New Lyapunov-Type Inequalities for Third-order Linear Differential Equations

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A second-order linear differential equation is
\[ x'' + q(t)x(t) = 0, \quad (1) \]
where \( q \in C([a, b], \mathbb{R}) \).

**Theorem 1**

Assume \( x(t) \) is a solution of Eq. (1) such that \( x(a) = x(b) = 0 \) and \( x(t) \neq 0 \) for \( t \in (a, b) \). Then
\[
\int_a^b |q(t)| \, dt > \frac{4}{b - a}. \quad (2)
\]

The inequality (2) is known as the Lyapunov inequality for Eq. (1).
The Lyapunov inequality was improved by Hartman for the more general equation

\[
\left( r(t) x' \right)' + q(t) x(t) = 0, \quad (3)
\]

where \( q, r \in C([a, b], \mathbb{R}) \) and \( r(t) > 0 \) for \( t \in [a, b] \).

**Theorem 2**

Assume \( x(t) \) is a solution of Eq. (3) such that \( x(a) = x(b) = 0 \) and \( x(t) \neq 0 \) for \( t \in (a, b) \). Then

\[
\int_a^b q^+(t) \, dt > \frac{4}{\int_a^b r^{-1}(t) \, dt}. \quad (4)
\]
The Lyapunov inequality was improved by Hartman for the more general equation

\[
(r(t)x')' + q(t)x(t) = 0, \tag{3}
\]

where \(q, r \in C([a, b], \mathbb{R})\) and \(r(t) > 0\) for \(t \in [a, b]\).

**Theorem 2**

Assume \(x(t)\) is a solution of Eq. (3) such that \(x(a) = x(b) = 0\) and \(x(t) \neq 0\) for \(t \in (a, b)\). Then

\[
\int_{a}^{b} q_+(t) \, dt > \frac{4}{\int_{a}^{b} r^{-1}(t) \, dt}. \tag{4}
\]

We note that (2) can be replaced by

\[
\int_{a}^{b} q_+(t) \, dt > \frac{4}{b - a}.
\]
Parhi and Panigrahi obtained the Lyapunov-type inequality for the third-order linear differential equation

\[ x''' + q(t)x(t) = 0 \quad (5) \]

**Theorem 3**

(a) Assume \( x(t) \) is a solution of Eq. (5) such that \( x(a) = x(b) = 0 \) and \( x(t) \neq 0 \) for \( t \in (a, b) \). If there exists a \( \xi \in [a, b] \) such that \( x''(\xi) = 0 \), then

\[ \int_a^b |q(t)| \, dt > \frac{4}{(b - a)^2}. \]
Parhi and Panigrahi obtained the Lyapunov-type inequality for the third-order linear differential equation

\[ x''' + q(t)x(t) = 0 \]  \( (5) \)

**Theorem 3**

(a) Assume \( x(t) \) is a solution of Eq. (5) such that \( x(a) = x(b) = 0 \) and \( x(t) \neq 0 \) for \( t \in (a, b) \). If there exists a \( \xi \in [a, b] \) such that \( x''(\xi) = 0 \), then

\[ \int_{a}^{b} |q(t)| \, dt > \frac{4}{(b - a)^2}. \]

(b) If \( x''(t) \neq 0 \) for \( t \in [a, b] \) and \( x(t) \) has three consecutive zeros such that \( a < b < c \), then

\[ \int_{a}^{c} |q(t)| \, dt > \frac{4}{(c - a)^2}. \]
We considered third-order half-linear differential equations of the form

\[
\left( \phi_{\alpha_2} \left( \left( \phi_{\alpha_1} (x') \right)' \right) \right)' + q(t) \phi_{\alpha_1 \alpha_2} (x) = 0, \tag{6}
\]

where \( q \in C(\mathbb{R}, \mathbb{R}) \), \( \phi_p (x) = |x|^p \text{sgn} x \), and \( \alpha_1, \alpha_2 > 0 \). For simplicity we denote \( \alpha = (\alpha_1 + 1) \alpha_2 \).
We considered third-order half-linear differential equations of the form

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where \( q \in C(\mathbb{R}, \mathbb{R}) \), \( \phi_p(x) = |x|^p \text{sgn}x \), and \( \alpha_1, \alpha_2 > 0 \). For simplicity we denote \( \alpha = (\alpha_1 + 1) \alpha_2 \).

