \right] (\xi) = 2^{\alpha |(k-j)|} \hat{\psi}(2^{-k} \xi) \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)
\]
\[
= (2^{-k |\xi|})^{-\alpha} \hat{\psi}(2^{-k} \xi) (2^{-j |\xi|})^{\alpha} \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)
\]
\[
= \mathcal{W}(\alpha)(2^{-k} \xi) \mathcal{V}(\alpha)(2^{-j} \xi) \hat{f}(\xi)
\]
\[
= \mathcal{F} \left[ \mathcal{W}_k^{(\alpha)} * \mathcal{V}_j^{(\alpha)} * f \right] (\xi)
\]
Hardy Space Estimates for Bilinear Operators

Linear Theory

\[ 2^{|\alpha| (k-j)} F \left[ Q_k Q_j \tilde{Q}_j f \right] (\xi) = 2^{|\alpha| (k-j)} \hat{\psi} (2^{-k} \xi) \hat{\phi} (2^{-j} \xi) \hat{f} (\xi) \]
\[ = (2^{-k} |\xi|)^{-|\alpha|} \hat{\psi} (2^{-k} \xi) (2^{-j} |\xi|)^{|\alpha|} \hat{\phi} (2^{-j} \xi) \hat{f} (\xi) \]
\[ = \hat{W}(\alpha) (2^{-k} \xi) \hat{V}(\alpha) (2^{-j} \xi) \hat{f} (\xi) \]
\[ = F \left[ W_k^{(\alpha)} \ast V_j^{(\alpha)} \ast f \right] (\xi) \]

\[ \sum_{j \in \mathbb{Z}} W_k^{(\alpha)} \ast V_j^{(\alpha)} \ast b = W_k^{(\alpha)} \ast (T_V^{(\alpha)} b), \text{ where } T_V^{(\alpha)} f = \sum_{j \in \mathbb{Z}} V_j^{(\alpha)} \ast f. \]
$2^{|\alpha|(k-j)} \mathcal{F} \left[ Q_k Q_j \tilde{Q} f \right] (\xi) = 2^{|\alpha|(k-j)} \hat{\psi}(2^{-k} \xi) \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)$

$= (2^{-k} |\xi|)^{-|\alpha|} \hat{\psi}(2^{-k} \xi) (2^{-j} |\xi|)^{|\alpha|} \hat{\phi}(2^{-j} \xi) \hat{f}(\xi)$

$= \mathcal{W}(\alpha)(2^{-k} \xi) \mathcal{V}(\alpha)(2^{-j} \xi) \hat{f}(\xi)$

$= \mathcal{F} \left[ \mathcal{W}_k^{(\alpha)} * \mathcal{V}_j^{(\alpha)} * f \right] (\xi)$

$\sum_{j \in \mathbb{Z}} \mathcal{W}_k^{(\alpha)} * \mathcal{V}_j^{(\alpha)} * b = \mathcal{W}_k^{(\alpha)} * (T_V^{(\alpha)} b), \text{ where } T_V^{(\alpha)} f = \sum_{j \in \mathbb{Z}} \mathcal{V}_j^{(\alpha)} * f.$

Then $T_V^{(\alpha)} \in CZO$ is a convolution operator bounded on $L^2$. 
\[ 2|\alpha|(k-j) \mathcal{F} \left[ Q_k Q_j \tilde{Q}_j f \right](\xi) = 2|\alpha|(k-j) \hat{\psi}(2^{-k} \xi) \hat{\phi}(2^{-j} \xi) \hat{f}(\xi) \]
\[ = (2^{-k}|\xi|)^{-|\alpha|} \hat{\psi}(2^{-k} \xi) (2^{-j}|\xi|)^{|\alpha|} \hat{\phi}(2^{-j} \xi) \hat{f}(\xi) \]
\[ = \mathcal{W}(\alpha)(2^{-k} \xi) \mathcal{V}(\alpha)(2^{-j} \xi) \hat{f}(\xi) \]
\[ = \mathcal{F} \left[ \mathcal{W}_k(\alpha) \ast \mathcal{V}_j(\alpha) \ast f \right](\xi) \]

