(7 pts) 1. Let $f(x)$ be a polynomial in $F[x]$ which has no multiple roots in any extension field of $F$. If $K$ is the splitting field of $f(x)$ over $F$, and $G = Gal(K/F)$, show that $f(x)$ is irreducible in $F[x]$ if and only if $G$ transitively permutes the roots of $f(x)$ in $K$. 
(7 pts) 2. Consider the matrix of rational entries \( A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \).

If \( R = \mathbb{Q}[A] \) is the ring of polynomials in \( A \) with rational coefficients, prove that \( R \) is a field. (Hint: Consider the homomorphism \( \mathbb{Q}[x] \to \mathbb{Q}[A] \), where \( x \mapsto A \).)
(7 pts) 3. Let $R$ be a commutative ring and $N$ be the set of all nilpotent elements in $R$. (An element $x$ in a ring is called nilpotent if $x^n = 0$ for some nonnegative integer $n$.)

(a) Show that $N$ is an ideal of $R$.

(b) Is the statement (a) still correct without the commutativity condition on $R$? Prove or give an example.
(7 pts) 4. Let $n > 1$ be a positive integer. Calculate the degree of the splitting field of $f(x) = x^n - 2$ over the field of rational numbers $\mathbb{Q}$. 
(8 pts) 5. Let $R$ be a ring. Prove that the following three conditions are equivalent for the left $R$-module $M$.

(i) Any increasing chain of submodules $M_1 \subseteq M_2 \subseteq \ldots$, of $M$ eventually stabilizes.

(ii) Any submodule of $M$ is finitely generated.

(iii) Any family of submodules of $M$ has a maximal member with respect to inclusion.
(8 pts) 6. Recall that a ring is called Noetherian if every ascending chain of ideals terminates. Show that any PID is Noetherian. Give an example of a Noetherian integral domain which is not a PID.
(8 pts) 7. Let $V$ be an $n$-dimensional vector space over a field $F$, and let

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$$

be a chain of subspaces of $V$, with $\dim(V_i/V_{i+1}) = 1$ for $i = 0, 1, \ldots, n - 1$. Suppose that $T : V \to V$ is a linear transformation satisfying $T(V_i) \subseteq V_{i+1}$ for all $i = 0, 1, \ldots, n - 1$. Compute the characteristic polynomial of $T$. 
(8 pts) 8. Let $M$ be a Noetherian $R$-module and let $\phi : M \to M$ be a surjective $R$-module homomorphism. Prove that $\phi$ is injective. (Hint: consider the increasing sequence of submodules $0 \subseteq \ker \phi \subseteq \ker \phi^2 \subseteq \ldots$)