Instructions:
Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, and also a calculator. This exam is worth 60 points.

The chart below indicates how many points each problem is worth.

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1. Find the reference angle for the following.

(a) \(-135^\circ\)

\[ \text{Third quadrant} \]

\[ \overline{-135^\circ} \]

\[ \overline{45^\circ = \text{reference angle}} \]

(b) \(\frac{8\pi}{3}\) radians

Note \(\frac{8\pi}{3} > 2\pi\) and \(\frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}\) radians

\[ \text{Second quadrant} \]

\[ \frac{8\pi}{3} \]

\[ \pi - \frac{2\pi}{3} = \frac{\pi}{3} \text{ radians is the reference angle} \]

2. Find the exact value.

(a) \(\cos(-135^\circ) = -\cos(45^\circ)\) = \(-\frac{\sqrt{2}}{2}\)

The cosine is negative in the third quadrant.

(b) \(\sin\left(\frac{8\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\) = \(\frac{\sqrt{3}}{2}\)

The sine is positive in the second quadrant.
3. Verify the identity.

\[
\frac{\sin(\theta)}{\csc(\theta) - \cot(\theta)} = 1 + \cos(\theta)
\]

\[
\frac{\sin(\theta)}{1 - \cos(\theta)} = \frac{\sin^2(\theta)}{1 - \cos(\theta)} = \frac{\sin^2(\theta)}{1 - \cos(\theta)} = \text{using the Pythagorean Theorem}
\]

\[
\frac{1 - \cos^2(\theta)}{1 - \cos(\theta)} = (1 - \cos(\theta))(1 + \cos(\theta)) = 1 + \cos(\theta).
\]

Another approach is to cross multiply first, and then

\[
(1 + \cos(\theta))(\csc(\theta) - \cot(\theta)) = \csc(\theta) - \cot(\theta) + \cos(\theta)\csc(\theta) - \cos(\theta)\cot(\theta)
\]

\[
= \frac{1}{\sin(\theta)} - \frac{\cos(\theta)}{\sin(\theta)} + \frac{\cos(\theta)}{\sin(\theta)} - \cos(\theta)\left(\frac{\cos(\theta)}{\sin(\theta)}\right)
\]

\[
= \frac{1 - \cos^2(\theta)}{\sin(\theta)} = \frac{\sin^2(\theta)}{\sin(\theta)} = \sin(\theta),
\]

Here we use the Pythagorean Theorem.
4. The hypotenuse of a right triangle has length 38 inches, and one leg of the triangle has length 13 inches. Find the angles in this triangle (in degrees, rounded to the nearest degree).

One angle satisfies \( \sin(\alpha) = \frac{13}{38} \) so that

\[ \alpha = \sin^{-1}\left(\frac{13}{38}\right) \approx 20^\circ \] (need a calculator for this)

The other angle is \( \beta = 90^\circ - \alpha \approx 90^\circ - 20^\circ = 70^\circ \)

(or use \( \beta = \cos^{-1}\left(\frac{13}{38}\right) \))

5. Find all solutions to the equation.

\[ \cos(x) = 2\sin(x)\cos(x) \]

\[ 0 = 2\sin(x)\cos(x) - \cos(x) \]

\[ 0 = \cos(x)\left(2\sin(x) - 1 \right) \]

Either \( \cos(x) = 0 \) or \( 2\sin(x) - 1 = 0 \)

\[ \sin(x) = \frac{1}{2} \]

\[ x = \frac{\pi}{2} + 2\pi n \quad \text{or} \quad x = \frac{\pi}{6} + 2\pi n \]

or \[ x = \frac{3\pi}{2} + 2\pi n \quad \text{or} \quad x = \frac{5\pi}{6} + 2\pi n \]

for some integer \( n \)
6. Let \( y = 2 \cos(3x + \frac{\pi}{2}) \).

(a) Find the amplitude.

\[
A = 2
\]

(b) Find the period.

\[
\text{Period} = \frac{2\pi}{3}
\]

(c) Find the phase shift.

\[
\text{Phase shift} = -\frac{\pi/3}{3} = -\frac{\pi}{9}
\]

(d) Graph the function.

If \( 0 \leq 3x + \frac{\pi}{2} \leq 2\pi \)
then \( -\frac{\pi}{2} \leq 3x \leq \frac{3\pi}{2} \)
and \( -\frac{\pi}{6} \leq x \leq \frac{\pi}{2} \) describes
one "cycle" of this graph.

Note if \( 3x + \frac{\pi}{2} = 0 \)
then \( x = -\frac{\pi}{6} \).
7. Find the solutions satisfying $0 \leq x < 2\pi$ radians.

$$\sin(2x - \frac{\pi}{2}) = \frac{1}{2}$$

$$2x - \frac{\pi}{2} = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} + 2n\pi, \quad \text{or} \quad \frac{\pi}{2} + \frac{5\pi}{6} + 2n\pi$$

$$2x = \frac{\pi}{2} + \frac{5\pi}{6} + 2n\pi$$

$$2x = \frac{4\pi}{3} + 2n\pi$$

$$x = \frac{\pi}{3} + n\pi, \quad \text{or} \quad x = \frac{2\pi}{3} + n\pi$$

Four solutions in the interval $0 \leq x < 2\pi$

$$\left\{ \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3} \right\}$$

8. The two angles $\alpha$ and $\beta$ both have terminal sides in the second quadrant, and $\cos(\alpha) = \frac{4}{5}$ and $\sin(\beta) = \frac{3}{4}$. Find the exact value of $\sin(\alpha - \beta)$.

$$\begin{align*}
\sin(\alpha) &= \frac{3\sqrt{7}}{8} \\
\sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
&= \left(\frac{3\sqrt{7}}{8}\right)\left(-\frac{\sqrt{7}}{4}\right) - \left(\frac{4}{5}\right)\left(\frac{3}{4}\right) \\
&= -\frac{21}{32} + \frac{3}{32} = -\frac{18}{32} = -\frac{9}{16}
\end{align*}$$
9. A man measures the angle of elevation from the ground to the top of a tower to be 30°. He then walks 50 feet towards the tower, and now measures the angle of elevation from the ground to the top of the tower to be 45°. Find the height of the tower.

\[
\frac{1}{\sqrt{3}} = \tan(30°) = \frac{x}{50 + x}
\]

\[
50 + x = \sqrt{3} \times x
\]

\[
50 = \sqrt{3} \times x - x = x\left(\sqrt{3} - 1\right)
\]

\[
x = \frac{50}{\sqrt{3} - 1} \text{ feet} 
\approx 68.3 \text{ feet}
\]

\[
x = \frac{50(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{50(\sqrt{3} + 1)}{3 - 1} = \frac{50(\sqrt{3} + 1)}{2} = 25(\sqrt{3} + 1)
\]

\[
x = 25\sqrt{3} + 25 \text{ feet}
\]