Structure of Cohomology of Line Bundles on G/B for Semisimple Groups

Short running title: Cohomology of Line Bundles

Zongzhu Lin
University of Massachusetts
Amherst, MA 01003

Let G be a connected and simply connected semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$, T a maximal torus of G and B a Borel subgroup containing T. Each weight in $X(T)$ determines a line bundle on the flag variety G/B. It turns out that cohomology of the line bundle is isomorphic to the derived functor of the induction functors from the category of B-modules to the category of G-modules for each 1-dimensional B-module defined by a weight in $X(T)$. Further, $\mathrm{Ind}_B^G \lambda = H^0(G/B, \lambda)$ turns out to be the dual of a Weyl module. One of the main problems is to calculate the characters of the irreducible G-modules. For $p = 0$, the character of the irreducible G-module of highest weight λ is given by Weyl's character formula and the G-module structures of the cohomology of line bundles are well understood. However, for $p > 0$, the story is quite different and many of the results remain conjectural.

Since characters of Weyl modules are given by Weyl's character formula, understanding the *structure* of $H^0(G/B, \lambda)$ turns out to be the main problem. It is also interesting to understand the structure of the higher cohomology, which might help us to understand H^0. H. Andersen has a series of papers toward the understanding of the higher cohomology, such as the simple socle of H^1 and filtrations of H^i [4]. In [5] , using the representations of infinitesimal subgroup schemes of G, he proved generically that $H^{\ell(w)}(G/B, w \cdot \lambda)$ has simple socle and simple head and their highest weights can be calculated. In [6], he proved that the socle series of $H^0(\lambda)$ comes generically from the induced module for an infinitesimal subgroup of G. Using alcove identification, Doty and Sullivan [11] studied the structure of the higher cohomology modules inside the bottom p^2-alcove generically and got a similar result on socles and heads. They proved that the Jantzen filtration for $H^0(\lambda)$ coincides with the radical filtration for the group of type B_2. In [12], they used the same method to prove that the second socle of $H^{\ell(w)}(w \cdot \lambda)$ comes from an induced module for an infinitesimal subgroup.

From Andersen's work [5] for socles and heads of $H^{\ell(w)}(w \cdot \lambda)$ and some other evidence of investigation into the higher cohomology, Humphreys [14] suggests that there should be a structural correspondence between $H^{\ell(w)}(w\cdot\lambda)$ and the induced module of a certain *extremal* weight from a W-twisted Borel subgroup. This is partly verified in this paper.

We start with the G_rT-module $Z_r(\lambda)$ which can be considered to be a submodule of the injective module $Q_r(\lambda)$. Note that $Q_r(\lambda) = Q_r(\lambda_0) \otimes p^r\lambda_1$ with $Q_r(\lambda_0)$ being a G-module (for p large). This helps us to pass from G_rT to G. The main result in section 2 is Theorem 2.8 which proves that the socle series of $H^{\ell(w)}(w \cdot \lambda)$ comes from that of $Z_r(w \cdot \lambda)$ for λ generic.

In section 3, we use the W-actions on $X(T)$ defined in 1.2 and derive relations between higher cohomology modules and induced modules. The same results for radical series are also derived (Theorem 3.6). Section 4 applies the results proved in section 3. Humphreys' conjecture is proved in Theorem 4.1 by using the W-twisted representations developed in 1.6. It turns out that the rigidities of $H^{\ell(w)}(w \cdot \lambda)$, $H^0(\lambda)$ and $Z_r(\lambda)$ are generically equivalent (Theorem 4.7). If one assumes the Lusztig conjecture, then all these modules are rigid for $r = 1$. As a feedback, the study of the higher cohomology modules gives more information on Weyl modules.

In section 5 the nongeneric structure is studied. Theorem 5.3 shows that when a weight is in a box with upper vertex in a higher chamber, it will have a lower nonvanishing extra cohomology. Theorem 5.4 gives the simple socle of $H^{\ell(w)}(w \cdot \lambda)$ if $w \cdot \lambda$ is far away from higher chambers.

