CALCULUS 1
Exam II
March 17, 2005

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Below you will find 10 problems, each worth 10 points. Solve the problems in the space provided. When writing a solution to a problem, show all work. No books or notes are allowed.

Problem 1. Find the derivative of \( f(x) = x\sqrt{x^2+1} \). (DO NOT SIMPLIFY.)

\[
f'(x) = \left[ \frac{d}{dx}(x) \right] \cdot \left[ (x^2+1)^{\frac{1}{2}} \right] + \left[ x \right] \cdot \left[ \frac{d}{dx}((x^2+1)^{\frac{1}{2}}) \right]
\]

\[
= 1 \cdot \left( x^2+1 \right)^{\frac{1}{2}} + x \cdot \left( \frac{1}{2} \cdot (x^2+1)^{-\frac{1}{2}} \cdot (2x) \right)
\]

\[
= \left( x^2+1 \right)^{\frac{1}{2}} + x^2 \cdot (x^2+1)^{-\frac{1}{2}}
\]

Problem 2. A rectangular page is to contain 54 square inches of print with 2 inches margin on top, and 1 inch margins on left, right, and bottom. Find the dimensions of the page that uses the least amount of paper.

Let \( x = \text{width of printed region} \)

\[ y = \text{height of printed region} \]

Given: \( xy = 54 \), \( x>0, y>0 \)

Want: \( \text{minimize } A = (x+2)(y+3) \)

\[ y = \frac{54}{x}, \quad A = f(x) \text{ where } \begin{cases} x>0 \end{cases} \]

\[ f(x) = (x+2)\left( \frac{54}{x} + 3 \right) = 54 + \frac{108}{x} + 3x + 6 \]

\[ f'(x) = 3 - \frac{108}{x^2} \]

CP: \(3 - \frac{108}{x^2} = 0\), \( x^2 = 36 \)

\[ x = 6 \text{ (Min))} \]

\[ x = -6 \text{ (Not in domain)} \]

\[ f(x) \text{ has abs. min } @ x = 6. \quad (y = \frac{54}{6} = 9) \]

Answer: \( \text{width } = x+2 = \frac{8}{3} \) \( \text{height } = y+3 = 12 \) \( \text{dimensions of paper}! \)
Problem 3. The function \( f(x) = x + 1 + \frac{1}{x - 1} \) has its first and second derivatives already computed: \( f'(x) = \frac{x^2 - 2x}{(x - 1)^2} \) and \( f''(x) = \frac{2}{(x - 1)^3} \).

(a) Indicate the intervals where \( f(x) \) is increasing/decreasing, and concave upward/downward.

\[
\begin{array}{c|cccccccc}
 x & -\infty & 0 & 1 & 2 & \cdots & 10 \\
 \hline
 f'(x) & ++ + + + & 0 & - - & 0 & + + + + + \\
f''(x) & - - - - - & + + + + + + + + + + \\
 f(x) & & & & & & \\
\end{array}
\]

\( x = 1 \) is not in domain. No inflection pts.

C.P.: \( x^2 - 2x = 0 \), \( x = 0, 2 \)

No inflection pts.

\( -\infty, 0 \), \( 0, 2 \) incr.

\( 0, 1 \), \( 1, 2 \) decr.

\( -\infty, 1 \)

\( 1, \infty \)

conc. up.

Inc.

Dec.

(b) Find all the asymptotes (horizontal/slant/vertical, if any).

W.A.: \( x = 1 \)

H.A.: NONE

SLANT: \( y = x + 1 \) (\( \Theta \) both ends)

(c) Sketch (use the white space on the right) the graph of \( f(x) \). Below list intercepts, critical points, and inflection points (if any).

\[
\begin{array}{c|c|c}
 x & f(x) & C.P. \\
 0 & 0 & (0, 0) \\
 2 & 4 & (2, 4) \\
\end{array}
\]

\( y \)-int: 0

\( x \)-int: 0

Inflection pts: NONE

Problem 4. Find the critical numbers of \( f(x) = \sqrt{x^2 - 2x} \).

\[
f'(x) = \left(x^2 - 2x\right)^{\frac{1}{3}}, \quad f''(x) = \frac{1}{3} \left(x^2 - 2x\right)^{-\frac{2}{3}} \cdot (2x - 2)
\]

\[
f'(x) = \frac{2}{3} \cdot \frac{x - 1}{(\sqrt{x^2 - 2x})^2}
\]

C.P.: \( f'(x) \) undefined: \( x^2 - 2x = 0 \) \( x = 0, 2 \)

\( f''(x) = 0 \) \( x = 1 \)
Problem 5. Given the equation $x^3 + y^2 = 5$, use implicit differentiation to find the derivative $dy/dx$ at the point $(1, -2)$.

$$
\frac{d}{dx} (x^3 + y^2) = \frac{d}{dx} (5)
$$

$$
3x^2 + 2y \frac{dy}{dx} = 0
$$

$$
3 - 4 \frac{dy}{dx} = 0
$$

$$
\frac{dy}{dx} = \frac{3}{4}
$$

Problem 6. The side of a square region decreases at a rate of 1 inch per second. What is the rate of change of the area, when the side is 20 inches?