\[ \sum_{j \in \mathbb{Z}} \mathcal{W}_k(\alpha) \ast \mathcal{V}_j(\alpha) \ast b = \mathcal{W}_k(\alpha) \ast (T_\mathcal{V}(\alpha) b), \text{ where } T_\mathcal{V}(\alpha) f = \sum_{j \in \mathbb{Z}} \mathcal{V}_j(\alpha) \ast f. \]

Then \( T_\mathcal{V}(\alpha) \in CZO \) is a convolution operator bounded on \( L^2 \).

Then \( T_\mathcal{V}(\alpha) \) is also bounded on \( BMO \), \( T_\mathcal{V}(\alpha) b \in BMO \).
\[ 2^{|\alpha| (k-j)} \mathcal{F} \left[ Q_k Q_j \tilde{Q} f \right] (\xi) = 2^{|\alpha| (k-j)} \hat{\psi} (2^{-k} \xi) \hat{\phi} (2^{-j} \xi) \hat{f} (\xi) \]

\[ = (2^{-k} |\xi|)^{-|\alpha|} \hat{\psi} (2^{-k} \xi) (2^{-j} |\xi|)^{|\alpha|} \hat{\phi} (2^{-j} \xi) \hat{f} (\xi) \]

\[ = \mathcal{W}^{(\alpha)} (2^{-k} \xi) \mathcal{V}^{(\alpha)} (2^{-j} \xi) \hat{f} (\xi) \]

\[ = \mathcal{F} \left[ \mathcal{W}_k^{(\alpha)} \ast \mathcal{V}_j^{(\alpha)} \ast f \right] (\xi) \]

\[ \sum_{j \in \mathbb{Z}} \mathcal{W}_k^{(\alpha)} \ast \mathcal{V}_j^{(\alpha)} \ast b = \mathcal{W}_k^{(\alpha)} \ast (T_V^{(\alpha)} b), \text{ where } T_V^{(\alpha)} f = \sum_{j \in \mathbb{Z}} \mathcal{V}_j^{(\alpha)} \ast f. \]

Then \( T_V^{(\alpha)} \in CZO \) is a convolution operator bounded on \( L^2 \).

Then \( T_V^{(\alpha)} \) is also bounded on \( BMO \), \( T_V^{(\alpha)} b \in BMO \),

\[ \sum_{k \in \mathbb{Z}} 2^{2 |\alpha| k} |Q_k [\Pi_b]_\alpha (x)|^2 \, dx \, \delta_{t=2^{-k}} \approx \sum_{k \in \mathbb{Z}} \left| \mathcal{W}_k^{(\alpha)} \ast (T_V^{(\alpha)} b) (x) \right|^2 \, dx \, \delta_{t=2^{-k}} \]

is a Carleson measure. Therefore \( \Pi_b \) is bounded on \( H^p \).
Littlewood-Paley/Carleson and Atom-to-Molecule Mapping

Atom-to-Molecule approach:

\[ T^*(x^\alpha) = 0 \ & \ T1 = 0 \ \Rightarrow \ T: \text{(atoms)} \rightarrow \text{(molecules)} \ \Rightarrow \ T : H^p \rightarrow H^p \]
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But one of the \( \iff \)'s fails because \( T : H^p \rightarrow H^p \nRightarrow T1 = 0 \)

(Since \( \Pi_b : H^p \rightarrow H^p \), but \( \Pi_b = b \))
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Littlewood-Paley/Carleson measure approach:

\[ T^*(x^\alpha) = 0 & [T]_\alpha \in BMO_{|\alpha|} \implies T : H^p \to H^p \]
Littlewood-Paley/Carleson and Atom-to-Molecule Mapping

