OPEN PROBLEMS RELATED TO OPERATOR ALGEBRAS ON  
$L^p$ SPACES

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ABSTRACT. We state a number of problems suggested by recent work on  
$L^p$ analogs of Cuntz algebras, UHF algebras, group C*-algebras, and crossed  
products.

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This is a collection of problems suggested by recent work on analogs of Cuntz  
algebras and UHF algebras acting on $L^p$ spaces, as well as group $L^p$ operator  
algebras and $L^p$ operator crossed products. It is a greatly expanded version of  
an earlier collection [23], which was written for a conference on graph algebras in  
algebra and in operator algebras. (Several of the problems in [23] have since been  
solved, and are not listed here.) The problems are grouped in sections of (sometimes  
loosely) related problems. Some of them, as noted, are already being worked on.  
The expected interest and difficulty varies widely, and is often very uncertain.

The main references for work already done are Section 5 of [1], [24], [25], [26],  
[27], and [32]. The arXiv preprints [24], [25], [26], and [27] are scheduled to be  
updated at some point, with minor changes to the first three and significant new

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material in [27]. (In particular, the full and reduced operator crossed products by an amenable group are the same.)

Essentially no proofreading has been done since the (rather shorter) version [23] of 28 April 2013. Many references are missing. Moreover, there are other sets of problems, on $L^p$ versions of AF algebras and on incompressible $L^p$ operator algebras, which are not included here. There are also individual missing problems. The amount and quality of the commentary and references is very uneven.

1. Generalizations of spatial $L^p$ analogs of Cuntz algebras

Problem 1.1. Extend results for $O^p_d$ to $O^p_∞$. (We do not expect all the equivalent conditions for a representation to be spatial to be still equivalent. But any representation $ρ$ for which $ρ(s_j)$ is a spatial partial isometry with reverse $ρ(t_j)$ for all $j$ should be “good”.)

Is $O^p_∞$ purely infinite and simple? What is its K-theory?

Problem 1.2. Extend results for $O^p_d$ to $L^p$ analogs of the extended Cuntz algebras $E^p_d$. (The same comment applies as in Problem 1.1.)

Problem 1.3. Suppose that $M_{n_1}(L_{d_1}) ∼= M_{n_2}(L_{d_2})$. (This is known to be equivalent to $M_{n_1}(O_{d_1}) ∼= M_{n_2}(O_{d_2})$.) Does it follow that $M_{n_1}(O^p_{d_1}) ∼= M_{n_2}(O^p_{d_2})$? (K-theory shows that the reverse implication holds.)

A particular example to consider, suggested by Gene Abrams, is whether $M_3(O^p_5)$ is isomorphic to $O^p_2$. In this case, the isomorphisms for Leavitt algebras and C*-algebras don’t send the standard generators to single words in the standard generators, which has the potential to cause problems with norms in the setting of $L^p$ operator algebras.

Problem 1.4. (This problem is being worked on by a graduate student, María Eugenia Rodríguez.) Extend results for $O^p_d$ to $L^p$ analogs of graph algebra, or at least subclasses (such as algebras of finite graphs, perhaps with no sources or no sinks, or such as Cuntz-Krieger algebras).

Problem 1.5. This problem is a followup to Problem 1.4. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$ (This matrix has been called “2−”.) Then $O_2 ∼= O_A$ but the sign of the determinant flow equivalence invariant is opposite to that for the matrix corresponding to $O_2$, namely $\det(1 - A^t) > 0$. Is $O^p_A$ isomorphic to $O^p_2$?

Problem 1.6. This is also a followup to Problem 1.4. Suppose $E$ and $F$ are graphs (perhaps in a suitable subclass), and $L^C(E) ∼= L^C(F)$. Does it follow that the spatial $L^p$ analogs of these algebras (providing they can be defined) are isomorphic?

Problem 1.7. The space $l^p$ embeds in $O^p_∞ ⊂ L(L^p(X, μ))$ in a very nice algebraic manner: as the closed linear span of the generating isometries. This was used to show that if $ϕ: O^p_∞ → L(E)$ is a continuous homomorphism, then $E$ contains a subspace isomorphic to $l^p$. In particular, this is true whenever there is a continuous homomorphism from $L(L^p(X, μ))$ to $L(E)$. 
Can one embed $L^p([0, 1])$ in a similar way in a subalgebra of $L(L^p([0, 1]))$? Does it then follow that if $\psi: L(L^p([0, 1])) \to L(E)$ is a continuous homomorphism, then $E$ contains a subspace isomorphic to $L^p([0, 1])$? When $p = 2$, there is such a construction, using creation and annihilation operators on the Fock space (actually, for any separable Hilbert space). However, one gets nothing new, since every separable Hilbert space is isometrically isomorphic to $l^2$. Presumably when $p \neq 2$, if this construction works (it might not be too hard to imitate the Fock space construction), one gets different algebras.

**Problem 1.8.** Let $p \in (1, \infty) \setminus \{2\}$, and let $s \in L(l^p(\mathbb{Z}_{>0}))$ be the unilateral shift. Let $T_p$ be the norm closed subalgebra of $L(l^p(\mathbb{Z}_{>0}))$ generated by $s$ and its reverse, the backwards shift. This algebra contains $K(l^p(\mathbb{Z}_{>0}))$. What is the quotient? It is a commutative unital Banach algebra generated by an invertible element and its inverse. Is it isomorphic to the closed subalgebra of $L(l^p(\mathbb{Z}))$ generated by the bilateral shift and its inverse? Is the maximal ideal space isomorphic to $S^1$? Which functions on the maximal ideal space are in the algebra?

(Something, but not much, is known about the closed subalgebra of $L(l^p(\mathbb{Z}))$ generated by the bilateral shift and its inverse.)

2. **Nonspatial representations and representations on other Banach spaces**

**Problem 2.1.** What sort of algebras does one get as $\overline{\rho(L_d)}$ for representations $\rho: L_d \to L(L^p(X, \mu))$ which are not spatial? Are they simple? Are they purely infinite? Are they amenable? Do they have the same K-theory?

**Problem 2.2.** The spatial $L^p$ Cuntz algebras seem to be “minimal” in some sense. Is there something that deserves to be called a “maximal” $L^p$ analog of $O_d$? If there is, what can one say about it? In particular, what about the questions in Problem 2.1?

**Problem 2.3.** What are the right spaces to choose when $p = \infty$? One could consider representations on any of $c_0 = \mathbb{C}_0(\mathbb{Z}_{>0})$, $l^\infty$, $L^\infty(X, \mu)$ for a $\sigma$-finite measure space $(X, \mathcal{B}, \mu)$, or $C(X)$ for a compact metric space $X$. (Lamperti’s Theorem fails for $p = \infty$, so the resulting theory might be quite different.)

**Problem 2.4.** Develop the theory for representations of $L_d$ on more general Banach spaces than $L^p(X, \mu)$. Which Banach spaces admit representations of $L_d$? Which ones admit representations $\rho$ which are contractive on generators, strongly forward isometric, or such that both $\rho$ and $\rho'$ are strongly forward isometric? (Some of these answers are probably known. Certainly not all separable Banach spaces admit representations at all. There is a recently constructed infinite dimensional separable Banach space on which every bounded linear map is of the form scalar plus compact operator.) Can one find representations of $L_d$ on nonisomorphic Banach spaces such that the closures of the ranges are nevertheless isomorphic (isometrically isomorphic) as Banach algebras? Can one find a representation $\rho: L_d \to L(E)$ such that $\overline{\rho(L_d)}$ is not simple? Can one find one such that $\overline{\rho(L_d)}$ has the “wrong” K-theory?