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1 Preliminaries

1.1 Basic Notation Let G be a connected and simply connected semisimple algebraic group over an algebraically closed field k of characteristic p. Throughout this paper, we fix a maximal torus $T \subset G$ and Borel subgroup $B \supset T$. Let R be the root system of G associated to the maximal torus T and R^+ be the set of positive roots such that B corresponds to the negative roots $-R^+ = R^-$. Let $S \subset R^+$ be the set of simple roots. $X(T)$ denotes the weight lattice of T and $W = N(T)/T$ the Weyl group which acts on $X(T)$. Let $\langle\ ,\ \rangle$ be a W-invariant inner product on $X(T) \bigotimes \mathbf{R}$ such that $s_\alpha\lambda = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$ for all $\lambda \in X(T) \bigotimes \mathbf{R}, \alpha \in R$ where s_α is the reflection defined by α and $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$ is the coroot of α. Denote by

$$\begin{aligned} X_+ &= \{\lambda \in X(T) : \langle\lambda, \alpha^\vee\rangle \geq 0, \forall\alpha \in S\} \\ &= \left\{\lambda \in X(T) : \langle\lambda, \alpha^\vee\rangle \geq 0, \forall\alpha \in R^+\right\} \end{aligned}$$

the set of dominant weights. There is a partial order "$\leq$" on $X(T)$ defined as follows: $\mu \leq \lambda$ iff $\lambda - \mu$ is a sum of positive roots. For $\lambda \in X_+, L(\lambda)$ denotes the simple G-module of highest weight λ.

For $\lambda \in X(T)$ we denote by λ (instead of k_λ) the 1-dimensional T-module which can be extended to a B-module through the decomposition $B = TU$ with U acting trivially on λ; here U is the unipotent radical of B. Denote by

$$H^i(E) = R^i\mathrm{Ind}_B^G E = H^i(G/B, E)$$

the i-th derived functor of the induction functor Ind_B^G for all B-modules E. For $E = \lambda$ we denote $H^i(\lambda) = R^i\mathrm{Ind}_B^G(\lambda) = H^i(G/B, \lambda)$. The following are standard results which can be found in [16](II, ch5).

- $H^0(\lambda) \neq 0$ iff $\lambda \in X_+$;
- $L(\lambda)$ is the (simple) socle of $H^0(\lambda)$ for all $\lambda \in X_+$, and these yield all the simple G-modules.

1.2 W-Actions The Weyl group W (as a reflection group on the real space $X(T) \otimes \mathbf{R}$) is generated by the simple reflections $\{s_\alpha \mid \alpha \in S\}$. Let

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha = \sum_{\alpha\in S}\omega_\alpha$$

where $\{\omega_\alpha \mid \alpha \in S\}$ is the set of fundamental weights. If $w(\lambda)$ denotes the usual action of W on $X(T) \otimes \mathbf{R}$, the "dot" action of W is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for all $\lambda \in X(T) \otimes \mathbf{R}$ and $w \in W$. The affine Weyl group W_p is the group generated by reflections $\{s_{\alpha,np} \mid \alpha \in R,\ n \in \mathbf{Z}\}$. Here the reflection $s_{\alpha,np}$ is defined by $s_{\alpha,np} \cdot \lambda = s_\alpha \cdot \lambda + np\alpha$ for all $\lambda \in X(T) \otimes \mathbf{R}$.

Let $\ell(w)$ be the minimal length of w expressed as a product of simple reflections in W and C be the fundamental alcove

$$C = \{\lambda \in X(T) \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p,\ \forall \alpha \in R_+\};$$

$$\bar{C} = \{\lambda \in X(T) \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p,\ \forall \alpha \in R_+\},$$

where $\bar{C}$ is a fundamental domain of the reflection group W_p. The following theorem can be found in Jantzen [16](II 5.5, 5.6).

Theorem *If $\lambda \in \bar{C}$ and $w \in W$, then*

$$H^i(w \cdot \lambda) = \begin{cases} L(\lambda), & \text{if } i = \ell(w) \text{ and } \lambda \in X_+, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this paper unless indicated else, we fix an integer $r \geq 1$, and define

$$X_r = \{\lambda \in X(T) \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle \leq p^r,\ \ \forall \alpha \in S\}$$

to be the set of r-restricted weights. In this way, any element $\mu \in X(T)$ has a unique decomposition $\mu = \mu_0 + p^r\mu_1$ with $\mu_0 \in X_r$ and $\mu_1 \in X(T)$. An element of $p^rX(T) - \rho$ is called a special point. If $\nu = p^r\nu_1 - \rho$ is a special point, then $W_\nu = \{y \in W_p \mid y \cdot \nu = \nu\}$ is a subgroup of W_p generated by $(s_\alpha)_\nu$ ($\alpha \in S$) such that $(s_\alpha)_\nu \cdot \mu = s_\alpha \cdot (\mu - p^r\nu_1) + p^r\nu_1$ ($\mu \in X(T)$). The natural map $W \to W_\nu$ by $w \mapsto w_\nu$ with

$$w_\nu \cdot (\mu) = w \cdot (\mu - p^r\nu_1) + p^r\nu_1 \qquad \forall \mu \in X(T),$$

is an isomorphism of groups.

