1. Let $A$, $B$, and $S$ be $n$-by-$n$ matrices over a field $F$ with $S$ invertible and $S^{-1}AS = B$.

(a) Prove that $A$ and $B$ have the same characteristic polynomial. [Hint: Use the Multiplication Theorem for determinants.]

(b) Prove that if $B$ is a diagonal matrix, then each diagonal entry in $B$ is an eigenvalue of $A$. 
2. Use 1.(b) to prove (by contradiction) that if \( A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \), then there is no \( S \in M_2(\mathbb{C}) \) for which \( S^{-1}AS \) is diagonal.

3. Let \( V \) be a complex inner product space, let \( T : V \to V \) be linear, and let \( \lambda \) be an eigenvalue of \( T \). Prove that

(a) \( T \) is unitary \( \Rightarrow |\lambda| = 1 \) and

(b) \( T \) is self-adjoint (i.e., Hermitian) \( \Rightarrow \lambda \in \mathbb{R} \).
4. Consider the Hermitian matrix \( H = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix} \). Find a unitary matrix \( U \) such that \( U'HU \) is diagonal. [The inner product in \( \mathbb{C}^2 \) is \((a, b) = a \cdot \bar{b}\).]

5. Find a spectral decomposition for the matrix \( H \) in problem 4. That is, find self-adjoint matrices \( E_1 \) and \( E_2 \) in \( M_2(\mathbb{C}) \) with \( E_1E_2 = E_2E_1 = 0 \), \( E_1 + E_2 = I \), and \( H = \alpha E_1 + \beta E_2 \) where \( \alpha \) and \( \beta \) are the eigenvalues of \( H \).