Which Banach spaces $E$ admit representations $\rho: L_d \to L(E)$ such that $\overline{\rho(L_d)}$ is simple, or has the “right” K-theory?

The following condition on a unital representation $\rho$ of $L_d$ on a Banach space $E$ seems to be something one might want to require for $\rho$ to be “reasonable”.

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Definition 2.5. Let $E$ be a Banach space, and let $\rho: L_d \to L(E)$ be a representation. We say that $\rho$ is strongly forward isometric if $\rho(s_j)$ is an isometry for every $j$ and for every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$, the element $\rho \left( \sum_{j=1}^d \lambda_j s_j \right)$ is a scalar multiple of an isometry.

Problem 2.6. A strongly forward isometric representation $\rho$ of $L_d$ on a Banach space $E$ defines a norm $\| \cdot \|_\rho$ on $\mathbb{C}^d$ by the formula, for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$,

$$\left\| \sum_{j=1}^d \lambda_j \rho(s_j) \xi \right\| = \| \lambda \|_\rho \| \xi \|$$

for all $\xi \in E$. Which norms on $\mathbb{C}^d$ can arise this way? What if one requires that Equation (2.1) hold for all $\lambda \in \mathbb{C}^d$ and $\xi \in E$, but drops the requirement that $\rho(s_j)$ be an isometry for all $j$? What if one restricts $E$ to lie in some special class of Banach spaces, such as all $L^p(X, \mu)$ for some fixed $p$? What if one adds the requirement that the representation $\rho'$ also be strongly forward isometric?

Problem 2.7. Fix a norm on $\mathbb{C}^d$ which corresponds to a strongly forward isometric representation $\rho$ as in Problem 2.6. Let $\sigma$ be some other strongly forward isometric representation of $L_d$ which yields the same norm on $\mathbb{C}^d$. Does it follow that there is an (isometric) isomorphism from $\rho(L_d)$ to $\sigma(L_d)$ which sends generators to generators? (This is true for the norm corresponding to spatial representations, provided one restricts to $L^p$ spaces of $\sigma$-finite measure spaces.)

Problem 2.8. A strongly forward isometric representation $\rho$ of $L_\infty$ (definition similar to Definition 2.5) on a Banach space $E$ defines a norm $\| \cdot \|_\rho$ on $\mathbb{C}^\infty$ by the formula, for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{C}^\infty$,

$$\left\| \sum_{j=1}^\infty \lambda_j \rho(s_j) \xi \right\| = \| \lambda \|_\rho \| \xi \|$$

for all $\xi \in E$. Completing $\mathbb{C}^\infty$ in this norm, we get a Banach space of sequences.

First, which Banach spaces can arise this way? What can one say about them?

Second, suppose $p$ is fixed. Consider a representation $\rho$ obtained in the following manner. Let $\rho_0$ be a spatial representation of $L_\infty$ on $L^p$ (or perhaps a more general strongly forward isometric representation on $L^p$ or on $L^p(X, \mu)$ for some $\sigma$-finite measure space $(X, \mathcal{B}, \mu)$), and let $\varphi: \rho_0(L_\infty) \to L(E)$ be a continuous homomorphism. Then take $\rho = \varphi \circ \rho_0$. Now which Banach spaces can arise? Also, what are the possibilities for $E$? We know that $E$ must have a subspace isomorphic to $L^p$. Can anything else be said?

3. $L^p$ analogs of UHF algebras and generalizations

Problem 3.1. For distinct values of $p_1$ and $p_2$ in $(1, \infty)$, and arbitrary finite $d_1$ and $d_2$, there is no nonzero continuous homomorphism from $\mathcal{O}^{(p_1)}_{d_1}$ to $\mathcal{O}^{(p_2)}_{d_2}$. There is always least one direction (from $p_1$ to $p_2$ or from $p_2$ to $p_1$) in which we know there can be no nonzero continuous homomorphism between spatial $L^p$ UHF algebras. What about the other direction?

Problem 3.2. (This problem is being worked on by Maria Grazia Viola.) Generalize the theory of spatial $L^p$ UHF algebras to spatial $L^p$ AF algebras, including K-theoretic classification, ideal structure, etc.
Problem 3.3. For $p \neq 2$, there are representations of $M_p^d$ on $L^p(X,\mu)$ which are contractive on the standard matrix units but are not isometric. Allowing these presumably gives more general $L^p$ UHF algebras than the spatial ones. If we require that the maps in the direct system still be contractive, do we actually get new algebras? If so, the K-theory must be the same, but are the algebras still simple? (This might follow quickly from results already proved.) Are they isomorphic, or isometrically isomorphic, to the spatial ones?

Problem 3.4. Study analogs of $L^p$ UHF algebras on other Banach spaces or other families of Banach spaces. If $(e_{j,k})^{d}_{j,k=1}$ is the standard system of matrix units in $M_d$, then the natural condition on a representation $\rho: M_d \to L(E)$ seems, at first sight, to be that $\|\rho(e_{j,k})\| \leq 1$ for all $j$ and $k$. When can one construct direct systems as for the UHF algebras (with or without this norm condition) such that all the maps in the system are contractive, or all the maps in the system are isometric? The K-theory of the resulting algebras must always be as expected, but when are they simple? When are two of them isomorphic?

Problem 3.5. For $p \in (1, \infty)$, the algebra $O_p^d$ is a corner in an $L^p$ operator reduced crossed product by $\mathbb{Z}$ of the tensor product of a spatial $L^p$ UHF algebra and $K(l_p)$. (One uses something a bit different from $K(l_p)$ when $p = 1$.) What if one uses a nonspatial $L^p$ UHF algebra? Are there nonspatial UHF algebras on which suitable actions exist? For which representations $\rho$ of $L_d$ can one get $\rho(L_d)$ as a crossed product in an analogous way?

4. Representation theory

Problem 4.1. For fixed $p \in [1, \infty)$, find all representations (not necessarily spatial) of $M_d$ on an $L^p$ space. Probably easier: Find all unital representations. For $p = 2$, they are all similar to *-homomorphisms. For $p \neq 2$, are all unital representations similar to spatial representations? Are all unital representations direct integrals, in some suitable sense, of representations of the form $a \mapsto sas^{-1}$, from $M_d$ to $M_d$?

Problem 4.2. Is there a useful coarse classification of representations of $L_d$ on spaces $L^p(X,\mu)$ which are contractive on generators? Can every representation of $L_d$ on $L^p(X,\mu)$ be decomposed in some way into strongly forward isometric representations? A direct sum decomposition is surely too much to hope for, but what about some kind of direct integral decomposition?

Perhaps one should restrict to strongly forward isometric representations to begin with. Can one usefully divide them into any more classes than spatial and nonspatial?

Problem 4.3. Classify isometric representations of $O_p^d$ on spaces of the form $L^p(X,\mu)$ up to a suitable equivalence relation. One should bear in mind that, since $O_d$ is a C*-algebra not of type I, the classification of its representations up to unitary equivalence is generally regarded as hopeless. There are several features for $p \neq 2$ which might lead to a different outcome. First, as shown by Lamperti’s Theorem, there are many fewer isometries on $L^p(X,\mu)$ for $p \neq 2$, and hence presumably fewer representations. On the other hand, this also makes it much more difficult for representations to be isometrically equivalent. Furthermore, even the spaces on which we are representing the algebras are not all isometrically isomorphic.