Now we define the following new actions of W on $X(T)$. Let $W_r = \{w_r \mid w \in W\}$ with two actions as follows

$$w_r(\mu) = w_r(\mu_0 + p^r\mu_1) = \mu_0 + p^rw(\mu_1); \tag{1}$$

$$w_r \cdot (\mu) = w_r \cdot (\mu_0 + p^r \mu_1) = \mu_0 + p^r w \cdot \mu_1, \tag{2}$$

for all $\mu \in X(T)$ and $w \in W$. From the definition one can easily see that $w_r \cdot (\mu) = w_r(\mu + p^r \rho) - p^r \rho$.

Furthermore for a special point $\nu = p^r \nu_1 - \rho$ we denote by

$$(w_r)_\nu \cdot \mu = w_r \cdot (\mu - p^r \nu_1) + p^r \nu_1$$

the action of the group $(W_r)_\nu$ on $X(T)$. Also we have an involution on $X(T)$ defined by $\lambda \longmapsto \lambda^* = -w_0(\lambda)$, where w_0 is the unique longest element in W (λ^* is not to be confused with the dual module of k_λ). With all the actions defined here one can easily check the following:

$$w_r\,(\mu + p^r \lambda) = w_r(\mu) + p^r w(\lambda); \tag{3}$$

$$w_r \cdot (\mu + p^r \lambda) = w_r \cdot \mu + p^r w(\lambda); \tag{4}$$

$$(w\lambda)^* = w_0 w w_0(\lambda^*); \tag{5}$$

$$(w \cdot \lambda)^* = w_0 w w_0 \cdot (\lambda^*); \tag{6}$$

$$(w_r \lambda)^* = (w_0 w w_0)_r\,(\lambda^*); \tag{7}$$

$$(w_r \cdot \lambda)^* = (w_0 w w_0)_r \cdot \lambda^*; \tag{8}$$

$$(w_\nu \cdot \lambda)^* = (w_0 w w_0)_{\nu^*} \cdot \lambda^*; \tag{9}$$

$$((w_r)_\nu \cdot \lambda)^* = ((w_0 w w_0)_r)_{\nu^*} \cdot \lambda^*. \tag{10}$$

With the notation introduced here, we recall:

$$H^i(\lambda)^* \cong H^{N-i}\,(w_0 \cdot \lambda^*) \quad \text{(Serre duality)}; \tag{11}$$

$$L(\lambda)^* \cong L(\lambda^*). \tag{12}$$

Here $N = \ell(w_0)$ and M^* is the dual module of M for a module M.

1.3 Frobenius Subgroups and Their Representations We denote $F : G \to G$ to be the Frobenius map. Let $G_r = \mathrm{Ker}(F^r)$ which is an infinitesimal subgroup scheme of G. Let G_rT and G_rB be the pull back of T and B respectively, and B_rT and B_r be the pullback T and 1 in B respectively. If E is a B_rT-module, then $\mathrm{Ind}_{B_r}^{G_r} E \cong \mathrm{Ind}_{B_rT}^{G_rT} E$ as G_r-module. Further if E is a B-module, $\mathrm{Ind}_{B_r}^{G_r} E \cong \mathrm{Ind}_{B_rT}^{G_rT} E$ as G_r-module. Thus $\mathrm{Ind}_{B_r}^{G_r} E$

can be extended to a G_rT-(or G_rB-)module if E is a B_rT-(or B-)module. One can see [16](II,ch 9) for details. For $\lambda \in X(T)$, as G_r-modules we have

$$Z_r(\lambda) = \mathrm{Ind}_{B_r}^{G_r}\lambda \cong \mathrm{Ind}_{B_rT}^{G_rT}\lambda \cong \mathrm{Ind}_{B}^{G_rB}\lambda.$$

Since $Z_r(\lambda)$ can be extended to a G_rT and G_rB-module in a natural way, we simply denote all three of them by $Z_r(\lambda)$ without using different notation; the meaning of $Z_r(\lambda)$ should be clear from the context. Let $L_r(\lambda)$ be the socle of $Z_r(\lambda)$, then $L_r(\lambda)$ is a simple module as G_r-, G_rT-, G_rB-module (we use $L_r(\lambda)$ for all these modules in the same way).

