CALCULUS II - FINAL EXAM
December 13, 2006

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin. You have 1 hour and 50 minutes.

\[ \int \sec x \, dx = \ln |\sec x + \tan x| + C \]

\[ \int \csc x \, dx = -\ln |\csc x + \cot x| + C \]

\[ \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left( \frac{x}{a} \right) + C \]

\[ \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C \]

\[ \int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \text{arcsec} \left( \frac{|x|}{a} \right) + C \]

\[ \int \sqrt{a^2 - u^2} \, du = \frac{1}{2} \left( u \sqrt{a^2 - u^2} + a^2 \arcsin \frac{u}{a} \right) + C \]

\[ \int \sqrt{u^2 \pm a^2} \, du = \frac{1}{2} \left( u \sqrt{u^2 \pm a^2} \pm a^2 \ln |u + \sqrt{u^2 \pm a^2}| \right) + C \]

Centroid for the region trapped between \( y = f(x) \), \( y = g(x) \), \( a \leq x \leq b \), (with \( \rho = 1 \)) \( M_x = \frac{1}{2} \int_a^b f(x)^2 - g(x)^2 \, dx \), \( M_y = \int_a^b x(f(x) - g(x)) \, dx \)

Maclaurin Series:

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^n+1 x^n}{n} \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \]
1. Evaluate the following integrals.

\[ (12) \quad a) \int \frac{\ln(x)}{x^2} \, dx \]

\[ (12) \quad b) \int \frac{x^2}{\sqrt{1-x^2}} \, dx \]

\[ (12) \quad c) \int \frac{x^3 + 2}{x^2 - x} \, dx \]
2. Let $R$ be the region trapped between $y = \cos x$ and $y = \sin x$ with $0 \leq x \leq \pi/4$.

(6) a) Find the area of the region $R$.

(8) b) Find $\bar{y}$, the $y$ coordinate of the centroid of $R$. (Do not calculate $\bar{x}$.)

Hint: $\cos(2x) = \cos^2(x) - \sin^2(x)$.

3. Evaluate the following limits or indicate that they diverge. Show all work.

(8) a) $\lim_{x \to 0} \frac{e^{3x} - 1 - 3x}{x \sin(x)}$

(8) b) $\lim_{x \to \infty} x^{1/x}$
4. Solve the initial value problem, \( \frac{dy}{dt} = \frac{e^y}{t}, \ y(1) = 2 \).

5. Find the interval of convergence of the power series \( \sum_{n=1}^{\infty} \frac{(x + 4)^n}{\sqrt{n} \cdot 3^n} \). (Make clear the status of any end points.)
6. Determine whether the following series converge or diverge. State clearly which test you are using and implement the test as clearly as you can. The answer for each problem is worth 2 points and the work you show 4 points.

(6) a) \( \sum_{n=3}^{\infty} \frac{\ln n}{n} \)

(6) b) \( \sum_{n=2}^{\infty} \frac{n}{n^3 - 3} \)

(6) c) \( \sum_{n=1}^{\infty} \sin \left( \frac{\pi n}{2} \right) \)

7. Let \( S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \).

(4) a) Mark the positions of the partial sums \( S_1, S_2, S_3 \) and the sum \( S \) on the number line below.

\[
\begin{array}{c}
-1 & 0 & 1
\end{array}
\]

(4) b) How many terms are required to estimate the sum \( S \) with an error less than .01?

(3) c) Evaluate the sum \( S \) by making use of a familiar series; see cover page.
8. Let $T_2(x)$ be the second degree Taylor polynomial for the function $f(x) = \sin(2x)$ centered at $a = \pi/4$ and $R_2(x) = f(x) - P_2(x)$.

\(8\) a) Calculate $T_2(x)$.

\(4\) b) Find an upper bound on the remainder $R_2(x) = f(x) - T_2(x)$ valid for any real number $x$, using Taylor’s inequality.

\(3\) c) $T_2(x)$ is the unique quadratic polynomial satisfying what three properties in terms of the graph of $f(x)$. (These are the defining properties of a Taylor polynomial.)

\(4\) 9. a) Use the geometric series formula to find a power series expansion for 
\[
\frac{1}{1 + x^2}
\]

\(4\) b) Deduce from part (a) a power series expansion for $\arctan x$. (Hint: See cover page.)

\(2\) c) Use part (b) to evaluate $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \ldots$
10. a) Use series given on the cover sheet to find the first three nonzero terms of the Maclaurin series for $e^x \ln(1 + x^2)$.

b) Use long division to find the first three nonzero terms of the Maclaurin series for

$$f(x) = \frac{1 + x + 2x^2 + 3x^3 + 4x^4 + \ldots}{1 - x + x^2}$$

11. Use a binomial expansion to find the first three nonzero terms of the Maclaurin series for $f(x) = \sqrt{1 + 3x}$.

12. Sketch the graph of the polar equation $r = 4 \cos(\theta)$, $0 \leq \theta \leq \pi$, and convert the equation to a rectangular equation in $x$ and $y$. What familiar shape is it?
13. Consider the curve with parametric equations \( x = e^{3t} \), \( y = \sin(2t) \).

(10) a) Find the equation of the tangent line to the curve at \( t = 0 \).

(10) b) Set up \textbf{but do not evaluate} an integral representing the length of the curve above with \( 0 \leq t \leq \pi \).

(10) 14. Set up \textbf{but do not evaluate} an integral representing the area bounded by one petal of the rose \( r = \sin(7\theta) \).