Show all your work in the space provided under each question.

1. Find a basis for the nullspace of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

2. Consider the polynomials $p_1(x) = 2$, $p_2(x) = x^2 + 1$, $p_3(x) = x^2 + 2x + 1$ in $\mathcal{P}_2 = \{p(x) : \deg p(x) \leq 2\}$ the set of all polynomials of degree less than or equal to 2. Are $p_1$, $p_2$, $p_3$ linearly independent? Show your computations.
3. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$. Compute $A^{-1}$.

4. Find the coordinates of the vector $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ with respect to the basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^3$. 
5. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(v) = Av$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. Find a basis for the image of $T$ and a basis for the nullspace of $T$.

6. Use the Gram-Schmidt process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by \[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\]
7. Compute $\det A$ where $A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

8. On $[-1, 1]$ define $f$ by $f(x) = 1$ if $x \geq 0$, $f(x) = -1$ if $x < 0$. Compute the best approximation to $f(x)$ by functions of the form $c_1 \sin \pi x + c_2 \sin 2\pi x$; that is, find $c_1$, $c_2$, which minimizes $\|f(x) - (c_1 \sin \pi x + c_2 \sin 2\pi x)\|$. 
9. Let $W$ be the subspace of $\mathbb{R}^4$ spanned by 
\[
\begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
\end{bmatrix}
\quad \text{and} \quad 
\begin{bmatrix}
2 \\
1 \\
2 \\
1 \\
\end{bmatrix}
\] 
Find $\text{proj}_W b$ where $b = 
\begin{bmatrix}
1 \\
1 \\
1 \\
0 \\
\end{bmatrix}.

10. Suppose $A$ is an $n \times n$ matrix and $\lambda$ is a fixed eigenvalue of $A$. Let 
\[W = \{v \in \mathbb{R}^n : Av = \lambda v\}.
\] 
Prove that $W$ is a subspace of $\mathbb{R}^n$. 
11. Find all the eigenvalues and a basis of each eigenspace for the matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}. \]

12. Find a matrix \( P \) so that \( P^{-1}AP \) is diagonal where \( A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}. \)
13. In a galaxy far, far away are two planets, Gykx and Mykx. Each year 40% of the residents of Gykx move to Mykx and 10% of the residents of Mykx move to Gykx; the remainder do not move. If this year there are exactly 1 million residents on each planet, what is the equilibrium distribution of residents on each planet?

14. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with the property that the equation $T(x) = b$ has a unique solution for every $b \in \mathbb{R}^m$. What is the rank of $T$? What can you say about the relative sizes of $m$ and $n$?
15.a) Find a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates the plane by $\frac{\pi}{6}$ radians around the origin.

b) Let $A$ be the matrix associated to $T$ as in (a). Is $A$ orthogonal? Give reasons to support your claim.