1. Let $\ell_1(\mathbb{C}) = \left\{ \{x_i\}_{i=1}^{\infty} \mid x_i \in \mathbb{C} \text{ and } \sum_{i=1}^{\infty} |x_i| < \infty \right\}$. For $\{x_i\}_{i=1}^{\infty} \in \ell_1(\mathbb{C})$ define

$$\|\{x_i\}_{i=1}^{\infty}\|_1 = \sum_{i=1}^{\infty} |x_i|.$$ 

This defines a norm on $\ell_1(\mathbb{C})$. (You may assume this.)

Fix a sequence $\{y_i\}_{i=1}^{\infty}$ with $\sup_i |y_i| = 1$. Define $T : \ell_1(\mathbb{C}) \to \mathbb{C}$ by $T(\{x_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} x_i y_i$.

Show that $T$ is linear and bounded. Compute $\|T\|$.

2. Consider the system of equations:

$$x^2 + y^2 + z^2 = 2$$

$$\sin(xyz) = 0.$$ 

Show that for $(x, y, z)$ in a neighborhood of $(1, 0, 1)$ these equations define $x$ and $y$ as functions of $z$.

3. a) Give an example of a function $f$ defined on an interval $[a, b]$ such that $f$ is not Riemann integrable but $|f|$ is Riemann integrable.

b) Give an example of a Riemann integrable function $f$ defined on an interval $[a, b]$ and a Riemann integrable function $g$ defined on an interval containing the range of $f$ such that the composition $g \circ f$ is not Riemann integrable.

4. Suppose $\alpha$ is increasing on $[a, b]$, $f \geq 0$ and $f, f^3 \in R(\alpha)$. Suppose also that $\int_{a}^{b} f^3 d\alpha = 0$. Prove that $\int_{a}^{b} f d\alpha = 0$.

5. Fix $0 < \varepsilon < 1$. Construct a closed set $K \subseteq [0, 1]$ such that $\lambda(K) > 1 - \varepsilon$ and $K$ contains no rationals.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that $g \circ f$ is measurable.
7. Suppose $f$ is a bounded measurable function on the interval $[-\pi, \pi]$. Prove

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0.$$

Hint: This is easy if $f$ is a step function. (This is known as the Riemann-Lebesgue lemma.)

8. Suppose $\{f_n\}$ is a sequence of functions in $L^1(\mathbb{R})$ with $\int_{\mathbb{R}} |f_n(x)| \, dx \leq 1$ for every $n$.

Suppose $\lim_{n \to \infty} f_n(x) = f(x)$ exists for every $x \in \mathbb{R}$.

Prove that $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |f(x)| \, dx \leq 1$.

Give an example to show that it is possible that $\int_{\mathbb{R}} |f_n(x)| \, dx = 1$ for every $n$ yet $\int_{\mathbb{R}} |f(x)| \, dx < 1$. 