Do all six. Throughout, \((X,A,\mu)\) is a measure space, and \(m\) denotes Lebesgue measure on \(\mathbb{R}\).

1. Let \(g\) be a bounded measurable function on \(\mathbb{R}\) which has the property that for every measurable set \(E\),
   \[
   \lim_{n \to \infty} \int_E g(nx) \, dx = 0. 
   \]
   Show that for every \(f \in L^1(\mathbb{R})\),
   \[
   \lim_{n \to \infty} \int_\mathbb{R} f(x) g(nx) \, dx = 0. 
   \]

2. Fix \(0 < \varepsilon < 1\). Construct a closed set \(K \subset [0,1]\) such that \(K\) contains no rationals and \(m(K) > 1 - \varepsilon\).

3. Let \(\varphi : X \to Y\) where \(Y\) is a topological space.
   
   (a) What is meant by saying \(\varphi\) is measurable?

   (b) Suppose \(\varphi\) is measurable and \(E\) is a Borel subset of \(Y\). Prove that \(\varphi^{-1}(E)\) is measurable.

4. Let \(\{f_n\}\) be a sequence of functions in \(L^1([0,1])\) such that \(f_n(t) \to 0\) for every \(t \in [0,1]\).
   
   Is it true that \(\int_0^1 f_n(t) \, dt \to 0\) ?

5. Let \(f\) and \(f_1, f_2, f_3 \ldots\) be measurable functions from \(X\) to \(\mathbb{C}\). For \(\delta > 0\) and \(n = 1, 2, 3 \ldots\) define \(S_n(\delta) = \{x \in X : |f(x) - f_n(x)| \geq \delta\}\).
   
   Prove that if \(\mu(X) < \infty\) and \(f_n \to f\) a.e. on \(X\) then \(\lim_{n \to \infty} \mu(S_n(\delta)) = 0, \forall \delta > 0\).

6. Suppose \(f_n \to f\) a.e. on \(X\), and \(\int_X |f_n|^2 \, d\mu < \infty\) for every \(n\), \(\int_X |f|^2 \, d\mu < \infty\).
   
   Suppose \(\int_X |f_n|^2 \, d\mu \to \int_X |f|^2 \, d\mu\) as \(n \to \infty\).
   
   Prove that \(\int_X |f_n - f|^2 \, d\mu \to 0\) as \(n \to \infty\).

   Hint: \(|f_n - f|^2 \leq 2(|f_n|^2 + |f|^2)\) (You should prove this.)