Short answer questions (8 points each):

1. Express the triple integral of the function $x^2 - y + z$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 8$. Do not evaluate the iterated integral.

\[
\int_{0}^{2} \int_{0}^{8-2x} \int_{0}^{8-2x-y} (x^2 - y + z) \, dz \, dy \, dx
\]

2. Is the vector field $\vec{F}(x, y, z) = (xy, xz, yz)$ conservative? If so, find a potential function. If not, demonstrating that it is not completely answers the question.

Test by finding curl:

\[
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & xz & yz
\end{vmatrix} = (x-x) \hat{i} - (0-0) \hat{j} + (z-x) \hat{k} = 0
\]

So it's \underline{not} conservative.
Other orders of integration for #1

\[
\iint_{R_1} (x^2 - y + z) \, dx \, dy
\]

\[
\iint_{R_2} (y^2 - z) \, dy \, dz
\]

\[
\iint_{R_3} (z^2 - x) \, dz \, dx
\]

\[
\iint_{R_4} (x^2 - y^2 - z^2) \, dx \, dy \, dz
\]
3. Let $C$ be the portion of the circle of radius 1 about the origin lying in the first quadrant beginning at $(0, 1)$ and ending at $(1, 0)$. Give a parametrization of $C$ and use it to express the line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

as an ordinary definite integral, where $\vec{F}$ is the vector field $\vec{F}(x, y) = (-1, x)$. Do not evaluate the resulting definite integral.

Parameter by $u = \pi t$

\[\vec{r}(t) = \langle \cos t, \sin t \rangle \quad t \in \left[0, \frac{\pi}{2}\right]\]

\[\vec{F}(\vec{r}(t)) = \langle -\sin t, \cos t \rangle\]

So

$$\int_C \vec{F} \cdot d\vec{r} = \int_{0}^{\frac{\pi}{2}} (-\sin t)(-\sin t) + (\cos t)(\cos t) \, dt$$

$$= \int_{0}^{\frac{\pi}{2}} \sin^2 t + \cos^2 t \, dt$$

4. Express the volume of the region described in the next sentence as an iterated integral by using cylindrical coordinates. Do not evaluate the iterated integral.

The region lies between the planes $z = 1$ and $z = -1$, inside the sphere of radius 4 about the origin and outside the cylinder of radius 2 about the $z$-axis.

$$2 \leq r \leq \sqrt{16 - z^2}$$

$$-1 \leq z \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$dV = r \, dr \, dz \, d\theta$$

$$V = \iiint_{R} dV = \int_{0}^{2\pi} \int_{-1}^{1} \int_{2}^{\sqrt{16 - z^2}} r \, dr \, dz \, d\theta$$
#3 with parametrization by \( t = y \)

\[
\vec{r}(t) = \left< \sqrt{1-t^2}, t \right> \quad t \in [0,1]
\]

\[
\vec{r}'(t) = \left< \frac{1}{2} (1-t^2)^{-1/2} \cdot (-2t), 1 \right>
\]

\[
= \left< \frac{-t}{\sqrt{1-t^2}}, 1 \right>
\]

\[
\vec{F}(\vec{r}(t)) = \left< -1, \sqrt{1-t^2} \right>
\]

\[
\int_C \vec{F} \cdot d\vec{r} = 1 \quad \int_0^1 \left( -1 \frac{t}{\sqrt{1-t^2}} + 1 \sqrt{1-t^2} \right) \, dt
\]

\[
= \int_0^1 \frac{t}{\sqrt{1-t^2}} + \sqrt{1-t^2} \, dt
\]
Yet more short answer questions.

5. Find the divergence of $\vec{Q}(x, y, z) = ye^x\hat{i} + ze^y\hat{j} + xe^z\hat{k}$, and use it to decide whether or not the vector field could represent the velocity of a flow in an incompressible fluid.

\[
\nabla \cdot \vec{Q} = \frac{\partial}{\partial x} ye^x + \frac{\partial}{\partial y} ze^y + \frac{\partial}{\partial z} xe^z \\
= 0 + 0 + 0 = 0
\]

Yes, it could represent velocities in an incompressible fluid flow.

