PLEASE READ THIS PAGE!!!

1. **Important notices:**
   - Each problem is worth the same total number of points (independent of the number of parts in a problem).
   - Please indicate which six problems (of the nine problems) should be graded by clearly circling either “YES” or “NO” after “GRADE THIS PROBLEM?” at the top of each page. No, you may not mix and match parts from multiple problems.
   - If you do not clearly indicate which six problems should be graded, the instructor will grade all problems which you have attempted and then will use your lowest six scores.
   - You are allowed to use the sheet of notes that you prepared for use with this exam (both sides of one 8\(\frac{1}{2}\)" × 11" sheet of handwritten notes; no attachments). These notes must be turned in with your exam. No other type of written materials is allowed. No calculators, computers, cell phones, PDAs, (and the like) should be used during the exam.

2. **Hints:**
   - You might want to quickly look over all of the questions and start by working the questions that are easiest for you.
   - Show your work! If you are applying a theorem, state which theorem.
   - We’ve discussed a number of techniques this semester. Sometimes converting an ODE to a system of ODEs or converting it to polar coordinates can be helpful.
   - You don’t get extra points for finishing early. If you have extra time, please consider checking your work over one more time before turning in your paper.

3. **Honor pledge:** Read the following statement and sign your name:

   *On my honor, as a student, I have neither given nor received unauthorized aid on this academic work.*

   Signature: ____________________________________________

4. Please make sure that your exam contains *ten pages*, including this one.

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1. Circle either “True” or “False” for each of the following. You are not required to give reasons for your answers. You might earn partial credit for an incorrect True / False answer if you include a brief reason for your answer.

a) True False: Suppose that you need to approximate a continuous function \( f(x) \) on \([-1, 1]\) by a polynomial \( p(x) \) of degree \( \leq n \). Then the best approximation in the sense of least squares is given by the first \( n + 1 \) terms of the Legendre series for \( f(x) \).

b) True False: There is no initial value problem that has an infinite number of distinct solutions.

c) True False: \( 1/p! = 1/\Gamma(p + 1) \) is defined for all \( p \in \mathbb{R} \) and is zero when \( p \) is a negative integer.

d) True False: If we are modeling a problem using Bessel’s equation and we know that our solution \( x(t) \) must be bounded near \( t = 0 \), then our solution must be of the form \( x(t) = c J_p(t) \) for some real constant \( p \geq 0 \).

e) True False: In a \( 2 \times 2 \) linear autonomous system of ODEs, the paths near a center are always circles.

f) True False: Suppose that you are analyzing the behavior of a \( 2 \times 2 \) nonlinear autonomous system of ODEs near a simple critical point. In your linearized system \( X' = AX \), \( trA = -3 \) and \( det A = 1 \). Then this critical point acts as a sink node.

g) True False: Suppose that you are analyzing the behavior of a \( 2 \times 2 \) nonlinear autonomous system of ODEs near a simple critical point. In your linearized system \( X' = AX \), \( trA = -3 \) and \( det A = 1 \). Then the paths for your nonlinear system near this critical point must be identical to those for your linearized system.

h) True False: Suppose that your \( 2 \times 2 \) autonomous system of ODEs has a closed path in the phase plane. Then your system has a periodic solution.
2. (a) State the Sturm comparison theorem.
   
   (b) Give an intuitive explanation of why this result seems reasonable.
   
   (c) How is this result different from the Sturm separation theorem?
3. Recall that for any $p \in \mathbb{R}$,

$$J_p(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+p}}{n!(p+n)!}$$

(a) Show that

$$\frac{d}{dx}[x^{-p}J_p(x)] = -x^{-p}J_{p+1}(x).$$

(b) Prove that the positive zeros of $J_p(x)$ and $J_{p+1}(x)$ alternate (e.g., are interlaced).

*Hint:* Use part (a). You might also find the following fact useful:

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x).$$
4. (a) Show that the change of variable $Bx = \frac{1}{u} \frac{du}{dt}$ transforms the ODE

$$\frac{dx}{dt} + Bx^2 = Ct^m$$

into

$$\frac{d^2u}{dt^2} - BCt^m u = 0.$$ 

(b) Apply this change of variables to the ODE

$$\frac{dx}{dt} = t^2 + x^2.$$ 

(c) Solve your ODE from (b) for $u(t)$.

*Hint:* In one of our homeworks, we showed that the change of variables $z = ax^b$ and $w = yx^c$ converts Bessel’s equation $z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + (z^2 - p^2) w = 0$ into

$$x^2 \frac{d^2u}{dx^2} + (2c + 1) x \frac{du}{dx} + [a^2 b^2 x^2 + (c^2 - p^2 b^2)] y = 0.$$ 

Here, $a$, $b$, and $c$ are constants.
5. For which of the following ODEs does an oscillatory solution exist for \( t > 0 \)? Justify your answer.

(a) \( \frac{d^2x}{dt^2} - 3x/(\sin(t) - 5) = 0 \)

(b) \( \frac{d^2x}{dt^2} - 3(t^2 + 1) x = 0 \)

(c) \( \frac{d^2x}{dt^2} - \frac{dx}{dt} + 3x = 0 \)
6. For which of the following ODEs does a periodic solution exist? Justify your answer.

(a) \( \frac{d^2x}{dt^2} - (x^2 + 1) \frac{dx}{dt} + 3(x^2 + 1) = 0 \)

(b) \( \frac{d^2x}{dt^2} + (5x^4 - x^2) \frac{dx}{dt} + x^3 = 0 \)

(c) \( \frac{d^2x}{dt^2} - \frac{dx}{dt} - 3(\frac{dx}{dt})^2 - 5x^3 = 0 \)
7. Consider the autonomous system of ODEs
\[
\begin{align*}
    x'(t) &= -2x + y + (x^2 + y^2)x \\
    y'(t) &= -x - 2y + (x^2 + y^2)y.
\end{align*}
\]
and answer the following:

(a) Verify that \((0, 0)\) is a simple critical point.

(b) Characterize this critical point (node, saddle, etc.) and discuss its stability.

(c) Does a periodic solution exist for this system of ODEs? Justify your answer.
8. Let $A = \begin{pmatrix} 2 & 0 \\ -2 & 4 \end{pmatrix}$.

(a) Find $e^{tA}$. Note that your answer will be real-valued. Show your work.

(b) Sketch the behavior of solutions of $X' = AX$ near $(0, 0)$ in the phase plane.
9. Prove that the initial value problem

\[
\frac{dx}{dt} = t^3 \cos(tx), \quad x(0) = 42
\]

has a unique solution for \(-1 \leq t \leq 1\).