(12 pts.) 1. Approximate the square root $\sqrt{47}$ in two ways:

(a) Using the tangent line to $y = \sqrt{x}$ at $x = 49$.

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$m = y'(49) = \frac{1}{2\sqrt{49}} = \frac{1}{14}$$

$$y_0 = f(x_0) = \sqrt{49} = 7$$

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

$$\sqrt{47} \approx 7 + \frac{1}{14}(47-49) = 7 - \frac{2}{14} = \frac{6}{7} = \frac{48}{7}$$

$$= 6.857142$$

(b) Using Newton’s Method for $f(x) = x^2 - 47$ with initial estimate $x_1 = 7$ (compute one step, that is, $x_2$ only).

$$f'(x) = 2x$$

$$f'(7) = 49 - 47 = 2$$

$$f'(7) = 2(7) = 14$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 7 - \frac{2}{14} = \frac{6}{7} = \frac{48}{7}$$

(Note: $\sqrt{47} \approx 6.857142$)
(24 pts.) 2. Find the derivative of each function. (Do not simplify.)

(a) \( x = 5^\cos(t) \)

\[
\frac{dx}{dt} = 5^{\cos(t)} \cdot \ln(5) \cdot (-\sin(t))
\]

Note: \( x = e \)

(b) \( y = \log_{10}(1 + \tan(2x)) \)

\[
\frac{dy}{dx} = \frac{1}{1 + \tan(2x)} \cdot \frac{1}{\ln(10)} \cdot \sec^2(2x) \cdot 2
\]

(c) \( z = \frac{\tan^{-1}(3x)}{x} \)

\[
\frac{dz}{dx} = \frac{x \left( \frac{1}{1 + 9x^2} \cdot 3 \right) - \tan^{-1}(3x)}{x^2}
\]

(d) \( y = x^x \)

\[
\ln y = \ln (x^x) = x \ln x \\
\frac{1}{y} \frac{dy}{dx} = \ln x + x \cdot \left( \frac{1}{x} \right)
\]

\[
\frac{dy}{dx} = y \left( \ln x + 1 \right) = x^x \left( \ln x + 1 \right)
\]
3. Find the second derivative of \( y = \sin(x^2) \).

\[
y' = \frac{dy}{dx} = \cos(x^2) \cdot 2x = 2x \cos(x^2) \\
y'' = \frac{d^2y}{dx^2} = 2 \cos(x^2) - 2x \sin(x^2) \cdot (2x) \\
= 2 \cos(x^2) - 4x^2 \sin(x^2)
\]

4. Find an equation of the tangent line at the point \((-2, 1)\) to the curve \(xy^3 + 2y^2 + 3x + 6 = 0\).

\[
y^3 + 3xy^2 \frac{dy}{dx} + 4y \frac{dy}{dx} + 3 = 0
\]

\[
\frac{dy}{dx} (3xy^2 + 4y) = -y^3 - 3
\]

\[
\frac{dy}{dx} = \frac{-y^3 - 3}{3xy^2 + 4y}
\]

\[
m = \frac{-1 - 3}{-6 + 4} = \frac{-4}{-2} = 2
\]

\[
y - 1 = 2(x + 2)
\]

5. Let \( x = t - 2\sqrt{t} \) be the position of a moving particle. Find the time \( t \) in the interval \( 1 \leq t \leq 4 \) when the instantaneous velocity is equal to the average velocity on this interval.

\[
\frac{dx}{dt} = 1 - 2(\frac{1}{2})t^{-1/2} = 1 - \frac{1}{\sqrt{t}} = \frac{1}{3}
\]

\[
\frac{2}{3} = \frac{1}{\sqrt{t}}
\]

\[
\frac{3}{2} = \sqrt{t}
\]

\[
t = \frac{9}{4}
\]
(10 pts.) 6. Let \( f(x) = 3x^4 + 4x^3 \).

(a) Find all critical numbers.

\[
\frac{f'(x)}{12x^3 + 12x^2} = 12x^2(x+1) = 0 \quad \text{when} \quad \begin{cases} x = 0 \\ x = -1 \end{cases}
\]

(b) Find the absolute maximum and the absolute minimum of this function on the interval \(-2 \leq x \leq 1\).

\[
\begin{array}{c|c}
 x & f(x) \\
 \hline
 -2 & 48 - 32 = 16 = \text{MAX} \\
 -1 & 3 - 4 = -1 = \text{MIN} \\
 0 & 0 \\
 1 & 3 + 4 = 7 \\
\end{array}
\]

The absolute maximum of 16 occurs at the endpoint \( x = -2 \).

The absolute minimum of -1 occurs at the critical number \( x = -1 \).

(12 pts.) 7. A 13 foot ladder is leaning against a wall. The bottom of the ladder slides across the ground away from the wall at a rate of 4 feet per second. Find the rate at which the top of the ladder slides down the wall at the instant when the bottom of the ladder is 5 feet from the wall.

At the instant when \( x = 5 \),

\[
25 + y^2 = 169
\]

\[
y^2 = 144
\]

\[
y = 12 \text{ feet}
\]

Note \( \frac{dx}{dt} = 4 \frac{\text{feet}}{\text{second}} \)

\[
\frac{dy}{dt} = -\frac{40}{24} = -\frac{20}{12} = -\frac{5}{3} \text{ feet per second}
\]
(8 pts.) 8. A spherical balloon with a volume \( V = \frac{4}{3} \pi r^3 \) is being filled with air. Find the rate of change of the volume with respect to the radius at the instant when the radius equals 20 centimeters.

\[
\frac{dV}{dr} = \frac{4}{3} \pi (3r^2) = 4\pi r^2
\]

When \( r = 20 \text{ cm} \), \( \frac{dV}{dr} = 4\pi (20)^2 \)

\[
= 4\pi (400) = 1600\pi \frac{\text{cm}^3}{\text{cm}} = 1600\pi \text{ cm}^2
\]

\[
= 5026.548 \text{ cm}^2
\]

(12 pts.) 9. A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight toward the wall at a speed of \( \frac{1}{2} \) meter per second, at what rate is the length of his shadow on the wall changing at the instant when he is 8 meters from the wall?

[Similar triangles diagram]

\[
y = \frac{24}{x} = 24 \cdot x^{-1}
\]

\[
\frac{dy}{dt} = -24 \cdot \frac{dx}{dt}
\]

\[
\frac{dx}{dt} = \frac{1}{2} \text{ meter} / \text{second}
\]

At the instant when \( 12-x=8 \), \( x=4 \),

\[
\frac{dy}{dt} = -24 \left( \frac{1}{2} \right) = -12 \text{ cm} / \text{second}
\]

\[
= \frac{3}{4} \text{ meter} / \text{second}
\]