6. Find the curl of the vector field $\vec{Q}(x, y, z) = ye^x\hat{i} + ze^y\hat{j} + xe^z\hat{k}$, and use it to decide whether the vector field is irrotational.

\[
\nabla \times \vec{Q} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
ye^x & ze^y & xe^z
\end{vmatrix}
\]

\[
= (xe^z - e^x)\hat{i} - (e^y - ye^z)\hat{j} + (ze^x - e^2)\hat{k}
\]

No, it is not irrotational.
Long questions (18 points each)

7. Let $R$ be the region lying in the first octant (all coordinates positive) and inside the sphere of radius 3 about the origin. Use spherical coordinates to express the following triple integral as an iterated integral and evaluate the resulting iterated integral.

$$
\iiint_R \frac{z}{x^2 + y^2 + z^2} \, dV
$$

$x^2 + y^2 + z^2 = \rho^2$

$\rho \cos \phi$

$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

$$
\frac{\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}{\rho^2} = \rho \sin \phi \, d\rho \, d\phi \, d\theta
$$

$$
\int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta
$$

Since in $\rho, \phi, \theta$ parameter space $R$ is a right solid and the integrand factor $1 - \rho^2$ function

$$
\int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta
$$

$$
= \int_0^{\pi/2} \int_0^{\pi/2} \left[ \frac{1}{2} \rho^2 \right]_0^3 \, d\phi \, d\theta
$$

$$
= \int_0^{\pi/2} \int_0^{\pi/2} \frac{9}{2} \, d\phi \, d\theta
$$

$$
= \frac{\pi}{2} \cdot \frac{9}{2} \cdot \frac{1}{2} = \frac{9 \pi}{8}
$$
8. Find the average value of the function \( w(x, y) = x^2 \) on the disk of radius 1 centered at the origin (in the \( xy \)-plane).

\[
\text{Area} = \pi \cdot 1^2 = \pi
\]

5. \( \text{Avg value} = \frac{1}{\pi} \iint \limits_{D} x^2 \, dA \)

polar coord seems good for this:

region is:

\[
0 \leq \theta \leq 2\pi \quad 0 \leq r \leq 1
\]

\[
\begin{align*}
5. & \quad \int \int \limits_{D} x^2 = \int \int \limits_{D} r^2 \cos^2 \theta \\
& \quad dA = r \, dr \, d\theta
\end{align*}
\]

\[
\begin{align*}
\text{Avg} &= \frac{1}{\pi} \int \int \int \limits_{D} r^2 \cos^2 \theta \, dr \, d\theta \\
& = \frac{1}{\pi} \int_{0}^{2\pi} \cos^2 \theta \, d\theta \int_{0}^{1} r^3 \, dr
\end{align*}
\]

\[
\begin{align*}
& = \frac{1}{\pi} \left[ \left. \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right|_{0}^{2\pi}, \frac{r^4}{4} \right|_{0}^{1} \\
& = \frac{1}{\pi} \left[ \left. \frac{2\pi}{2} + 0 - (0 - 0) \right] \cdot \left( \frac{1}{4} - 0 \right) \\
& = \frac{1}{\pi} \cdot \pi \cdot \frac{1}{4} = \frac{1}{4}
\end{align*}
\]
#8 A rectangular (0,0) region:

\[-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} -1 \leq x \leq 1\]

So,

\[
A_{\text{avg}} = \frac{1}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 \, dy \, dx
\]

\[
= \frac{1}{\pi} \int_{-1}^{1} x^2 \left[ y \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx
\]

\[
= \frac{1}{\pi} \int_{-1}^{1} 2x^2 \sqrt{1-x^2} \, dx = \frac{2}{\pi} \int_{-1}^{1} x^2 \sqrt{1-x^2} \, dx
\]

by form integral table

\[
= \frac{2}{\pi} \left[ \frac{-x}{4} (1-x^2)^{3/2} + \frac{1}{8} (x\sqrt{1-x^2} + \arcsin x) \right]_{x=-1}^{x=1}
\]

\[
= \frac{4}{\pi} \left[ -\frac{1}{4} \cdot 0 + \frac{1}{8} \left( 1 \cdot 1 + \frac{\pi}{2} \right) \right]
\]

\[
= \frac{4}{\pi} \cdot \frac{1}{8} \cdot \frac{\pi}{2} = \frac{1}{4}
\]