$A = \text{area}, \ x = \text{side}$

Given:

$$
\begin{align*}
\frac{dx}{dt} &= -1 \\
x &= 20 \\
A &= x^2
\end{align*}
$$

Want: $\frac{dA}{dt}$

$$
\frac{dA}{dt} = 2x \frac{dx}{dt} = 2 \cdot 20 \cdot (-1) = -40 \text{ (in/\text{sec})}
$$

Problem 7. Find the absolute extrema of the function $f(x) = x^3 - 12x$, in the interval $[-3, 1]$.

$$
\begin{align*}
f'(x) &= 3x^2 - 12 \\
\text{C. P.: } 3x^2 - 12 &= 0, \ x^2 = 4, \ x = \pm 2
\end{align*}
$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>-11</td>
</tr>
<tr>
<td>-2</td>
<td>16</td>
</tr>
</tbody>
</table>

Abs max @ $-2$ is 16

Abs min @ 1 is -11
Problem 8. How many zeros does the function \( f(x) = x^3 - 1200x + 1 \) have?
\[
f'(x) = 3x^2 - 1200; \quad \text{C.P.:} \quad 3x^2 - 1200 = 0 \quad \Rightarrow \quad x = -20, x = 20
\]
Look for changes of signs in values of \( f(x) \) at C.P. & endpoints:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-∞</th>
<th>zero</th>
<th>-20</th>
<th>zero</th>
<th>20</th>
<th>zero</th>
<th>∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-∞</td>
<td>change</td>
<td>16001</td>
<td>change</td>
<td>-15999</td>
<td>change</td>
<td>∞</td>
</tr>
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</table>

**Answer:** three zeros

Problem 9. Find the intervals where the function \( f(x) = \frac{x^2 + x + 4}{x + 1} \) is increasing/decreasing.
Find relative maximum/minimum points.
\[
f'(x) = \frac{(2x + 1)(x+1) - (x^2 + x + 4)(1)}{(x+1)^2} = \frac{2x + 2x - 3}{(x+1)^2}
\]
C.P.: \( x^2 + 2x - 3 = 0 \) \( \Rightarrow \quad \frac{1}{8} < x = -3 \)
Neither \( f(x) \) nor \( f'(x) \) defined \( @ \quad x = -1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-∞</th>
<th>-3</th>
<th>1</th>
<th>∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>++++</td>
<td>0</td>
<td>---</td>
<td>++++</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>rel max</td>
<td>rel min</td>
<td></td>
<td></td>
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\( \text{incr:} \quad (-∞, -3), (1, ∞) \)
\( \text{decr:} \quad (-3, -1), (-1, 1) \)

Problem 10. Consider the function \( f(x) = x^2 + 4 \sin x \), defined on \([0, 2\pi]\). Find the inflection points, and the intervals where \( f(x) \) is concave upward/downward.
\[
f'(x) = 2x + 4 \cos x; \quad f''(x) = 2 - 4 \sin x
\]
Candidates for I.P.: \( f''(x) = 0 \), \( 2 - 4 \sin x = 0 \)
\( \sin x = \frac{1}{2} \) \( \Rightarrow \quad \frac{\pi}{6}, \frac{5\pi}{6} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>\frac{\pi}{6}</th>
<th>\frac{5\pi}{6}</th>
<th>\frac{2\pi}{6}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(x) )</td>
<td>++++</td>
<td>0</td>
<td>---</td>
<td>++++</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>concave I.P. conc. I.P conc. downward upward</td>
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So: \( f(x) \) conc. up on \( \left( 0, \frac{\pi}{6} \right), \left( \frac{5\pi}{6}, 2\pi \right) \)
conc. down on \( \left( \frac{\pi}{6}, \frac{5\pi}{6} \right) \); inflection pts. \( \left( \frac{\pi}{6}, \frac{5\pi}{6} \right) \)