**Theorem 4**

Assume \( x(t) \) is a solution of Eq. (6) with \( x(a) = x(b) = 0 \) and \( x(t) \neq 0 \) for all \( t \in (a, b) \). Suppose there is a \( \xi \in [a, b] \) such that \( (\phi_{\alpha_1} (x'))' (\xi) = 0 \). Then

\[
\int_a^\xi q_-(t) dt + \int_\xi^b q_+ (t) dt > \left( \frac{2}{b-a} \right)^\alpha. \tag{7}
\]
Theorem 5

Assume $x(t)$ is a solution of Eq. (6) with $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for all $t \in (a, b) \cup (b, c)$. Then either

$$\max_{\xi \in [a,b]} \left\{ \int_{a}^{\xi} q_-(t) dt + \int_{\xi}^{b} q_+(t) dt \right\} > \left( \frac{2}{b-a} \right)^\alpha$$

or

$$\max_{\xi \in [b,c]} \left\{ \int_{b}^{\xi} q_-(t) ds + \int_{\xi}^{c} q_+(t) dt \right\} > \left( \frac{2}{c-b} \right)^\alpha .$$

As a result

$$\max_{\xi \in [a,c]} \left\{ \int_{a}^{\xi} q_-(t) dt + \int_{\xi}^{c} q_+(t) dt \right\} > \left( \frac{2}{c-a} \right)^\alpha .$$
A third-order linear differential equation is

\[ x''' + q(t)x(t) = 0. \tag{8} \]

Here \( \alpha_1 = \alpha_2 = 1 \) and \( \alpha = 2 \).

**Corollary 6**

(a) Assume \( x(t) \) is a solution of Eq. (8) with \( x(a) = x(b) = 0 \) and \( x(t) \neq 0 \) for \( t \in (a, b) \). Suppose there is a \( \xi \in [a, b] \) such that \( x''(\xi) = 0 \). Then

\[
\int_a^\xi q_-(s)ds + \int_\xi^b q_+(s)ds > \frac{4}{(b - a)^2}.
\]

(b) Assume \( x(t) \) is a solution of Eq. (8) with \( x(a) = x(b) = x(c) = 0 \) and \( x(t) \neq 0 \) for all \( t \in (a, b) \cup (b, c) \). Then

\[
\max_{\xi \in [a, c]} \left\{ \int_a^\xi q_-(s)ds + \int_\xi^c q_+(s)ds \right\} > \frac{4}{(c - a)^2}.
\]
Theorem 7

Assume Eq. (8) has a solution \( x(t) \) satisfying

\[
x(a) = x(b) = x(c) = 0 \quad \text{and} \quad x(t) \neq 0 \quad \text{for} \quad t \in (a, b) \cup (b, c).
\]  

(9)

Then one of the following holds:

(a) \[ \int_a^c q_-(t)dt > \frac{8}{(c - a)^2}, \]

(b) \[ \int_a^c q_+(t)dt > \frac{8}{(c - a)^2}, \]

(c) \[ \int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{8}{(c - a)^2}. \]

As a result,

\[ \int_a^c |q(t)| dt > \frac{8}{(c - a)^2}. \]
Proof. We have \(|x(t_1)| = \max \{|x(t)| : t \in [a, b]\} \) and \(|x(t_2)| = \max \{|x(t)| : t \in [b, c]\}\). As a result,

\[ x'(t_1) = x'(t_2) = 0. \] (10)

Let

\[ G(t, s) = \frac{1}{t_2 - t_1} \begin{cases} (s - t_1)(t_2 - t), & t_1 \leq s \leq t \leq t_2; \\ (t - t_1)(t_2 - s), & t_1 \leq t \leq s \leq t_2. \end{cases} \]

Then \(G(t, s)\) is the Green’s function for the BVP

\[ -y''' = r(t), \quad y(t_1) = y(t_2) = 0. \] (11)

Hence the solution \(y(t)\) satisfies

\[ y(t) = \int_{t_1}^{t_2} G(t, s)r(s)ds. \] (12)
Substituting $y(t) := x'(t)$ and $r(t) = q(t)x(t)$, we get

$$x'(t) = \int_{t_1}^{t_2} G(t, s)q(s)x(s)ds.$$ 

It follows that

$$x(t) = \int_b^t \int_{t_1}^{t_2} G(\tau, s)q(s)x(s)dsd\tau = \int_{t_1}^{t_2} \left( \int_b^t G(\tau, s)d\tau \right) q(s)x(s)ds. \quad (13)$$