One possibility is to ask for a classification only up to approximate isometric equivalence. For $p = 2$, any two representations on a separable Hilbert space are
approximately unitarily equivalent, by Voiculescu’s Theorem. But the analog might not be true for \( p \neq 2 \). Another possibility is to restrict to representations which are free or approximately free. A third possibility is to consider only representations which are part of a family which varies continuously with \( p \), giving perhaps a representation of a continuous field of the Banach algebras \( \mathcal{O}_d^p \) on a continuous field of the Banach spaces \( L^p(X, \mu) \). (We have not checked that one even gets a continuous field in either case, although it seems likely.)

5. General theory of \( L^p \) operator algebras

Problem 5.1. Is there an abstract characterization of Banach algebras which are isometrically (or isomorphically) representable on a Banach space of the form \( L^p(X, \mu) \)? Or is there an abstract characterization of a particular class of such algebras? A characterization might involve extra structure. For example, if \( A \subset L(L^p(X, \mu)) \), then \( M_n(\mathcal{A}) \subset L(L^p(\{1, 2, \ldots, n\} \times X)) \), so \( A \) has an \( L^p \) analog of operator space structure.

Something weaker is known: see Corollary 1.5.2.2 of [17].

Problem 5.2. Let \((X, \mathcal{B}, \mu)\) be a measure space, let \( A \subset L(L^p(X, \mu)) \) be a closed subalgebra, and let \( I \subset A \) be a closed ideal, possibly assuming extra conditions. Does it follow that there is a measure space \((Y, \mathcal{C}, \nu)\) such that \( A \) is isometrically isomorphic to a closed subalgebra of \( L(L^p(Y, \nu)) \)?

This seems hard. A much weaker theorem is known for \( L^p \) operator spaces [20]. A somewhat weaker result is known for algebras: the problem has a positive solution if one considers the class of subalgebras of the algebras of bounded operators on closed subspaces of \( L^p \) spaces instead of the algebras of bounded operators on \( L^p \) spaces. See Corollary 1.5.2.3 of [17]. However, the statement we have in mind may well not be true.

Here is a special case.

Problem 5.3. Let \( p \in [1, \infty) \setminus \{2\} \). Is there a measure space \((Y, \mathcal{C}, \nu)\) such that the \( L^p \) Calkin algebra \( L(L^p(\mathbb{Z}))/K(L^p(\mathbb{Z})) \) is isometrically isomorphic to a subalgebra of \( L(L^p(Y, \nu)) \)? Is there a measure space \((Y, \mathcal{C}, \nu)\) such that the \( L^p \) Calkin algebra \( L(L^p([0, 1]))/K(L^p([0, 1])) \) is isometrically isomorphic to a subalgebra of \( L(L^p(Y, \nu)) \)? If so, can \((Y, \mathcal{C}, \nu)\) be chosen to be \( \sigma \)-finite?

Other than appealing to the fact that quotients of \( C^* \)-algebras are \( C^* \)-algebras, and the Gelfand-Naimark Theorem, I know of no way to prove this in the case \( p = 2 \). (The Calkin algebra \( L(L^2(\mathbb{Z}))/K(L^2(\mathbb{Z})) \) can’t be represented on a separable Hilbert space, since it contains uncountably many mutually orthogonal nonzero projections.)

Problem 5.4. Let \( I \) be a directed set, and let \((A_i)_{i \in I}, (\varphi_{ij})_{i \leq j}\) be a direct system of \( L^p \) operator algebras with contractive homomorphisms. Is the direct limit \( A = \lim_{\rightarrow} A_i \) again an \( L^p \) operator algebra?

One might want to assume that the maps in the system are isometric, or consider the matricially normed case and assume that they are completely contractive or completely isometric. The answer is yes if the maps are isometric and there is a collection of compatible contractive linear (not necessarily multiplicative) left
inverses $T_{i,j}: A_j \rightarrow A_i$ for $i, j \in I$ with $i \leq j$. One might also restrict to direct limits over $\mathbb{Z}_{\geq 0}$ instead of over a general directed set.

The case of infinite tensor products can be handled as was done for UHF algebras in [25]. For the general $C^*$ case, where the result is true, the only proof I know, even for direct systems over $\mathbb{Z}$ and with injective homomorphisms, but without assuming the existence of linear left inverses depends on the abstract characterization of $C^*$-algebras (Gelfand-Naimark Theorem).

We have defined a spatial $L^p$ operator tensor product $A \otimes_p B$ of $L^p$ operator algebras $A \subset L(L^p(X, \mu))$ and $B \subset L(L^p(Y, \nu))$, when $\mu$ and $\nu$ are $\sigma$-finite. Even for norm closed but nonselfadjoint algebras of operators on Hilbert spaces, the tensor product depends on how the algebras are represented. (In the case $M_n \otimes_p B$, it depends on how $B$ is represented.) See the discussion after Example 1.15 of [27]. However, if $A \subset L(L^p(X, \mu))$ and $B \subset L(L^p(Y, \nu))$ are given the obvious $L^p$ matricially normed structures, then using any other completely isometric representations on $L^p$ spaces yields the same completed tensor product $A \otimes_p B$ up to completely isometric isomorphism. This follows from Theorem 3.2 of [1] and Proposition 2.8 of [21].

**Problem 5.5.** Is there a useful $L^p$ analog of the maximal tensor product of $C^*$-algebras? If so, are there any general conditions under which the spatial and maximal tensor products are the same? (It would be really great if this followed from Banach algebra amenability of both factors or $p$-nuclearity of one factor. A result with Banach algebra amenability as a hypothesis is probably much too much to hope for. It can’t follow from Banach algebra amenability of just one factor.)

6. Tensor products of $L^p$ analogs of Cuntz algebras

**Problem 6.1.** Let $d_1, d_2 \in \{2, 3, \ldots\}$. Is the $L^p$ tensor product $O_{d_1} \otimes_p O_{d_2}$ simple? For $d \in \{2, 3, \ldots\}$, is the $L^p$ tensor product of $O_{d_1}$ with a spatial $L^p$ UHF algebra simple?

Are these tensor products purely infinite?

**Problem 6.2.** We assume that tensor products of the type considered in Problem 6.1 are in fact simple and products purely infinite. Elliott’s Theorem asserts that $O_2 \otimes O_2 \cong O_2$. More generally, the classification of purely infinite simple separable nuclear $C^*$-algebras satisfying the Universal Coefficient Theorem provides many isomorphisms of tensor products of Cuntz algebras and also of their tensor products with certain other algebras. For example, $O_3 \otimes O_4 \cong O_2$. (However, $O_3 \otimes O_3$ is not a Cuntz algebra.) If $D$ is the UHF algebra of type $2^\infty$, then $D \otimes O_2$ and $D \otimes O_3$ are both isomorphic to $O_2$, but $D \otimes O_4 \cong O_4$.

None of these isomorphisms is valid in the purely algebraic situation. It is known that the tensor product of two Leavitt algebras is never a Leavitt algebra. Leavitt algebras are finitely generated, while the tensor product of one of them with the algebraic analog of the UHF algebra of type $2^\infty$ is never finitely generated, so this kind of tensor product can also never be a Leavitt algebra.

