1. Circle either "True" or "False" for each of the following (two points each):
   a) True (False): In Pólya's four steps, you restate the problem in your own words in the "looking back" step.
      (Understand the problem) step
   b) True (False): The information about football games is extraneous for solving the following problem: "In two years, Zachary will be 25. He hasn't missed a Wildcat home football game in five years. How old is Zachary now?" The information about football games does not tell us anything about Zachary's age which we've asked to find.
   c) True (False): When you claim that the last digit of the product of two consecutive numbers can never be a three, checking a few examples is not enough to show that your claim is true.
   d) True (False): $(a^n)^m = a^{m+n}$ is always true when $a$, $m$, and $n$ are counting numbers. Recall that the counting numbers are $\{1, 2, 3, \ldots\}$. $(a^n)^m = a^{m+n} = a^m \cdot a^n$
   e) True (False): Every numeration system that we have studied this semester has a symbol representing the number zero. (See 2.3 part 5) (Yes, Roman numeration has no symbol for zero)
   f) True (False): $10 = 110_2$ $110_2 = (1 \cdot 2^2) + (1 \cdot 2) + (0 \cdot 1) = 4 + 2 = 6 \neq 10$
   g) True (False): The multiplicative identity for the whole numbers is 0. (Is the identity for multiplication)

2. Find the missing term in each of the sequences below. Show your work. Also circle the sequence type. (two points for the missing term, one point for the sequence type.)
   a) $2, 5, 8, 11, 14, \boxed{17}$ Arithmetic Geometric Neither
      
Consecutive terms differ by 3, so this is an arithmetic sequence with common difference $d = 3$. Next term is $14 + 3 = 14 + 3 = 17$.

   b) $1, 2, 3, 3, 3, 4, \boxed{4}$ Arithmetic Geometric Neither
      
$2 - 1 = 1$ no common difference Pattern is one 1, two 2's, three 3's $2 - 1 = 1$ no common ratio Expanding four 4's, so next term is 4.

   c) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \boxed{\frac{1}{81}}$ Arithmetic Geometric Neither
      
$\frac{1}{3} \div \frac{1}{3} = \frac{1}{3}$ common ratio is 1/3, so this is a geometric sequence.

   d) $3, 7, 15, 31, 63, \boxed{127}$ Arithmetic Geometric Neither
      
No common difference. Notice that the sequence of differences is geometric (common ratio is 2). The next difference should be $(32 \cdot 2) = 64$, so the next term of our sequence is $63 + 64 = 127$. Also, sequence is not geometric since $7/3 \neq 15/4$ (if $7/3 = 15/4$, then $7 \cdot 4 = 3 \cdot 15$ or $9 = 45$; not true) Alternative approach Notice pattern $n^{th}$ term = $(2 \cdot (n-1)^{st} + 1$ and check that it is consistent with the sequence given.


3. Our universe is $U = \{a, b, d, e, f, g, h, i, j, k, m\}$. Let $A = \{a, b, d, e, f\}$, $B = \{d, e, g, h, i\}$, and $C = \{e, f, g, h, j, k\}$.

a) Draw a Venn diagram representing this information. (four points)

b) List the elements in $(A \cup B) \cap C$. (four points)

- $A \cup B = \{a, b, d, e, f, g, h, i\}$
- $(A \cup B) \cap C = \{e, f, g, h\}$

(c) List the elements in $A \cup B \cup C$. (four points)

4. Write the base 10 numeral 1,230,056 in expanded notation. (five points)

$1,230,056 = (1 \times 10^6) + (2 \times 10^5) + (3 \times 10^4) + (0 \times 10^3) + (5 \times 10^2) + (6 \times 10^1)$

5. Which base six numeral follows 5456? Explain your reasoning. (five points)

- $545_6 + 1 = 540_6 + (5_6 + 1) = 540_6 + 10_6 = 550_6$

Alternate approach:

- $545_6 = 5$ flats $+ 4$ longs $+ 5$ units
- Notice: $5$ units + $1$ unit = $6$ units = $1$ long
- So $545_6 + 1 = 5$ flats $+ 4$ longs $+ (5$ units + $1$ unit)$ = 5$ flats $+ 4$ longs $+ 1$ long
- $545_6 + 1 = 5$ flats $+ 4$ longs $= 550_6$
6. Convert the following numerals to base 10 notation. Show your work. (each conversion is worth five points)

a) MCMXLVI (Recall that for Roman numerals, M = 1000, D = 500, C = 100, L = 50, X = 10, V = 5, and I = 1.)

\[
\begin{align*}
\text{MCMXLVI} & = 1000 + (1000-100) + (50-10) + 5 + 1 \\
& = 1000 + 900 + 40 + 6 = 1946
\end{align*}
\]

b) \(135_8\)

\[
\begin{array}{c|c|c|c}
8^2 & 8^1 & 8^0 \\
1 & 3 & 5 & \hline
\end{array}
\]

\[
135_8 = (1 \times 8^2) + (3 \times 8^1) + (5 \times 8^0)
\]

\[
= 1 \times 64 + 3 \times 8 + 5 \times 1
\]

\[= 64 + 24 + 5 = 88 + 5 = 93
\]

7. a) Illustrate the addition of 2 and 7 on the whole number line. (five points)

![Whole number line illustration]

b) Illustrate \(9 - 2\) on the whole number line using the missing addend approach. (five points)

![Whole number line illustration]

**Missing Addend:** Next \(x\) so that \(2 + x = 9\)

8. We can write \((39 \cdot 37) + (37 \cdot 61)\) as the product of two whole numbers (namely, \(100 \cdot 37\)) by using the properties of the whole number system under addition and multiplication. Justify each step of the simplification shown below (i.e., give a reason why each "=" sign holds without explicitly calculating both sides.) (five points)

\[
(39 \cdot 37) + (37 \cdot 61) = (39 + 61) \cdot 37 \quad \text{commutative property (multiplication)}
\]

\[
= (39 + 61) \cdot 37 \quad \text{distributive property (multiplication over addition)}
\]

\[
= (39 + (1 + 60)) \cdot 37 \quad \text{addition fact \(61 = 1 + 60\)}
\]

\[
= ((39 + 1) + 60) \cdot 37 \quad \text{associative property (addition)}
\]

\[
= (40 + 60) \cdot 37 \quad \text{addition fact \(39 + 1 = 40\)}
\]
9. A small diner has recently surveyed all of its regular customers to find out which flavors of ice cream these customers enjoy. The results of the survey are shown in the Venn diagram below. The diner has 60 regular customers.

![Venn Diagram](image)

a) How many of the regular customers like all three flavors of ice cream? (two points)

10 customers

b) How many like both vanilla and chocolate ice cream? Show your work. (two points)

\[ 5 + 10 = 15 \text{ customers} \]

c) How many of the regular customers don't like any of the three ice cream flavors shown? Explain your reasoning. (two points)

The Venn diagram shows \(10 + 5 + 10 + 6 + 15 + 3 + 2\) = 51 customers who like one or more of the ice cream flavors. There are 60 regular customers. So, \(60 - 51 = 9\) customers don't like any of the three ice cream flavors.

d) Which of the three flavors of ice cream is liked by the greatest number of the regular customers? Explain your reasoning. (five points)

Vanilla: 10 like vanilla only
5 like vanilla and choc, but not strawberry
10 like all 3 flavors
6 like vanilla and strawberry, but not chocolate
31 like vanilla

Similarly, \((15 + 5 + 10 + 3) = 33\) like chocolate
\((2 + 6 + 10 + 3) = 21\) like vanilla

So, chocolate ice cream is liked by the greatest number of the regular customers.
10. We will define a function $f$ on \( \{0, 1, 2, 3, \ldots\} \) by the following rule: $f(x)$ is the remainder when you divide $x$ by 4 using the division algorithm. For example, $f(5) = 1$, because $5 \div 4 = (1 \cdot 4) + 1$. Also, $f(12) = 0$, since $12 \div 4 = (3 \cdot 4) + 0$.

a) What is the domain of $f$? (two points)

\[ \{0, 1, 2, 3, \ldots\} \rightarrow \text{(whole numbers)} \]

b) What is the range of $f$? Explain your answer. (four points)

By the division algorithm for whole numbers, the possible remainders when we divide by 4 are $0, 1, 2, 3$. So $f$ can only take these values.

\[
\begin{align*}
  f(0) &= 0 & \text{since } 0 &= (0 \cdot 4) + 0 \\
  f(1) &= 1 & \text{since } 1 &= (0 \cdot 4) + 1 \\
  f(2) &= 2 & \text{since } 2 &= (0 \cdot 4) + 2 \\
  f(3) &= 3 & \text{since } 3 &= (0 \cdot 4) + 3
\end{align*}
\]

The range of $f$ is $\{0, 1, 2, 3\}$

c) Is $f$ a 1-1 correspondence between $\{2, 4, 6, 8\}$ and $\{0, 1, 2, 3\}$? Explain your answer. (four points)

\[
\begin{align*}
  f(2) &= 2 \\
  f(4) &= 0 & (4 \div 4 = 1) \\
  f(6) &= 2 & (6 \div 4 = 1) \\
  f(8) &= 0 & (8 \div 4 = 2)
\end{align*}
\]

Not a 1-1 correspondence: Four different reasons (one is enough)

1. No $x \in \{2, 4, 6, 8\}$ with $f(x) = 1$
2. No $x \in \{2, 4, 6, 8\}$ with $f(x) = 3$
3. $f(2) = f(6) = 2$
4. $f(4) = f(8) = 0$

11. a) Is the set $A = \{1\}$ closed under division? Explain why or why not. (three points)

\[
\begin{align*}
  1 \div 1 &= 1 \\
  1 &\in A
\end{align*}
\]

(we need not worry about $3 \div 1$, $1 \div 3$ and the like, because $A = \{1\}$ and 3 is not in $A$)

b) Is there a subset of the counting numbers $\{1, 2, 3, \ldots\}$ that contains more than one element and is also closed under division? Note that the elements must be different. List the elements of the subset or explain why this can't be done. (three points)

No. Let our subset be $B$. If $a \in B$, then $a \div a = 1$ must be in $B$. But if $a$ and $1 \in B$, then $1 \div a \in B$.

Since $B$ must contain at least two different counting numbers, $a \neq 1$. So $1 \div a$ is not a whole number and thus is not in $B$. So $B$ cannot be closed under division.

Alternate approach Since $B$ contains at least two different counting numbers, we can choose $a, b \in B$ with $a < b$. Since $b$ is a counting number, $b \neq 0$ and $a \div b$ is defined. But $a \div b$ is not a whole number! So $a \div b$ is not in $B$ and we have shown that $B$ cannot be closed under division.