Evaluate at \( x = 1 \)}
9. Find the average value of the function \( w(x, y) = x^2 \) on the circle of radius 1 centered at the origin (in the xy-plane). (Just on the circle, not the whole disk - that was the previous question.)

\[
\text{length of curve} = \text{circumference} = 2\pi \cdot 1 = 2\pi
\]

So, avg value = \[
\frac{1}{2\pi} \int_C x^2 \, ds
\]

principle by 

\[\vec{r}(t) = \langle \cos t, \sin t \rangle \quad t \in \mathbb{R}, \quad 2\pi \]

\[\vec{r}'(t) = \langle -\sin t, \cos t \rangle
\]

\[|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1
\]

So, 

\[ds = dt
\]

\[x^2 = \cos^2 t
\]

\[
\text{Avg value} = \frac{1}{2\pi} \int_0^{2\pi} \cos^2 t \, dt
\]

\[
= \left. \frac{1}{2\pi} \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right] \right|_0^{2\pi} = \frac{1}{2}\int_0^{2\pi} \frac{\sin 2\theta}{4} 
\]

\[
= \frac{1}{2\pi} \left[ \frac{2\pi}{2} + 0 - (0 + 0) \right]
\]

\[
= \frac{1}{2}
\]
10. A large magnetic sheet (idealized as) occupying the plane \( z = 4 \) (in a convenient coordinate system for the problem with distance in meters) exerts a force of \( \frac{1}{(4-z)^2} \) Newtons on an iron particle located at \((x,y,z)\) when \( z < 4 \). If the force causes the iron particle, constrained in a helical tube, to move from \((3,0,0)\) to \((3,0,2)\) along the curve given parametrically by \( \vec{r}(t) = (3 \cos t, 3 \sin t, t/\pi) \). Find the work done by the force in Nm.

If \( \vec{F} = \frac{1}{(4-z)^2} \mathbf{F} \)

\[
\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & 3 & 0 \end{vmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{(4-z)^2} \\
\]

so its conservative. Find a potential function:

\[
f = \int (4-z)^{-2} \, dz = -(4-z)^{-1} + C(x,y) \]

\( f \) only \( f = -(4-z)^{-1} \) works since is \( x \) and \( y \)

\( \phi \) tvb are 0

So \( W = \int_{C} \vec{F} \cdot d\vec{r} = f(3,0,2) - f(3,0,0) \)

\[
= -(4-2)^{-1} - -(4-0)^{-1} \\
= - \frac{1}{2} + \frac{1}{4} \\
= \frac{1}{4} \text{ Nm} 
\]
by finding the line integral directly.

**Curve**

\[ \vec{r}(t) = \left< 3 \cos t, 3 \sin t, \frac{t}{\pi} \right>, \quad t \in [0, 2\pi] \]

\[ \vec{r}'(t) = \left< -3 \sin t, 3 \cos t, \frac{1}{\pi} \right> \]

\[ F(\vec{r}(t)) = \frac{1}{(4 - \frac{t}{\pi})^2} \vec{k} \]

So \( \pi \)

\[ W = \int_{0}^{2\pi} \left< 0, 0, (4 - \frac{t}{\pi})^{-2} \right> \cdot \left< -3 \sin t, 3 \cos t, \frac{1}{\pi} \right> dt \]

\[ = \int_{0}^{2\pi} \frac{1}{\pi} (4 - \frac{t}{\pi})^{-2} \, dt \]

\[ = \int_{0}^{2\pi} -u^{-2} \, du \]

\[ = \int_{4}^{2} u^{-2} \, du = -u^{-1} \bigg|_{4}^{2} \]

\[ = -\frac{1}{4} - (-\frac{1}{2}) = \frac{1}{4} \text{ Nm} \]