What happens for the $L^p$ analogs of these algebras? One can first test by computing the K-theory. For $C^*$-algebras, one uses normally uses the Künneth formula, but more primitive methods are available for the examples given. Presumably one gets the same K-theory as in the $C^*$-algebra case. Isomorphism of the algebras, however, seems much more problematic. 
7. General theory of $L^p$ operator crossed products

In the following problems, $F^p(G,A,\alpha)$ is the full $L^p$ operator crossed product and $F^p_1(G,A,\alpha)$ is the reduced $L^p$ operator crossed product. Recall that if $G$ is amenable, then $\kappa_r: F^p(G,A,\alpha) \to F^p_1(G,A,\alpha)$ is an isometric isomorphism.

**Problem 7.1.** Let $p \in (1,\infty) \setminus \{2\}$, let $G$ be a second countable locally compact group, and assume that the map $\kappa_r: F^p(G) \to F^p_1(G)$ is an isometric isomorphism. Does it follow that $G$ is amenable, as in the C*-algebra case?

The restriction $p \neq 1$ is necessary, since both $F^1(G)$ and $F^1_1(G)$ can be canonically identified with $L^1(G)$. There might be a C* proof in the literature which is easily adapted (possibly using positive definite functions), but there might not be, and the problem might be hard.

**Problem 7.2.** Let $p \in (1,\infty) \setminus \{2\}$, let $G$ be a second countable locally compact group, and let $(G,A,\alpha)$ be an isometric nondegenerately representable separable $G$-$L^p$ operator algebra. Is the map $\kappa_r: F^p(G,A,\alpha) \to F^p_1(G,A,\alpha)$ necessarily surjective? Is it possible for this map to be injective but not an isomorphism?

I don’t know how to start. It seems unlikely that the map is injective if $A = \mathbb{C}$ and $G$ isn’t amenable, but I don’t know. If $G = F_n$ with $n \geq 2$, then $F^p(G)$ has a closed ideal $I$ of codimension 1 (coming from the one dimensional trivial representation), while $F^p_1(G)$ is simple. However, for $p \neq 2$, I don’t see any obvious reason that $\kappa_r(I)$ can’t be dense in $F^p_1(G)$.

**Problem 7.3.** For $p = 1$, it is known that the map $\kappa_r: F^1(G) \to F^1_1(G)$ is always an isometric isomorphism, even when $G$ is not amenable. (This map is essentially the identity map on $L^1(G)$.) What about $\kappa_r: F^1(G,A,\alpha) \to F^1_1(G,A,\alpha)$ for an isometric action of $G$ on an $L^1$ operator algebra $A$?

It might be easy to prove that $\kappa_r: F^1(G,A,\alpha) \to F^1_1(G,A,\alpha)$ is always an isometric isomorphism, or it might be false.

**Problem 7.4.** Let $G$ be a second countable locally compact abelian group. Is the maximal ideal space of $F^p(G)$ isomorphic to $\hat{G}$? Is the Gelfand transform injective?

This is known for all $p$ when $G$ is discrete, and for $p = 1$ and $p = 2$ for arbitrary locally compact abelian $G$. The best guess is that the proof for discrete $G$ needs to be modified to account for the fact that the group elements are not in $F^p(G)$, in ways that can be seen by looking at the proofs of the general case for $p = 1$ and $p = 2$.

**Problem 7.5.** Let $G$ be a second countable locally compact abelian group which is not compact. Let $p \in (1,\infty) \setminus \{2\}$. Does $F^p(G)$ have spectral synthesis?

Spectral synthesis for a commutative Banach algebra $A$ means, in effect, that all the closed ideals of $A$ come from open subsets of the maximal ideal space of $A$. Here, for $G$ discrete (and presumably for general locally compact $G$; see Problem 7.4), it means that closed ideals of $F^p(G)$ come from open subsets of $\hat{G}$.

We have $F^1(G) = L^1(G)$. It is known that if $G$ is not compact, then $L^1(G)$ does not have spectral synthesis. In fact, spectral synthesis fails badly. According
to Theorem 42.21 of [14] (see Definition 39.9 of [14] for the terminology), and
the additional statements in 42.26 of [14], the algebra $L^1(G)$ has intractable ideal
structure for every locally compact but noncompact abelian group $G$. On the other
hand, $F^2(G) = C_0(\hat{G})$, which trivially has spectral synthesis. What happens for
other values of $p$?

There are several open questions related to functoriality.

**Problem 7.6.** Fix a second countable locally compact group $G$. Take the category
of $G$-$L^p$ operator algebras to consist of all $L^p$ operator algebras with continuous
isometric actions of $G$ and equivariant contractive algebra homomorphisms. Does
$(G, A, \alpha) \mapsto F_p(G, A, \alpha)$ define a functor to $L^p$ operator algebras with contrac-
tive algebra homomorphisms? Does $(G, A, \alpha) \mapsto F_p^r(G, A, \alpha)$ define a functor to
$L^p$ operator algebras with contractive algebra homomorphisms?

Let $(G, A, \alpha)$ and $(G, B, \beta)$ be such algebras, and let $\varphi: A \rightarrow B$ be an equivariant
contractive algebra homomorphism. Since the crossed products were defined using
nondegenerate representations, naively one wants to know that whenever $(w, \pi)$ is
a nondegenerate (regular) covariant representation of $(G, B, \beta)$, then $(w, \pi \circ \varphi)$ is
a nondegenerate (regular) covariant representation of $(G, B, \beta)$. This is certainly
true of $A$, $B$, and $\varphi$ are unital, or, more generally, if $\varphi(A)$ contains an approximate
identity for $B$. But it will often fail if $\varphi$ is the inclusion of a direct summand or,
more generally, the inclusion of a proper ideal, especially one which isn’t essential.

There would be no issue if one did not restrict to the use of nondegenerate represen-
tations in the construction of crossed products. This motivates the following
question.

**Problem 7.7.** In the development of both full and reduced $L^p$ operator crossed
products in [27], what happens if one drops the requirements that the algebras be
nondegenerately representable and that the representations be nondegenerate?

In the case of C*-algebras, this change makes no difference. The essential reason
is that every *-representation of a C*-algebra on a Hilbert space can be decom-
posed as the direct sum of a nondegenerate representation on one Hilbert space
and the zero representation on another Hilbert space. Moreover, this decomposi-
tion is preserved by the constructions used in [27] (integrated form of a covariant
representation and regular covariant representation associated with a representa-
tion). In general, even for nonselfadjoint operator algebras on Hilbert spaces, no
such decomposition is possible. Consider the algebra

\[(7.1) \quad B = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{C} \right\} \subset M_2^p. \]

If $A$ is unital and $\pi: A \rightarrow L(L^p(X, \mu))$ is a nonzero degenerate contractive
representation, then $e = \pi(1)$ is an idempotent in $L(L^p(X, \mu))$ with $\|e\| = 1$. The-
orem 3 in Section 17 of [19] implies that there is a measure space $(Y, \mathcal{C}, \nu)$ such
that $eL^p(X, \mu)$ is isometrically isomorphic to $L^p(Y, \nu)$. It probably follows that for
unital algebras, the crossed product is unchanged if one allows degenerate represen-
tations. However, there might not be any subset $Y \subset X$ such that $eL^p(X, \mu)$ can
be identified with $L^p(Y, \mu)$ in any natural way. Furthermore, there is no guarantee
that we don’t have $\|1 - e\| > 1$. 
If $A$ is nonunital but has an approximate identity, one can hope for something
similar. As far as I know, nothing in this direction is known, but Banach space
theorists may know something.

Full crossed products of $C^*$-algebras preserve exact sequences.

**Theorem 7.8** (Lemma 2.8.2 of [22]). Let $G$ be a locally compact group. Let

$$0 \longrightarrow J \xrightarrow{\alpha} A \xrightarrow{\kappa} B \longrightarrow 0$$

be an exact sequence of $G$-algebras, with actions $\gamma$ on $J$, $\alpha$ on $A$, and $\beta$ on $B$. Then

the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \xrightarrow{\iota} C^*(G, A, \alpha) \xrightarrow{\kappa} C^*(G, B, \beta) \longrightarrow 0$$

of crossed products and induced maps is exact.

**Problem 7.9.** Is there an analog of Theorem 7.8 for $L^p$ operator crossed products?

There are many potential difficulties. To begin with, inclusions of ideals are
among the kinds of homomorphisms which give trouble with functoriality for crossed
products defined using nondegenerate representations. See the discussion after
Problem 7.6. The algebra $B$ in (7.1) is a perfectly good ideal in the subalgebra of
$M_2^z$ consisting of the upper triangular matrices.

The proof of injectivity of $\iota$ in Theorem 7.8 uses the fact that a nondegenerate
*-representation of $J$ on a Hilbert space has a unique extension to a representation
of $A$. The proof I know for existence depends on properties of positive elements of
$C^*$-algebras.

The standard proof of surjectivity of $\kappa$ in Theorem 7.8 actually just shows that
$\kappa$ has dense range (this is essentially immediate) and relies on $C^*$ theory to deduce
that $\kappa$ is surjective. The analog of this $C^*$ result is known to fail for $L^p$ operator
algebras (even for nonselfadjoint $L^2$ operator algebras).

The proof that $\text{Ker}(\kappa) \subset \text{Ran}(\iota)$ relies on knowing that
$C^*(G, A, \alpha)/C^*(G, J, \gamma)$ is a $C^*$-algebra. In the $L^p$ operator case, even if we have gotten this far, we do not
know that this quotient is an $L^p$ operator algebra.

**Problem 7.10.** Let $p \in [1, \infty)$, let $A$ be a simple $L^p$ operator algebra, let $G$ be
a countable group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action such that $\alpha_g$ is outer for every $g \in G \setminus \{1\}$. Does it follow that $F_p^p(G, A, \alpha)$ is simple?

For $C^*$-algebras, the answer is yes. See Theorem 3.1 of [18]. See [10] for some
earlier related work. The proofs in of [18] rely heavily on positivity. In the purely
algebraic situation, for finite groups and unital rings, the result is true, by Theo-
rem 4 of [4]. This implies the result asked for in the problem when $G$ is finite and $A$ is unital.

(The purely algebraic case with an arbitrary group appears as Proposition 1.1(a)
of [13]. The entire proof is “The following is proved by a classical shortening proof.”)

8. $L^p$ OPERATOR CROSSED PRODUCTS AND DYNAMICAL SYSTEMS

**Problem 8.1.** Let $X$ be a compact metric space, and let $h: X \rightarrow X$ be a minimal
homeomorphism. Define $\alpha \in \text{Aut}(C(X))$ by $\alpha(f) = f \circ h^{-1}$ for $f \in C(X)$. As in the
$C^*$ case, abbreviate $F^p(Z, C(X), \alpha)$ to $F^p(Z, X, h)$. (Since the group is amenable,
this is the same as the reduced $L^p$ operator crossed product.)

Can there ever be a nonzero continuous homomorphism from $F^{p_1}(Z, X_1, h_1)$ to
$F^{p_2}(Z, X_2, h_2)$ with $p_1 \neq p_2$ and $h_1$ and $h_2$ both minimal?
First suppose that $X$ is the Cantor set. Using Rokhlin towers in $X$, one can find unital subalgebras of $F^p(Z, X, h)$ which are isomorphic to finite direct sums of finite matrix algebras. For choices of $p_1$ and $p_2$ for which the methods used in [25] for $L^p$ UHF algebras work, one can hope to adapt those methods. One difficulty: the norms on the matrix algebras might well not be the spatial norms.

In general, copying Rokhlin tower methods will give finite direct sums of finite matrix algebras over (commutative) nonunital algebras, making the situation harder to deal with.

**Problem 8.2.** Let $h : X \rightarrow X$, and let $F^p(Z, X, h)$ be as in Question 8.1. What information about $h$ can one recover from the isomorphism class or isometric isomorphism class of $F^p(Z, X, h)$? For example, if $X$ is the Cantor set, and $h_1, h_2 : X \rightarrow X$ are minimal homeomorphisms, then $C^*(Z, X, h_1) \cong C^*(Z, X, h_2)$ if and only if $h_1$ and $h_2$ are strongly orbit equivalent. (This is the Giordano-Putnam-Skau Theorem.) But for minimal homeomorphisms $h_1 : X_1 \rightarrow X_1$ and $h_2 : X_2 \rightarrow X_2$, in which both $X_1$ and $X_2$ are compact manifolds of dimension at least 2, it seems to be easy to have $C^*(Z, X_1, h_1) \cong C^*(Z, X_2, h_2)$ when the dynamics of $h_1$ and $h_2$ are quite different, and even when $X_1$ and $X_2$ are quite different.

One expects it to be less likely that, for example, $F^p(Z, X_1, h_1) \cong F^p(Z, X_2, h_2)$ than $C^*(Z, X_1, h_1) \cong C^*(Z, X_2, h_2)$, since $L^p$ operator algebras are apparently more rigid than C*-algebras. In particular, the methods used to prove that all spatial representations of the Leavitt algebra $L_d$ generate isometrically isomorphic $L^p$ operator algebras can probably be adapted to show that if $h : X \rightarrow X$ is a minimal homeomorphism then the copy of $Z$ in $F^p(Z, X, h)$ is isometrically isomorphic to $F^p(Z)$, not to $C(S^1)$. (One will want the fact (proved in one of my scratch TeX files) that contractive representations of $C(X)$ on $L^p(Y, \nu)$, for $p \neq 2$ and a $\sigma$-finite measure space $\nu$, must act via multiplication operators.)

It thus seems that the outcome here is likely to be different from the C* case when $p \neq 2$.

Even for very elementary examples (free nonminimal actions of finite groups), there seems to be a difference from the C* case when $p \neq 2$. Fix $n \in \mathbb{Z}_{>0}$, and consider the action of $G = \mathbb{Z}/n\mathbb{Z}$ on $S^1$ generated by rotation by $2\pi/n$, that is, the homeomorphism $h(\zeta) = e^{2\pi i/n} \zeta$ for $\zeta \in S^1$. The crossed product C*-algebra $C^*(G, S^1, h)$ is computed explicitly in [28]. As is well known, it is just $C(S^1, M_n)$, but, as the computation shown, the “natural” outcome of the calculation is the algebra $B$ defined as follows. Set

$$s = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \in M_n.$$  

The define

$$B = \{ f \in C([0, 1], M_n) : f(0) = sf(1)s^* \}.$$  

The proof that $B \cong C(S^1, M_n)$ uses a continuous path in the unitary group $U(M_n)$ from 1 to $s$. 
We suppose that $F^p(G, S^1, h)$ is isometrically isomorphic to
$$B_p = \{ f \in C([0, 1], M_p^n) : f(0) = sf(1)s^* \}.$$ But for $p \neq 2$, there is no continuous path in the isometry group of $M_p^n$ from 1 to $s$. This suggests that $F^p(G, S^1, h)$ is not isometrically isomorphic to $C(S^1, M_p^n)$, although they are certainly isomorphic.