Atom-to-Molecule approach:

\[ T^*(x^\alpha) = 0 \text{ and } T1 = 0 \Rightarrow T : \text{(atoms)} \rightarrow \text{(molecules)} \Rightarrow T : H^p \rightarrow H^p \]

But one of the \( \Leftarrow \)'s fails because \( T : H^p \rightarrow H^p \nRightarrow T1 = 0 \)

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Littlewood-Paley/Carleson measure approach:

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\[ T^*(x^\alpha) = 0 \text{ and } [[T]]_\alpha \in BMO_{|\alpha|} \Rightarrow T : H^p \rightarrow H^p \]
A continuous bilinear operator $T$ from $\mathcal{S} \times \mathcal{S}$ into $\mathcal{S}'$ is a BCZO if

$$\langle T(f_1, f_2), g \rangle = \int_{\mathbb{R}^{3n}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2$$

when $\text{supp}(f_1) \cap \text{supp}(f_2) \cap \text{supp}(g) = \emptyset$, and

$$\partial_\alpha x \partial_\beta y_1 \partial_\gamma y_2 K(x, y_1, y_2) \leq C (|x - y_1| + |x - y_2|)^{-\frac{n}{2} - (|\alpha| + |\beta| + |\gamma|)}$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_{\geq 0}$.
A continuous bilinear operator $T$ from $S \times S$ into $S'$ is a BCZO if

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when $\text{supp}(f_1) \cap \text{supp}(f_2) \cap \text{supp}(g) = \emptyset$, where

$$\partial_x^\alpha \partial_{y_1}^\beta \partial_{y_2}^\gamma K(x, y_1, y_2) \leq C(|x - y_1| + |x - y_2|)^{-\left(2n + |\alpha| + |\beta| + |\gamma|\right)}$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$. 

Theorem (Christ-Journé, Grafakos-Torres)

Let $T \in \text{BCZO}$. Then $T \in \text{WBP}$, $T^* \in \text{BMO}$ for $j = 0, 1, 2$ if and only if $T$ is bounded from $L^2 \times L^2$ into $L^1$. 
Bilinear Calderón-Zygmund Operators

A continuous bilinear operator $T$ from $S \times S$ into $S'$ is a BCZO if

$$\langle T(f_1, f_2), g \rangle = \int_{\mathbb{R}^{3n}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2$$

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for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$.

**Theorem (Christ-Journé, Grafakos-Torres)**

Let $T \in \text{BCZO}$. Then $T \in \text{WBP}$, $T^*j(1, 1) \in \text{BMO}$ for $j = 0, 1, 2$ if and only if $T$ is bounded from $L^2 \times L^2$ into $L^1$. 
A trivial example of a bilinear Calderón-Zygmund operator is the product operator

\[ P(f_1, f_2)(x) = f_1(x)f_2(x). \]
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This allows one to identify reasonable boundedness properties to expect for bilinear Calderón-Zygmund operators. For example, if \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \) with \( 0 < p_1, p_2, p \leq 1 \), then \( P \) satisfies

- \( L^{p_1} \times L^{p_2} \rightarrow L^p \) (Hölder’s inequality)
- \( H^{p_1} \times H^{p_2} \rightarrow L^p \) (Grafakos-Kalton)
- \( H^{p_1} \times H^{p_2} \nrightarrow H^p \)
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- \( H^{p_1} \times H^{p_2} \not\to H^p \)

Even the product operator \( P \) does not satisfy the mapping properties we are looking for, \( H^{p_1} \times H^{p_2} \not\to H^p \) when \( p \leq 1 \).
A trivial example of a bilinear Calderón-Zygmund operator is the product operator

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This allows one to identify reasonable boundedness properties to expect for bilinear Calderón-Zygmund operators. For example, if \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \) with \( 0 < p_1, p_2, p \leq 1 \), then \( P \) satisfies

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- \( H^{p_1} \times H^{p_2} \nRightarrow H^p \)

Even the product operator \( P \) does not satisfy the mapping properties we are looking for, \( H^{p_1} \times H^{p_2} \nRightarrow H^p \) when \( p \leq 1 \).

Let \( f \in H^1 \) be nonzero and real-valued. Then \( P(f, f)(x) = f(x)^2 \geq 0 \).

Hence \( P(f, f) \notin H^{1/2} \) and \( P : H^1 \times H^1 \nRightarrow H^{1/2} \).
Consider a bilinear Calderón-Zygmund operator $T$ that does map $H^{p_1} \times H^{p_2}$ to $H^p$ for $0 < p_1, p_2, p \leq 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. 
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- $f_1 \in H^{p_1}$ has oscillatory and regularity properties of order $\frac{1}{p_1}$
- $f_2 \in H^{p_2}$ has oscillatory properties and regularity of order $\frac{1}{p_2}$
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- $T(f_1, f_2) \in H^p$ has better oscillatory and regularity properties
  
  (of order $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$)
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- $f_1 \in H^{p_1}$ has oscillatory and regularity properties of order $\frac{1}{p_1}$
- $f_2 \in H^{p_2}$ has oscillatory properties and regularity of order $\frac{1}{p_2}$
- $T(f_1, f_2) \in H^p$ has better oscillatory and regularity properties (of order $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$)

If $T$ is to be bounded $H^{p_1} \times H^{p_2} \rightarrow H^p$, then $T$ must be cancellation and regularity improving in some sense.
Theorem (H.-Lu)

Let $T \in BCZO$ be bounded $L^2 \times L^2 \rightarrow L^1$ and $L \geq 0$. If $T$ satisfies

$$- \int_{\mathbb{R}^n} T(\psi_1, \psi_2)(x)x^\alpha dx = 0 \text{ for all } |\alpha| \leq n + 2(L + 1)$$
Theorem (H.-Lu)

Let $T \in BCZO$ be bounded $L^2 \times L^2 \rightarrow L^1$ and $L \geq 0$. If $T$ satisfies

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- $[[T]]_{\alpha, \beta} \in BMO_{|\alpha| + |\beta|}$ for all $|\alpha|, |\beta| \leq L$,
Theorem (H.-Lu)

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then $T$ is bounded from $H^{p_1} \times H^{p_2}$ into $H^p$ for all $\frac{n}{n+L+1} < p_1, p_2 \leq 1$ and $\frac{n}{2n+L+1} < p \leq 1$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. 
Theorem (H.-Lu)

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then $T$ is bounded from $H^{p_1} \times H^{p_2}$ into $H^p$ for all $\frac{n}{n + L + 1} < p_1, p_2 \leq 1$ and $\frac{n}{2n + L + 1} < p \leq 1$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Related: Grafakos-Kalton $T : H^{p_1} \times H^{p_2} \rightarrow L^p$, Hu-Meng $T : H^{p_1} \times H^{p_2} \rightarrow H^p$ for $p \approx 1$, Grafakos-Torres, Bényi, and Bényi-Maldonado-Nahmod-Torres.
Theorem (H.-Lu)

Let $T \in BCZO$ be bounded $L^2 \times L^2 \to L^1$ and $L \geq 0$. If $T$ satisfies

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- $[[T]]_{\alpha, \beta} \in BMO_{|\alpha| + |\beta|}$ for all $|\alpha|, |\beta| \leq L$,

then $T$ is bounded from $H^{p_1} \times H^{p_2}$ into $H^p$ for all $\frac{n}{n+L+1} < p_1, p_2 \leq 1$ and $\frac{n}{2n+L+1} < p \leq 1$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Related: Grafakos-Kalton $T : H^{p_1} \times H^{p_2} \to L^p$, Hu-Meng $T : H^{p_1} \times H^{p_2} \to H^p$ for $p \approx 1$, Grafakos-Torres, Bényi, and Bényi-Maldonado-Nahmod-Torres.

Define $[[T]]_{\alpha, \beta}(x) = \int_{\mathbb{R}^{2n}} K(x, y_1, y_2)(x - y_1)^\alpha (x - y_2)^\beta dy_1 dy_2$. 
A collection of bilinear operators \( \{ \Theta_k \} \) given by

\[
\Theta_k(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2
\]
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\[
\Theta_k(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2)f_1(y_1)f_2(y_2)dy_1 dy_2
\]

is in \( BLPSO(N, L) \) if

\[
|D_{y_1}^\alpha D_{y_2}^\beta \theta_k(x, y_1, y_2)| \lesssim 2^k(2n+|\alpha|+|\beta|)(1 + 2^k|x - y_1| + 2^k|x - y_2|)^{-N}.
\]

for \( |\alpha|, |\beta| \leq L \).
A collection of bilinear operators \{\Theta_k\} given by
\[
\Theta_k(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) f_1(y_1) f_2(y_2) \, dy_1 \, dy_2
\]
is in BLPSO(N, L) if
\[
|D_\alpha^\alpha D_\beta^\beta \theta_k(x, y_1, y_2)| \lesssim 2^k(2^n + |\alpha| + |\beta|)(1 + 2^k|x - y_1| + 2^k|x - y_2|)^{-N}.
\]
for $|\alpha|, |\beta| \leq L$. The square function $S_\Theta$ is defined by
\[
S_\Theta(f_1, f_2)(x)^2 = \sum_{k \in \mathbb{Z}} |\Theta_k(f_1, f_2)(x)|^2
\]
Lebesgue space theory: Maldonado, Naibo, H., Grafakos, Oliveira, Chaffee, among others.
A collection of bilinear operators \( \{ \Theta_k \} \) given by

\[
\Theta_k(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2)f_1(y_1)f_2(y_2)dy_1 \, dy_2
\]

is in \( BLPSO(N, L) \) if

\[
|D_{y_1}^\alpha D_{y_2}^\beta \theta_k(x, y_1, y_2)| \lesssim 2^k(2^n + |\alpha| + |\beta|)(1 + 2^k|x - y_1| + 2^k|x - y_2|)^{-N}.
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Lebesgue space theory: Maldonado, Naibo, H., Grafakos, Oliviera, Chaffee, among others.

For \( T \) as in the last theorem, \( \Theta_k(f_1, f_2) = Q_k T(f_1, f_2) \) is a \( BLPSO(N, L) \) and \( S_\Theta \) is bounded \( H^{p_1} \times H^{p_2} \to L^p \).
Theorem (H.-Lu)

Let $L \geq 0$. If $\{\Theta_k\} \in \text{BLPSO}$ and

$$d\mu_{\alpha, \beta}(x, t) = |[[[\Theta_k]]_{\alpha, \beta}(x)|^2 \delta_{t=2^{-k}} \, dx$$

is a Carleson measure for $|\alpha|, |\beta| \leq L$, then $S_\Theta$ is from $H^{p_1} \times H^{p_2}$ into $L^p$ for all $\frac{n}{2n+L+1} < p \leq 1$ and $\frac{n}{n+L+1} < p_1, p_2 \leq 1$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}.$$
Theorem (H.-Lu)

Let $L \geq 0$. If $\{\Theta_k\} \in BLPSO$ and

$$d\mu_{\alpha,\beta}(x, t) = \|[[\Theta_k]]_{\alpha,\beta}(x)\|^2 \delta_{t=2^{-k}} \, dx$$

is a Carleson measure for $|\alpha|, |\beta| \leq L$, then $S_\Theta$ is from $H^{p_1} \times H^{p_2}$ into $L^p$ for all $\frac{n}{2n+L+1} < p \leq 1$ and $\frac{n}{n+L+1} < p_1, p_2 \leq 1$ satisfying

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This is related to the BCZO theory by taking $\Theta_k(f_1, f_2) = Q_k T(f_1, f_2)$. 
Theorem (H.-Lu)

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This is related to the BCZO theory by taking $\Theta_k(f_1, f_2) = Q_k T(f_1, f_2)$.

$$2^{(|\alpha|+|\beta|)k} Q_k [[T]]_{\alpha,\beta}(x) = [[Q_k T]]_{\alpha,\beta}(x) = [[\Theta_k T]]_{\alpha,\beta}(x).$$
Theorem (H.-Lu)

Let $L \geq 0$. If $\{\Theta_k\} \in BLPSO$ and

$$d\mu_{\alpha,\beta}(x, t) = \left| \left[ \left[ \Theta_k \right] \right]_{\alpha,\beta}(x) \right|^2 \delta_{t=2^{-k}} \, dx$$

is a Carleson measure for $|\alpha|, |\beta| \leq L$, then $S_{\Theta}$ is from $H^{p_1} \times H^{p_2}$ into $L^p$ for all $\frac{n}{2n+L+1} < p \leq 1$ and $\frac{n}{n+L+1} < p_1, p_2 \leq 1$ satisfying

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This is related to the BCZO theory by taking $\Theta_k(f_1, f_2) = Q_k T(f_1, f_2)$.

$$2^{(|\alpha|+|\beta|)k} Q_k \left[ T \right]_{\alpha,\beta}(x) = \left[ Q_k T \right]_{\alpha,\beta}(x) = \left[ \Theta_k T \right]_{\alpha,\beta}(x).$$

$$\left[ T \right]_{\alpha,\beta} \in BMO_{|\alpha|+|\beta|} \iff \left[ Q_k T \right]_{\alpha,\beta} \text{ generates a Carleson measure.}$$
Remarks on Proving $Q_k T(f_1, f_2)$ is a BLPSO

$$Q_k T(f_1, f_2)(x) = \langle T(f_1, f_2), \psi_k^x \rangle$$
Remarks on Proving $Q_k T(f_1, f_2)$ is a BLPSO

$$Q_k T(f_1, f_2)(x) = \langle T(f_1, f_2), \psi^x_k \rangle$$

$$= \sum_{j_1, j_2 \in \mathbb{Z}} \langle T(Q_{j_1} f_1, Q_{j_2} f_2), \psi^x_k \rangle$$
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$$= \sum_{j_1, j_2 \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \left\langle T(\psi^{y_1}_{j_1}, \psi^{y_2}_{j_1}), \psi^x_k \right\rangle f_1(y_1) f_2(y_2) dy_1 dy_2$$
Remarks on Proving $Q_k T(f_1, f_2)$ is a *BLPSO*

\[
Q_k T(f_1, f_2)(x) = \langle T(f_1, f_2), \psi_k^x \rangle
\]

\[
= \sum_{j_1, j_2 \in \mathbb{Z}} \langle T(Q_{j_1} f_1, Q_{j_2} f_2), \psi_k^x \rangle
\]

\[
= \sum_{j_1, j_2 \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \left\langle T(\psi_{j_1}^{y_1}, \psi_{j_1}^{y_2}), \psi_k^x \right\rangle f_1(y_1) f_2(y_2) dy_1 dy_2
\]

\[
\theta_k(x, y_1, y_2) = \sum_{j_1, j_2 \in \mathbb{Z}} \left\langle T(\psi_{j_1}^{y_1}, \psi_{j_1}^{y_2}), \psi_k^x \right\rangle
\]
Remarks on Proving $Q_k T(f_1, f_2)$ is a *BLPSO*

$$Q_k T(f_1, f_2)(x) = \left< T(f_1, f_2), \psi^x_k \right>$$

$$= \sum_{j_1, j_2 \in \mathbb{Z}} \left< T(Q_{j_1} f_1, Q_{j_2} f_2), \psi^x_k \right>$$

$$= \sum_{j_1, j_2 \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \left< T(\psi^{y_1}_{j_1}, \psi^{y_2}_{j_1}), \psi^x_k \right> f_1(y_1) f_2(y_2) dy_1 dy_2$$

$$\theta_k(x, y_1, y_2) = \sum_{j_1, j_2 \in \mathbb{Z}} \left< T(\psi^{y_1}_{j_1}, \psi^{y_2}_{j_1}), \psi^x_k \right>$$

Need to show:

$$|\partial^{|\alpha|+|\beta|}_{y_1, y_2} \theta_k(x, y_1, y_2)| \lesssim 2^{(|\alpha|+|\beta|)k} 2^{2kn} (1 + 2^k |x - y_1| + 2^k |x - y_2|)^{-N}$$
Remarks on Proving $Q_k T(f_1, f_2)$ is a BLPSO

\[ Q_k T(f_1, f_2)(x) = \langle T(f_1, f_2), \psi_k^x \rangle = \sum_{j_1, j_2 \in \mathbb{Z}} \langle T(Q_{j_1} f_1, Q_{j_2} f_2), \psi_k^x \rangle = \sum_{j_1, j_2 \in \mathbb{Z}} \int_{\mathbb{R}^{2n}} \langle T(\psi_{y_1}^{j_1}, \psi_{y_2}^{j_1}), \psi_k^x \rangle f_1(y_1) f_2(y_2) \, dy_1 \, dy_2 \]

\[ \theta_k(x, y_1, y_2) = \sum_{j_1, j_2 \in \mathbb{Z}} \langle T(\psi_{y_1}^{j_1}, \psi_{y_2}^{j_1}), \psi_k^x \rangle \]

Need to show:

\[ |\partial_{y_1, y_2}^{\alpha, \beta} \theta_k(x, y_1, y_2)| \lesssim 2^{(|\alpha| + |\beta|)k} 2^{2kn} (1 + 2^k |x - y_1| + 2^k |x - y_2|)^{-N} \]

This requires \[ \int_{\mathbb{R}^n} T(\psi_1, \psi_2)(x)x^\alpha \, dx = 0 \] for \( |\alpha| \leq 2(L + 1) + n \).
A Paraproduct to Replace $P$

Recall: $P(f_1, f_2)(x) = f_1(x)f_2(x)$ is not bounded from $H^{p_1} \times H^{p_2} \to H^p$. 
A Paraproduct to Replace $P$

Recall: $P(f_1, f_2)(x) = f_1(x)f_2(x)$ is not bounded from $H^{p_1} \times H^{p_2} \to H^p$.

Define

$$
\Pi(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k f_2)(x),
$$

where $P_k$ is a nice approximation to the identity and $Q_k P_k = Q_k$. 

Likewise $\Pi(1, f_2)(x) = f_2(x)$. 

Recall: \( P(f_1, f_2)(x) = f_1(x)f_2(x) \) is not bounded from \( H^{p_1} \times H^{p_2} \to H^p \).

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\]
where \( P_k \) is a nice approximation to the identity and \( Q_k P_k = Q_k \).

It follows that \( \int_{\mathbb{R}^n} \Pi(\psi_1, \psi_2)(x)x^\alpha \, dx = 0 \) and \( [[\Pi]]_{\alpha, \beta} = 0 \).
A Paraproduct to Replace $P$

Recall: $P(f_1, f_2)(x) = f_1(x)f_2(x)$ is not bounded from $H^{p_1} \times H^{p_2} \to H^p$.

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It follows that $\int_{\mathbb{R}^n} \Pi(\psi_1, \psi_2)(x)x^\alpha dx = 0$ and $[[\Pi]]_{\alpha, \beta} = 0$.

