Midterm #1
Due at the start of class (2:30 pm) on Monday, February 27, 2006.

Late papers will only be accepted under truly extenuating and documentable circumstances. Please do not let this exam take over your life. You might want to set an upper limit on the amount of time you actively spend working on the exam, say 12 hours (but that’s up to you). While you have a long time window for working this exam, please don’t wait until the last minute to start working on this exam! If you wait until the last minute, you miss out on those wonderful “aha” moments that happen after a problem has been rattling around in the back of your mind for a few days and you suddenly have an idea for a solution. Good luck!

- There are nine problems below. Please turn in six problems for grading. At least one of these problems should be chosen from problems 6 and 7 and at least one of these problems should be chosen from problems 8 and 9. No, you may not mix and match parts from different problems.

- This exam is open book / open notes. This means that you may use the required textbook for Math 634, your notes from Math 633 and Math 634, handouts from Math 633 and Math 634, and your graded homeworks and midterms from Math 633 and Math 634. You may not consult other materials (such as other books, journals, or use the internet) to get help with these problems.

- Please reference any results you use (“by the zzz theorem” or “by Section 11.2 #97 in HW #50”). You may only apply results that we have covered in Math 633 and 634 (ending with Section 11.2). You are not permitted to use results that we haven’t covered. Results from problems that have not been assigned are not available as facts for you to apply. Neither are results concerning uniform convergence, Taylor series, L’Hospital’s rule, etc.

- You may not discuss (conversation, email, etc.) this work with anyone, with the exception of asking the instructor questions about this exam during class. The Advanced Help Sessions are off-limits for advanced calculus questions until the exam is due and the instructor’s office hours are also cancelled until the exam is due.

- Please write your answers in “Claim / Proof” or “Claim / Counterexample” format.

- You must include and sign the honor pledge on your paper:

  _On my honor, as a student, I have neither given nor received unauthorized aid on this academic work._

  Signature: __________________________________________

  (The exam problems are listed on the back of this page.)
1. Let \( \{c_k\} \) be a convergent sequence of real numbers. Suppose that \( c_k \geq d \) for all \( k \in \mathbb{N} \).

(a) Using the \( \epsilon - N \) definition of convergence of a sequence, show that \( \lim_{k \to \infty} c_k \geq d \).

(b) Does this conclusion still hold if we only require that \( c_k > d \) for all \( k \in \mathbb{N} \)? Discuss. In particular, is it possible for \( \lim_{k \to \infty} c_k = d \)? If so, give an example. If not, show that \( \lim_{k \to \infty} c_k > d \).

2. Let \( \{a_k\} \) be a sequence of real numbers.

(a) Show that if \( \sum_{k=1}^{\infty} a_k = \ell \), then \( \sum_{k=1}^{\infty} a_{2k} + a_{2k-1} = \ell \). (Hint: Consider the sequence of partial sums for both series. Is one a subsequence of the other?)

(b) Show that the converse does not hold. (Hint: Consider \( a_k = (-1)^k \).)

3. Let \( \{a_k\} \) be a bounded sequence of real numbers. For each \( k \in \mathbb{N} \), let \( f_k(x) = a_1 + a_2 x + \frac{a_3 x^2}{2} + \ldots + \frac{a_{k+1} x^k}{k!} \).

Does the sequence \( \{f_k : \mathbb{R} \to \mathbb{R}\} \) converge pointwise on \( \mathbb{R} \)? Justify your answer.

4. A sequence of points \( \{u_k\} \) in \( \mathbb{R}^n \) is said to be a Cauchy sequence provided for each \( \epsilon > 0 \), there exists \( K(\epsilon) \in \mathbb{N} \) such that whenever \( k \geq K(\epsilon) \) and \( \ell \geq K(\epsilon) \), we have \( \text{dist}(u_k, u_\ell) < \epsilon \).

(a) Show that \( \{u_k\} \) is a Cauchy sequence iff each component sequence is a Cauchy sequence.

(b) Show that a sequence in \( \mathbb{R}^n \) converges iff it is a Cauchy sequence. (Hint: This was proved in Section 9.1 for sequences in \( \mathbb{R} \).)

5. Let \( A \subseteq \mathbb{R}^n \). Let the closure of \( A \), denoted by \( \text{cl} A \), be defined as \( \text{cl} A = \text{int} A \cup \partial A \).

(a) Show that \( A \subseteq \text{cl} A \).

(b) Show that \( A = \text{cl} A \) iff \( A \) is closed in \( \mathbb{R}^n \).

6. Let \( A \) be a nonempty subset of \( \mathbb{R}^n \) and let \( a \in A \). Suppose that \( f : A \to \mathbb{R}^m \) satisfies the \( \epsilon - \delta \) definition of continuity at \( a \). Using the two definitions of continuity, show that \( f \) satisfies the sequential definition of continuity at \( a \).

7. Suppose that \( \mathcal{O} \subseteq \mathbb{R}^n \) is open and nonempty and suppose that \( f : \mathcal{O} \to \mathbb{R}^m \) is continuous. Suppose that \( u \in \mathcal{O} \) and that \( f(u) \neq 0 \). Show that there exists an open ball \( \mathcal{B} \) about \( u \) and such that \( \|f(v)\| > \|f(u)\|/3 \) for all \( v \in \mathcal{B} \).

8. Fix \( v \in \mathbb{R}^n \) and let \( f(u) = \langle u, v \rangle \). Is \( f : \mathbb{R}^n \to \mathbb{R} \) uniformly continuous? Justify your answer using the \( \epsilon - \delta \) definition or a counterexample.

9. Suppose that \( A \subseteq \mathbb{R}^n \) is nonempty and bounded. Suppose that \( f : A \to \mathbb{R}^m \) is uniformly continuous.

(a) Show that \( f(A) \) is bounded.

(b) If we assume that \( f \) is continuous (instead of uniformly continuous) on \( A \), will \( f(A) \) always be bounded?