**Problem 8.3.** Is it in fact true that $F^p(\mathbb{Z}/n\mathbb{Z}, S^1) \neq C(S^1, M_p^n)$? Generalize. For a free action of a finite group $G$ on a compact metric space $X$, how much information about the action can be recovered from $F^p(G, X)$?

**Problem 8.4.** Let $X = \mathbb{Z}/n\mathbb{Z}$, and let $\mathbb{Z}$ act on $X$ by translation. Identify $F^p(G, X)$. Is it isometrically isomorphic to $C(S^1, M_p^n)$, as happens in the $C^*$ case? Generalize.

The computation for the $C^*$ case is also done explicitly in [28].

**Problem 8.5.** Let $h : X \to X$, and let $F^p(\mathbb{Z}, X, h)$ be as in Question 8.1. Suppose $X$ is the Cantor set. Does it follow that the invertible elements of $F^p(\mathbb{Z}, X, h)$ are dense? That is, does $F^p(\mathbb{Z}, X, h)$ have stable rank one? This is known for $p = 2$. If $X = S^1$ and $h$ is an irrational rotation, does it follow that the invertible elements of $F^p(\mathbb{Z}, X, h)$ are dense? This is also known for $p = 2$.

The method of proof of the first $C^*$ result for real rank zero for (some) irrational rotation algebras (see [5]) can possibly be adapted to stable rank one. If so, it should work for the $L^p$ operator irrational rotation algebras as well. The proofs of more definitive results (even in the special cases explicitly mentioned in Problem 8.5; see [33] and [34]) seem likely to be harder to adapt—perhaps much harder.

In the case $p = 2$, stable rank one holds much more generally. For $p = 2$, both the special examples in Problem 8.5 also have real rank zero. A unital $C^*$-algebra has real rank zero if and only if it is an exchange ring, by Theorem 7.2 of [2], and the definition of an exchange ring (see the beginning of Section 1 of [2]) makes sense for general unital rings. So it seems reasonable to ask the following:

**Problem 8.6.** In the examples of Problem 8.5, is $F^p(\mathbb{Z}, X, h)$ an exchange ring?

I don’t know where to start on this one.

**Problem 8.7.** Let $p \in [1, \infty)$. Let $\alpha : G \to \text{Aut}(A)$ be an isometric action of a locally compact abelian group on an $L^p$ operator algebra. We have constructed a dual action
$$\hat{\alpha} : \hat{G} \to \text{Aut}(F^p(G, A, \alpha)),$$ Is there an analog of Takai duality [37] for the crossed products by this action?

As a special case, suppose $A = F^p(G)$, which is (at least when $G$ is discrete) a subalgebra (with a different norm) of $C_0(\hat{G})$, and where the action of $\hat{G}$ is by translation on $\hat{G}$. Does one get the compact operators on $L^p(\hat{G})$ (matrices if $G$ is finite)?

What happens for the related crossed product gotten by letting $\hat{G}$ act on $C_0(\hat{G})$ itself via the translation action on $\hat{G}$? More generally, suppose a second countable locally compact group $H$ (playing the role of $\hat{G}$) acts on itself by translation, and thus on $C_0(H)$. What is $F^p(H, C_0(H))$? Can $F^p(H, C_0(H))$ be different? (For $H$ discrete, it is known that the full and reduced crossed products are both $L(p(H))$!
when \( p \neq 1 \), and for \( p = 1 \) they are the closure of the operators on \( l^1(H) \) which have finite matrices with respect to the standard basis of \( l^1(H) \).

When algebras of compact operators on \( L^p \) spaces are involved, bear in mind that \( K(l^p(\mathbb{Z})) \) is presumably not isomorphic to \( K(l^p([0,1])) \) for \( p \neq 2 \).

**Problem 8.8.** Let \( G \) be a countable discrete group, and let \( X \) be an essentially free minimal compact metrizable \( G \)-space. Does it follow that \( F^p_\infty(G,X) \) is simple?

The answer is yes if the action is free (Theorem 5.6 of [27]) and in the \( C^* \) case (the corollary to Theorem 1 of [3]). The proof in [3] relies on positivity.

9. **K-theory of \( L^p \) operator crossed products and related problems**

**Problem 9.1.** Is there an analog for the crossed products \( F^p(\mathbb{R}, A, \alpha) \) of the Connes isomorphism theorem [8] which computes the K-theory \( K_\ast(C_\ast(\mathbb{R}, A, \alpha)) \) in the \( C^* \) case?

Following what was done in [27] for the Pimsner-Voiculescu exact sequence for crossed products by \( \mathbb{Z} \), it may well be that the best approach is to reduce the problem to the smooth crossed product case considered in [29] (based on methods of [11]).

**Problem 9.2.** Let \( n \in \{2, 3, \ldots, \} \), and let \( p \in [1, \infty) \). Let \( A \) be the reduced \( L^p \) operator algebra of the free group on \( n \) generators. Is the invertible group of \( A \) dense in \( A \)? That is, does \( A \) have stable rank one? This is known to be true when \( p = 2 \).

The \( C^* \) proof [9] is quite nontrivial, and the \( L^p \) operator case might be harder (if the result is even true).

**Problem 9.3.** Let \( n \in \{2, 3, \ldots, \} \), let \( F_n \) be the free group on \( n \) generators, and let \( p \in [1, \infty) \). Does \( F^p(F_n) \) have nontrivial projections? Does \( F^p_\infty(F_n) \) have nontrivial projections?

It is known that \( C_\ast^\ast(F_n) \) has no nontrivial projections. This result is usually obtained as a corollary of the computation of the K-theory of \( C_\ast^\ast(F_n) \) (Problem 9.4 below), and one expects the K-theory of \( F^p(F_n) \) and \( F^p_\infty(F_n) \) to be the same as in the \( C^* \) case. Adapting this, or related, K-theory computations to the \( L^p \) operator case is likely to be much harder than it was for crossed products by \( \mathbb{Z} \).

There may be a more direct route to at least the case of \( F^p_\infty(F_n) \) in Problem 9.3. The article [6] is a short elementary \( C^* \) proof that \( C_\ast^\ast(F_n) \) has no nontrivial projections, not explicitly using K-theory, and which might be adaptable to the \( L^p \) operator case. It is also conceivable that there is a homomorphism from \( F^p(F_n) \) to \( C_\ast^\ast(F_n) \), which would make this result trivial.

**Problem 9.4.** Let \( n \in \{2, 3, \ldots, \} \), let \( F_n \) be the free group on \( n \) generators, and let \( p \in [1, \infty) \). What is \( K_\ast(F^p(F_n)) \)? What is \( K_\ast(F^p_\infty(F_n)) \)? More generally, is there a version in the \( L^p \) operator context of the Pimsner-Voiculescu exact sequence [31] for reduced \( C^* \) crossed products by \( F_n \)?

The original computation of \( K_\ast(C_\ast^\ast(F_n)) \) was done in [31]. Methods from [30] or from [16] and the references there might be more easily adaptable to the case \( p \neq 2 \). The methods of [29] are unlikely to be helpful.

Recall the equivariant K-theory of Banach algebras, developed in Chapter 2 of [22]. The main result of [15] is that for a compact group \( G \) and an action
\( \alpha: G \to \text{Aut}(A) \) on a C*-algebra \( A \), one has a natural isomorphism \( K^G_\ast(A) \cong K_\ast(C^*(G,A,\alpha)) \).

**Problem 9.5.** Let \( G \) be a second countable compact group, and let \( A \) be an \( L^p \) operator algebra with a continuous isometric action of \( G \). Is there a natural isomorphism \( K^G_\ast(A) \cong K_\ast(F^p(G,A,\alpha)) \)?

Since equivariant K-theory has long exact sequences, such a natural isomorphism would imply that there is at least a K-theoretic analog of Theorem 7.8.

**Problem 11.2** below is also related to crossed products: it asks how the K-theory of certain crossed products varies with \( p \).

### 10. \( p \)-Nuclearity

The following definition is contained in Proposition 5.1(a) of [1].

**Definition 10.1.** Let \((X,\mathcal{B},\mu)\) be a measure space, and let \( A \subset L(L^p(X,\mu)) \) be a norm closed subalgebra. We say that \( A \) is \( p \)-nuclear if for every finite set \( F \subset A \) and every \( \varepsilon > 0 \), there exist \( n \in \mathbb{Z}_{>0} \) and \( p \)-completely contractive maps \( S: A \to M^p_n \) and \( T: M^p_n \to A \) such that \( \| (T \circ S)(a) - a \| < \varepsilon \) for all \( a \in F \).

When \( p = 2 \) and \( A \) is a C*-algebra, Theorem 1.1 of [36] shows that \( p \)-nuclearity is equivalent to nuclearity. Proposition 5.1(a) of [1] states that if \( G \) is discrete amenable, then \( F^p(G) \) is \( p \)-nuclear. (The result is stated in terms of \( F^p_r(G) \), but for amenable \( G \) it is the same algebra. It also excludes the case \( p = 1 \), but the proof seems still to work in that case. My guess is that \( p = 1 \) is excluded throughout [1] because a number of other proofs don’t work in that case.)

**Problem 10.2.** If \( G \) is locally compact and amenable, does it follow that \( F^p(G) \) is \( p \)-nuclear?

**Problem 10.3.** If \( G \) is a discrete group and \( F^p(G) \) is \( p \)-nuclear, does it follow that \( G \) is amenable?

One might need to exclude \( p = 1 \).

**Problem 10.4.** Let \( p \in [1,\infty) \setminus \{2\} \), let \( G \) be a countable amenable group, and let \( (G,A,\alpha) \) be an isometric nondegenerately representable separable \( G \)-\( L^p \) operator algebra. If \( A \) is \( p \)-nuclear, does it follow that \( F^p(G,A,\alpha) \) is \( p \)-nuclear?

It might be easy to adapt the proof of Proposition 5.1(a) of [1].

The conditional expectations constructed in [25] can be used to show that spatial \( L^p \) UHF algebras are \( p \)-nuclear, although this is not done in the current version. Because of the crossed product decomposition in [27], the following is almost a special case of Problem 10.4.

**Problem 10.5.** Let \( p \in [1,\infty) \setminus \{2\} \), and let \( d \in \{2,3,\ldots\} \). Is \( O^p_d \) \( p \)-nuclear?

Recall that a C*-algebra is amenable if and only if it is nuclear. (See Corollary 2 of [7] and Theorem 3.1 of [12]. Both papers make substantial use of von Neumann algebra methods.) This fact suggests the following problems.

**Problem 10.6.** Let \((X,\mathcal{B},\mu)\) be a measure space, and let \( A \subset L(L^p(X,\mu)) \) be a norm closed subalgebra. Suppose \( A \) is \( p \)-nuclear. Does it follow that \( A \) is an amenable Banach algebra?
I do not know the answer to this problem even for $p = 2$, because norm closed subalgebras of $L(L^2(X,\mu))$ need not be selfadjoint and therefore need not be C*-algebras. In view of the open question of whether a separable amenable $L^2$ operator algebra is similar to a C*-algebra, in the context of Problem 10.6 one might ask whether a 2-nuclear $L^2$ operator algebra is similar to a C*-algebra, or even is a C*-algebra. (It is possible that this follows quickly from well known results.)

The analog of the other implication for C*-algebras would be as follows. Let $(X,\mathcal{B},\mu)$ be a measure space, and let $A \subset L(L^p(X,\mu))$ be a norm closed subalgebra. Suppose $A$ is an amenable Banach algebra. Does it follow that $A$ is $p$-nuclear? We suspect that this is not really the interesting question, because $p$-nuclearity seems to be sensitive to the precise norm on the algebra. For example, suppose that $z = \text{diag}(1,2) \in M_2$, and we define a norm on $M_2$ by $\|a\| = \max(\|a\|_p, \|zaz^{-1}\|_p)$. This makes $M_2$ an $L^p$ operator algebra, but it seems almost certain that, with this norm, $M_2$ is not $p$-nuclear. Here are two related questions which attempt to correct for this issue.

**Problem 10.7.** Let $(X,\mathcal{B},\mu)$ be a measure space, and let $A \subset L(L^p(X,\mu))$ be a norm closed subalgebra. Suppose $A$ is a 1-amenable Banach algebra. Does it follow that $A$ is $p$-nuclear?

**Problem 10.8.** Let $(X,\mathcal{B},\mu)$ be a measure space, and let $A \subset L(L^p(X,\mu))$ be a norm closed subalgebra. Suppose $A$ is an amenable Banach algebra. Does it follow that $A$ isomorphic to a $p$-nuclear $L^p$ operator algebra?

If this proves to be difficult, we suggest the following interesting special cases.

**Problem 10.9.** Do the nonamenable $L^p$ UHF algebras constructed in [26] fail to be $p$-nuclear?

More generally:

**Problem 10.10.** If $A$ is an $L^p$ UHF algebra which is $p$-nuclear, does it follow that $A$ is spatial?

While it does not directly involve $p$-nuclearity, the following problem, related to Problem 10.3, seems relevant. It might have a fairly immediate solution based on known facts. (The answer is yes for $p = 1$.)

**Problem 10.11.** If $G$ is a discrete group and $F^p_p(G)$ is an amenable Banach algebra, does it follow that $G$ is amenable?

Suppose that in Definition 10.1, we replace $M_n^p$ with $M_n$ with any norm making it a (unital) $L^p$ operator algebra. Call the result “weak $p$-nuclearity”. (A better name is needed if anyone actually does anything with this concept.)

**Definition 10.12.** Let $(X,\mathcal{B},\mu)$ be a measure space, and let $A \subset L(L^p(X,\mu))$ be a norm closed subalgebra. We say that $A$ is (unital) weakly $p$-nuclear if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist $n \in \mathbb{Z}_{>0}$, a $\sigma$-finite measure space $(Y,\mathcal{C},\nu)$, a (unital) subalgebra $F \subset L(L^p(Y,\nu))$ algebraically isomorphic to $M_n$, and $p$-completely contractive maps $S: A \to F$ and $T: F \to A$ such that $\|(T \circ S)(a) - a\| < \varepsilon$ for all $a \in F$.

It is easy to see that the nonamenable $L^p$ UHF algebras considered in Section 4 of [26] are unitally weakly $p$-nuclear, so this condition does not imply amenability. However, it might exclude algebras like the upper triangular matrices.
11. Problems about what happens when $p$ varies

**Problem 11.1.** Suppose $1 \leq p_1 \leq p_2 \leq p_3 < \infty$. Can one get $O_{d_2}^p$ from $O_{d_1}^p$ and $O_{d_1}^p$ by Banach space interpolation?

One should probably start with $M_d^p$. The question is then whether the norm on $M_d^{p_2}$ can be gotten from those on $M_d^{p_1}$ and $M_d^{p_3}$ by Banach space interpolation. Here one can allow $p = \infty$.

The following question makes sense anyway, but might well be related to Banach space interpolation.

**Problem 11.2.** Let $X$ be a compact metric space, and let $G$ be a countable discrete group which acts on $X$. Then $C(X)$ is an $L^p$ operator algebra for all $p \in [1, \infty)$. As in the $C^*$ case, abbreviate $F_p^p(G, C(X))$ to $F_p^p(G, X)$. Is the $K$-theory of $F_p^p(G, X)$ independent of $p$?

If $G = \mathbb{Z}$, there is a Pimsner-Voiculescu exact sequence which computes $K_*(F_p^p(\mathbb{Z}, X))$ in terms of $K^*(X)$, up to an extension problem, just as for $C^*$-algebras. In this case, Problem 11.2 asks whether that extension problem has the same solution for all $p \in [1, \infty)$.

The following question was raised by Zhuang Niu.

**Problem 11.3.** Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and for $p \in [1, \infty)$ let $A_p^0$ be a choice for the $p$-irrational rotation algebra which varies “continuously” with $p$. (There are at least two possibilities, probably different. One is the universal algebra generated by two bijective isometries $u$ and $v$ on an $L^p$ space such that $vu = e^{2\pi i \theta} uv$. Another is the $L^p$ operator crossed product $F_p^p(\mathbb{Z}, S^1, h_\theta)$ with $h_\theta : S^1 \to S^1$ being the rotation $h_\theta(\zeta) = e^{2\pi i \theta} \zeta$ for $\zeta \in S^1$. How does the spectrum $\text{sp}(u + u^{-1} + v + v^{-1})$ vary with $p$?

When $p = 2$, the spectrum $\text{sp}(u + u^{-1} + v + v^{-1})$ is known to be a Cantor set.

A related problem has been considered, and suggests that the answer might be interesting. Here, $p$ is allowed to be in $[1, \infty]$. Let $n \in \{2, 3, \ldots\}$, and let $F_n$ be the free group on $n$ generators $\gamma_1, \gamma_2, \ldots, \gamma_n$. Represent $F_p^p(F_n)$ on $l^p(F_n)$ in the standard way, for $g \in F_n$ let $\delta_g$ be the corresponding standard “basis” vector in $l^p(F_n)$, and let $R_p$ be the “radial subalgebra”, consisting of all $a \in F_p^p(F_n)$ such that $a\delta_1 \in l^p(F_n)$ is a function whose value $(a\delta_1)(g)$ at $g \in F_n$ depends only on the length of $g$ as a reduced word. (Functions with this kind of dependence are called “radial”.) Then $R_p$ is a maximal abelian subalgebra of $F_p^p(F_n)$ (Theorem 4.3 of [35]), generated by $b = \sum_{j=1}^n (u_{\gamma_j} + u_{\gamma_j}^{-1})$. Moreover, for $p \in [1, 2]$, and with $\omega = \sqrt{2n - 1}$,

$$\text{sp}(b) = \{ \lambda \in \mathbb{C} : |\lambda - 2\omega| + |\lambda + 2\omega| \leq 2\omega^{2/p} + 2\omega^{2-2/p} \}$$

(Theorem 3.1 of [35]). There is also a formula for $||b||$, given without proof in the remark at the end of Section 3 of [35], namely $||b|| = \omega^{2/p} + \omega^{2-2/p}$. 

**Problem 10.13.** Consider the subalgebra of $M_n^p$ consisting of all upper triangular matrices. Is it (unitally) weakly $p$-nuclear?

If the answer is no, what special properties do the (unitally) weakly $p$-nuclear $L^p$ operator algebras have?
12. Miscellaneous problems

Problem 12.1. What happens to $O_d^p$ with real scalars? (The K-theory will be different, but this already happens when $p = 2$. We used complex scalars at one crucial place in the proof of equivalence of many of the conditions for a representation to be spatial. Are the results really different?)

Problem 12.2. The algebras $O_d^p$ are purely infinite and simple. We defined a simple unital Banach algebra to be purely infinite if for every $x \in A \setminus \{0\}$ there exist $x, y \in A$ such that $xay = 1$. How many of the consequences of pure infiniteness of C*-algebras carry over to this situation? The group $K_0(A)$ is isomorphic to the set of Murray-von Neumann equivalence classes of nonzero idempotents in $A$. (This follows from the proof of Corollary 5.15 of [25].) Is $K_1(A)$ isomorphic to $\text{inv}(A)/\text{inv}_0(A)$? As a special case: Is the invertible group of $O_d^p$ connected?

Problem 12.3. Our nonisomorphism results are all results on nonisomorphism as Banach algebras. What are the Banach space isomorphism and isometry classifications of the closures of the ranges of representations of Leavitt algebras on $L^p$ spaces? (For $\frac{1}{p} + \frac{1}{q} = 1$, the algebras $O_d^p$ and $O_d^q$ are isometrically antiisomorphic.)

Problem 12.4. Let $p \in (1, \infty) \setminus \{2\}$. Here are several possible ways to produce $L^p$ analogs of the hyperfinite factor of type II$_1$ acting on a separable Hilbert space. For $p = 2$, they all give the same algebra.

1. Let $G$ be a countable amenable ICC (infinite conjugacy classes) group. Represent the $L^p$ operator group algebra $F^p(G)$ on $L^p(G)$ via the regular representation, and take its weak operator closure. (This algebra is called $PM_p(G)$ in [1]. The algebra $F_p^p(G)$ is called $PF_p(G)$ there. We are relying on the fact that amenability of $G$ implies that $F^p(G) \rightarrow F_p^p(G)$ is an isomorphism.)

2. Start with a spatial $L^p$ UHF algebra $D$, with its unique trace $\tau$. Write down the explicit formula for the case $p = 2$ for the representation obtained by applying the Gelfand-Naimark-Segal construction to $\tau$. Can it be adapted to give a representation of $D$ on an $L^p$ space? If so, take the weak operator closure of the image of this representation.

3. Choose a free ergodic measure preserving action of a countable amenable group on a standard probability space $XBM$. Form the regular representation $(\nu, \pi)$ of $(G, L^\infty(X, \mu))$ coming from the representation of $L^\infty(X, \mu)$ on $L^p(X, \mu)$ as multiplication operators. It acts on $L^p(G \times X)$. Take the weak operator closed subalgebra of $L(L^p(G \times X))$ generated by the union of the ranges of $\nu$ and $\pi$.

Do these constructions all give isomorphic (in a suitable topological sense) algebras? (Even within each construction, it the algebra one gets could depend on the choice of the ingredients.) What is the K-theory of these algebras? (For $p = 2$, one gets $K_1 = 0$ and $\tau$ induces an isomorphism from $K_0$ to $\mathbb{R}$.) In the construction (2), at least one should get a representation of the algebraic infinite tensor product. If the norm closure isn’t the appropriate spatial $L^p$ UHF algebra, what is it? Is it amenable?
References

OPEN PROBLEMS


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