Without loss of generality, we may assume $x(t)$ satisfies one of following cases:

I. $x(t) > 0$ on $(a, b) \cup (b, c)$ and $x(t_1) \geq x(t_2)$;

II. $x(t) > 0$ on $(a, b) \cup (b, c)$ and $x(t_1) < x(t_2)$;

III. $x(t) > 0$ on $(a, b)$, $x(t) < 0$ on $(b, c)$, and $x(t_1) \geq -x(t_2)$;

IV. $x(t) > 0$ on $(a, b)$, $x(t) < 0$ on $(b, c)$, and $x(t_1) < -x(t_2)$.

In the sequel, we denote $m = \max\{|x(t_1)|, |x(t_2)|\}.$
Case I: In this case, $m = x(t_1)$. Then (13) with $t = t_1$ shows that

$$m = \int_{t_1}^{t_2} \left( \int_{t_1}^{b} G(t, s)dt \right) (-q(s))x(s)ds.$$

Using the facts that $G(t, s) \geq 0$ on $[t_1, t_2] \times [t_1, t_2]$, $0 \leq x(t) \leq m$ and $x(t) \neq m$ on $[t_1, t_2]$, and $-q(t) \leq q_-(t)$, we have

$$m < m \int_{t_1}^{t_2} \left( \int_{t_1}^{t_2} G(t, s)dt \right) q_-(s)ds.$$

Note that for $s \in [t_1, t_2]$

$$\int_{t_1}^{t_2} G(t, s)dt \leq \frac{1}{2}(s - a)(c - s)$$
Third-order Linear Case

Hence we obtain

\[ 1 < \frac{1}{2} \int_{t_1}^{t_2} (s - a)(c - s)q_-(s)ds \leq \frac{1}{2} \int_a^c (s - a)(c - s)q_-(s)ds, \]

We note that \( 4(s - a)(c - s) \leq (c - a)^2 \). Then

\[ \int_a^c q_-(t)dt > \frac{8}{(c - a)^2} \]

The proof for the rest is similar. We omit the details.
Third-order Linear Case

We first generalize the previous results to the following differential equation

\[(p(t)x''')' + q(t)x = 0, \quad (14)\]

where \(p, q \in C([a, c], \mathbb{R})\) and \(p(t) > 0\) for \(t \in [a, c]\). We denote \(P = \max\{p(t) : t \in [a, c]\}\).

**Theorem 8**

Assume Eq. (14) has a solution \(x(t)\) satisfying (9). Then one of the following holds:

(a) \[\int_a^c \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) q_-(t) dt > \frac{2}{P},\]

(b) \[\int_a^c \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) q_+(t) dt > \frac{2}{P},\]

(c) \[\int_a^b \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) q_-(t) dt + \int_c^b \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) q_+(t) dt > \frac{2}{P}.\]
We next consider the equation

$$x'''' + f(t)x'' + h(t)x = 0,$$

where $f, h \in C([a, c], \mathbb{R})$. We denote

$$p(t) = \exp \left( \int_a^t f(s) ds \right) \quad \text{and} \quad P = \exp \left( \max \left\{ \int_a^t f(s) ds : t \in [a, c] \right\} \right).$$

**Theorem 9**

Assume Eq. (15) has a solution $x(t)$ satisfying (9). Then one of the following holds:

(a) $$\int_a^c \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) p(t) h_-(t) dt > \frac{2}{P},$$

(b) $$\int_a^c \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) p(t) h_+(t) dt > \frac{2}{P},$$

(c) $$\int_a^b \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) p(t) h_-(t) dt$$

$$+ \int_t^c \left( \int_a^t \frac{d\tau}{p(\tau)} \right) \left( \int_t^c \frac{d\tau}{p(\tau)} \right) p(t) h_+(t) dt > \frac{2}{P}.$$
Finally, we consider the general third-order linear equation

\[ x''' + f(t)x'' + g(t)x' + h(t)x = 0, \]  

where \( f, g, h \in C([a, c], \mathbb{R}) \). We denote \( p(t) = \exp(\int_a^t f(s)ds) \).

**Theorem 10**

Assume Eq. (16) has a solution \( x(t) \) satisfying (9). Then

\[ \int_a^c \left( |g(t)| + (c - a)|h(t)| \right) p(t)dt > \frac{4}{\int_a^c p^{-1}(t)dt}. \]  

(17)
References


Thank You For Attending!!!