CALCULUS II - EXAM 2
March 6, 2007

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin. You have 65 minutes.

\[ \int \sec x \, dx = \ln | \sec x + \tan x | + C \quad \int \csc x \, dx = - \ln | \csc x + \cot x | + C \]
\[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left( \frac{x}{a} \right) + C \quad \int \sqrt{a^2 - u^2} \, du = \frac{1}{2} \left( u \sqrt{a^2 - u^2} + a^2 \arcsin \frac{u}{a} \right) + C, \]
\[ \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C \quad \int \sqrt{u^2 \pm a^2} \, du = \frac{1}{2} \left( u \sqrt{u^2 \pm a^2} \pm a^2 \ln | u + \sqrt{u^2 \pm a^2} | \right) + C \]

1. Evaluate the improper integrals or show that they diverge. Make careful use of limit notation.

(8) a) \[ \int_{1}^{4} \frac{dx}{\sqrt{4-x}} \]

(8) b) \[ \int_{3}^{\infty} \frac{1}{x-2} - \frac{1}{x-1} \, dx \]
(12) 2. Calculate the length of the catenary curve $y = \cosh x$, $a \leq x \leq b$.

Given: $\frac{d}{dx} \cosh x = \sinh x$, $\frac{d}{dx} \sinh x = \cosh x$, $\cosh^2(x) - \sinh^2(x) = 1$. 
3. a) Set up an integral for the area of the surface obtained by rotating the curve $y = \ln x$, $1 \leq x \leq 2$, around the $y$-axis.

b) Evaluate the integral in part (a). You may use the integral formulas on page 1.
4. Find the centroid of the region bounded by the curves \( y = x^2 \) and \( y = 1 \). Use symmetry where possible. Recall, for the region trapped between \( y = f(x) \), \( y = g(x) \), \( a \leq x \leq b \), with uniform density \( \rho = 1 \) we have 

\[
M_x = \frac{1}{2} \int_a^b f(x)^2 - g(x)^2 \, dx, \quad M_y = \int_a^b x(f(x) - g(x)) \, dx.
\]

5. Find the center of mass for the region \( S \cup T \) (the union of \( S \) and \( T \)) where \( S \) is a square of density \( \rho \) with vertices \((0, 0), (2, 0), (0, 2), (2, 2)\) and \( T \) is a triangle of density \( 2\rho \) with vertices at \((0, 0), (-3, 0), (0, -3)\). (Start by drawing a picture of the region.)
6. The population $P = P(t)$ of a colony satisfies the differential equation

$$\frac{dP}{dt} = \frac{1}{1000} P(1000 - P)$$

a) For what values of $P$ is the population increasing?

b) If the initial population is $P(0) = 2$, at what rate is the population growing at time $t = 0$?

c) Make a sketch of the graph of $P(t)$ given the initial condition $P(0) = 2$. You do not need to solve the differential equation to do this. Indicate clearly the behavior of $P(t)$ as $t \to \infty$.

7. Solve the following differential equation with initial condition.

$$\frac{dy}{dx} = (1 + y)e^{2x} \quad y(0) = 0.$$
8. a) Make a rough sketch of the curve \( x = 1 - \cos(t) \), \( y = \sin^2(t) \), \( 0 \leq t \leq \pi \).
Place an arrow on the curve to show the direction you travel with increasing \( t \). (It may help to do part (b) first.)

b) Convert the equation in part (a) to a cartesian equation in \( x, y \).

9. An object travels twice counterclockwise around a circle of radius 2 centered at (1,2) starting from the point (3,2). Give a set of parametric equations for the curve it follows.