Then $\Pi$ is bounded $H^{p_1} \times H^{p_2} \to H^p$. 
A Paraproduct to Replace $P$

Recall: $P(f_1, f_2)(x) = f_1(x)f_2(x)$ is not bounded from $H^{p_1} \times H^{p_2} \rightarrow H^p$.

Define

$$
\Pi(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k f_2)(x),
$$

where $P_k$ is a nice approximation to the identity and $Q_k P_k = Q_k$.

It follows that $\int_{\mathbb{R}^n} \Pi(\psi_1, \psi_2)(x)x^\alpha dx = 0$ and $[\Pi]_{\alpha, \beta} = 0$.

Then $\Pi$ is bounded $H^{p_1} \times H^{p_2} \rightarrow H^p$.

Also $\Pi(f_1, f_2) \approx P(f_1, f_2)$ in the sense that

$$
\Pi(f_1, 1) = \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k 1) = \sum_{k \in \mathbb{Z}} Q_k P_k f_1 = \sum_{k \in \mathbb{Z}} Q_k f_1 = f_1
$$
A Paraproduct to Replace $P$

Recall: $P(f_1, f_2)(x) = f_1(x)f_2(x)$ is not bounded from $H^{p_1} \times H^{p_2} \to H^p$.

Define

$$\Pi(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k f_2)(x),$$

where $P_k$ is a nice approximation to the identity and $Q_k P_k = Q_k$.

It follows that $\int_{\mathbb{R}^n} \Pi(\psi_1, \psi_2)(x) x^\alpha dx = 0$ and $[[\Pi]]_{\alpha, \beta} = 0$.

Then $\Pi$ is bounded $H^{p_1} \times H^{p_2} \to H^p$.

Also $\Pi(f_1, f_2) \approx P(f_1, f_2)$ in the sense that

$$\Pi(f_1, 1) = \sum_{k \in \mathbb{Z}} Q_k(P_k f_1 \cdot P_k 1) = \sum_{k \in \mathbb{Z}} Q_k P_k f_1 = \sum_{k \in \mathbb{Z}} Q_k f_1 = f_1$$

Likewise $\Pi(1, f_2) = f_2$. 
For \( b \in BMO \), define

\[
\Pi_b(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(\tilde{Q}_k b \cdot P_k f_1 \cdot P_k f_2)(x)
\]
A Bilinear Bony Paraproduct

For \( b \in BMO \), define

\[
\Pi_b(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(\tilde{Q}_k b \cdot P_k f_1 \cdot P_k f_2)(x)
\]

\( Q_k \) and \( P_k \) can be chosen so that \( \Pi_b(1, 1) = b \). Also \( \Pi_b^*(x^\alpha, \psi) = 0 \)

\[
2^{(|\alpha| + |\beta|)k} Q_k[[\Pi_b]]_{\alpha, \beta}(x) = W_k * (T_V^{(\alpha, \beta)}b)(x),
\]

where \( T_V^{(\alpha, \beta)} \) is bounded on \( BMO \).
A Bilinear Bony Paraproduct

For $b \in BMO$, define

$$\Pi_b(f_1, f_2)(x) = \sum_{k \in \mathbb{Z}} Q_k(\tilde{Q}_k b \cdot P_k f_1 \cdot P_k f_2)(x)$$

$Q_k$ and $P_k$ can be chosen so that $\Pi_b(1, 1) = b$. Also $\Pi^{-1}_b(x^\alpha, \psi) = 0$

$$2^{|\alpha|+|\beta|} k Q_k[[\Pi_b]]_{\alpha, \beta}(x) = W_k \ast \left( T_V^{(\alpha, \beta)} b \right)(x),$$

where $T_V^{(\alpha, \beta)}$ is bounded on $BMO$.

Then $[[\Pi_b]]_{\alpha, \beta} \in BMO_{|\alpha|+|\beta|}$ and $\Pi_b$ is bounded $H^{p_1} \times H^{p_2} \to H^p$ for appropriate $0 < p_1, p_2, p \leq 1$. 


Thank You