(10 pts.) 1.(a) Show that exactly 4 vector subspaces of the plane \( \mathbb{R}^2 \) are invariant under the linear transformation \( T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \in M_2(\mathbb{R}) \).

(b) Give any 2 × 2 real matrix \( S \in M_2(\mathbb{R}) \) such that exactly 3 vector subspaces of the plane \( \mathbb{R}^2 \) are invariant under \( S \).
(10 pts.) 2. The complex numbers \( \mathbb{C} \) are a vector space over the field of real numbers \( \mathbb{R} \). Let \( T : \mathbb{C} \to \mathbb{C} \) be the map \( T(z) = (3 + 2i)z \) for \( z \in \mathbb{C} \). Prove that \( T \) is linear transformation, and compute the determinant \( \det(T) \).
(10 pts.) 3. Let $F \subseteq K$ be a field extension, and assume $\alpha \in K$ is an algebraic element of odd degree. Prove that $\alpha^2$ is an algebraic element of odd degree, and that $F(\alpha^2) = F(\alpha)$. 
(12 pts.) 4. Let $T : V \to W$ be a surjective linear map of vector spaces over a field $F$. Let $B \subseteq W$ be a vector subspace, and define $A = T^{-1}(B) = \{ v \in V \mid T(v) \in B \}$. Prove that the quotient vector space $V/A$ is isomorphic to the quotient $W/B$. 
(12 pts.) 5. (a) Let $A$ and $B$ be $2 \times 2$ matrices, $A$ and $B \in M_2(F)$. Explain why the matrices $AB$ and $BA$ have the same characteristic polynomial.

(b) Let $S : V \to V$ and $T : V \to V$ be two linear transformations of a finite dimensional vector space $V$ over a field $F$. Prove that $\lambda \in F$ is an eigenvalue of the composition $S \circ T$ if and only if $\lambda$ is an eigenvalue of $T \circ S$. 
(12 pts.) 6. Let $V$ be a finite dimensional inner product space over the complex numbers. Let $f : V \to \mathbb{C}$ be a linear functional. Prove there exists a vector $u_0 \in V$ such that $f(v) = \langle v, u_0 \rangle$ for all $v \in V$. 
(12 pts.) 7. (a) Prove that $f(x) = x^4 - 2x^2 - 1$ is irreducible in $\mathbb{Q}[x]$. (Hint: Compute $f(x + 1)$.)

(b) Show that the splitting field $K$ for $f(x) = x^4 - 2x^2 - 1$ over the rational numbers $\mathbb{Q}$ is of degree $[K : \mathbb{Q}] = 8$. 
(12 pts.) 8. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space $V$ over a field $F$. Assume the minimal polynomial $p(x) \in F[x]$ for $T$ factors as $p(x) = f(x)g(x)$ in $F[x]$, and assume that $f(x)$ and $g(x)$ are relatively prime (so there exist polynomials $h(x)$ and $k(x)$ in $F[x]$ such that $f(x)h(x) + g(x)k(x) = 1$). Note that $f(T)$ and $g(T)$ are linear transformations of $V$. Prove that $V$ is the direct sum $V = \ker (f(T)) \oplus \ker (g(T))$. 