Let M be a G_rB module. Then $Soc_{G_r}M = Soc_{G_rT}M$ which is a G_rB-submodule of M as well. In fact, one knows that

$$Soc_{G_r}M = \bigoplus_{\lambda \in X_r} L_r(\lambda) \otimes \mathrm{Hom}_{G_r}\left(L_r(\lambda), M\right).$$

Here $\mathrm{Hom}_{G_r}(L_r(\lambda), M)$ has a B-module structure. Since G_r is normal in G_rB and $L_r(\lambda)$ has a G_rB structure, $Soc_{G_r}M$ extends to a G_rB-module. Note that the normality of G_r in G_rB implies that $Soc_{G_rT}M \subseteq Soc_{G_r}M$. However $\mathrm{Hom}_{G_r}(L_r(\lambda), M)$ is semisimple as G_rT-module. Therefore $Soc_{G_rT}M \supseteq Soc_{G_r}M$ and we have the equality.

From the above argument one can see that $Soc^i_{G_r}Z_r(\lambda)$ is always a G_rB-module for all $i \geq 1$; here $Soc^i_{G_r}Z_r(\lambda)$ is the i-th socle of $Z_r(\lambda)$. Let M be module and L a simple module (for whatever group). We denote $[M : L]$ to be the number of the simple quotients isomorphic to L if M has a composition series. The following basic facts can be found in [16](II, ch9).

$$L_r(\lambda) \text{ has a } G\text{-module structure if } \lambda \in X_r; \tag{1}$$

$$L_r(\lambda) = L_r(\lambda_0) \otimes p^r\lambda_1,\ Z_r(\lambda) = Z_r(\lambda_0) \otimes p^r\lambda_1; \tag{2}$$

$$L_r(\lambda) = Soc_{G_r}Z_r(\lambda),\ L_r\left(2\left(p^r-1\right)\rho - \lambda\right)^* \cong Z_r(\lambda)/Rad_{G_r}Z_r(\lambda); \tag{3}$$

$$L_r(\lambda)^* = L_r(-w_0\lambda) \otimes p^r\left(w_0(\lambda_1) - \lambda_1\right),\ Z_r(\lambda)^* = Z_r\left(2\left(p^r-1\right)\rho - \lambda\right); \tag{4}$$

$$\left[Z_r(\lambda) : L_r\left(\mu_0 + p^r\mu_1\right)\right] = \left[Z_r(w \cdot \lambda) : L_r\left(\mu_0 + p^r w \cdot \mu_1\right)\right]; \tag{5}$$

$$H^i(L_r(\mu_0 + p^r\mu_1)) = H^i\left(L_r(\mu_0) \otimes p^r\mu_1\right) = L(\mu_0) \otimes H^i(\mu_1)^{(r)}. \tag{6}$$

Here $w \in W$, $\lambda = \lambda_0 + p^r\lambda_1, \mu = \mu_0 + p^r\mu_1 \in X(T)$ with $\lambda_0, \mu_0 \in X_r$, and $M^{(r)}$ denotes the r-th Frobenius twist in (6) for a G-module M.

Proposition ([16],II 9.14) *If $Z_r(\lambda)$ has all its composition factors of the form $L_r(\mu)$ with $\mu_1 \in \bar{C}$, then for all $w \in W$ and $\mu \in X_+$:*

$$H^i(w \cdot \lambda) = 0, \quad \textit{if} \quad i \neq \ell(w); \tag{7}$$

$$[H^{\ell(w)}(w \cdot \lambda) : L(\mu)] = [Z_r(w \cdot \lambda) : L_r(w_r \cdot \mu)]. \tag{8}$$

This proposition follows easily from (5) and (6) and Theorem 1.2 by using the spectral sequence for $\mathrm{Ind}_B^G ? \cong \mathrm{Ind}_{G_rB}^G \circ \mathrm{Ind}_B^{G_rB} ?$ and the exactness of $\mathrm{Ind}_B^{G_rB}$.

1.4 Injective Modules Following [16](II, ch11), let $Q_r(\lambda)$ be the injective hull of $L_r(\lambda)$ in the category of G_rT-modules. $Q_r(\lambda)$ is also the injective hull of $L_r(\lambda)$ in the category of G_r-modules. So we will not distinguish them (avoiding the complicated notation in [16]).

Proposition ([16] II 11.4) *Let $\lambda \in X(T)$. Then the G_rT-module $Q_r(\lambda)$ admits a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = