CALCULUS II - FINAL EXAM
December 14, 2005

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin.

\[
\int a^x \, dx = \frac{1}{\ln a} a^x + C \quad \int \sin x \, dx = -\cos x + C
\]

\[
\int \cos x \, dx = \sin x + C \quad \int \tan x \, dx = -\ln |\cos x| + C
\]

\[
\int \cot x \, dx = \ln |\sin x| + C \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

\[
\int \csc x \, dx = -\ln |\csc x + \cot x| + C \quad \int \sec^2 x \, dx = \tan x + C
\]

\[
\int \csc^2 x \, dx = -\cot x + C \quad \int \sec x \tan x \, dx = \sec x + C
\]

\[
\int \csc x \cot x \, dx = -\csc x + C \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + C
\]

\[
\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \quad \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \text{arcsec}\left(\frac{|x|}{a}\right) + C
\]

\[
\int \sqrt{a^2 - u^2} \, du = \frac{1}{2} \left( u\sqrt{a^2 - u^2} + a^2 \arcsin\left(\frac{u}{a}\right) \right) + C,
\]

\[
\int \sqrt{u^2 + a^2} \, du = \frac{1}{2} \left( u\sqrt{u^2 + a^2} \pm a^2 \ln |u + \sqrt{u^2 + a^2}| \right) + C
\]

\[
\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx
\]

\[
\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, \ n \neq 1
\]

\[
\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, \ n \neq 1
\]
1. Calculate the following derivatives. Do not simplify.

(8) a) \( \frac{d}{dx} \tan(\sqrt{x \ln(x^2 + 1)}) \)

(8) b) \( \frac{d}{dx} (\sin x)^{1/x} \) (Logarithmic differentiation can be used if you like.)

(10) 2. Find the slope of the curve \( xe^y + y^3 = x^2 + 8 \) at \((0, 2)\).

(10) 3. Solve the initial value problem, \( \frac{dp}{dt} = (1 + p^2)t \), \( p(0) = 1 \).
4. Evaluate the following integrals.

\[ \int \frac{x^3 + 1}{x^2 - 1} \, dx = \]

\[ \int \ln(x) \, dx = \]

\[ \int \frac{dx}{(1 + x^2)^{1/2}} = \]

\[ \int \sec^3 x \, dx = \]
5. Evaluate the following limits or indicate that they diverge. Show all work.

(6) a) \( \lim_{x \to 0} \frac{3x^2 + x}{\sin(2x)} \)

(6) b) \( \lim_{x \to \infty} \sin\left(\frac{1}{x}\right)e^x \)

(7) 6. Evaluate the improper integral or state that it diverges. \( \int_0^2 \frac{1}{\sqrt{2 - x}} \, dx \)
Use proper limit notation.

(7) 7. Evaluate the sum \( \sum_{k=3}^{\infty} \frac{1}{k^2 - 2k} \)

8. Let \( S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} \).

(4) a) Mark the positions of the partial sums \( S_1, S_2, S_3 \) and the sum \( S \) on the number line below.

(4) b) How many terms are required to estimate the sum \( S \) with an error less than .01?
9. Determine whether the following series converge or diverge. State clearly which test you are using and implement the test as clearly as you can. The answer for each problem is worth 2 points and the work you show 6 points.

(8) a) \[ \sum_{n=2}^{\infty} \frac{n^2 - 2}{\sqrt{n} \cdot (n^3 + n)} \]

(8) b) \[ \sum_{n=1}^{\infty} e^{1/n} \]

(8) c) \[ \sum_{n=1}^{\infty} \sin(n)e^{-n} \]

(10) 10. Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} (-1)^n \frac{(x + 5)^n}{n3^n} \). (Make clear the status of any end points.)
11. Let \( P_2(x) \) be the second Taylor polynomial for the function \( f(x) = \cos(2x) \) centered at \( c = \pi/2 \). 

(6) a) Calculate \( P_2(x) \).

(4) b) Find an upper bound on the remainder \( R_2(x) = f(x) - P_2(x) \) using the Taylor remainder theorem.

(2) c) Give a precise verbal description of the graph of \( P_2(x) \) in comparison to the graph of \( f(x) \). (This is the defining property of a Taylor polynomial.)

12. a) Given \( \ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \), find the first three nonzero terms of the Maclaurin series for the following functions:

a) \( x \ln(1 + x^2) = \)

b) \( \frac{\ln(1 + x)}{1 - x + x^3} = \)
13. Consider the curve with parametric equations \( x = 2\sin(t) , \ y = 3\cos(t) , \ t \geq 0 \).

\[(5)\]  a) Find the slope of the curve at \((\sqrt{2}, \frac{3}{2}\sqrt{2})\), \((t = \pi/4)\).

\[(5)\]  b) Find the corresponding rectangular equation for the curve by eliminating the parameter \( t \).

\[(10)\]  14. Find the length of the curve \( x = t^2 , \ y = 2t , \ 0 \leq t \leq 1 \).
15. Convert the polar equation \( r = 4 \sin(\theta) \) to a rectangular equation in \( x \) and \( y \) and sketch the graph below.

16. Find the area bounded by one petal of the rose \( r = \sin(2\theta) \).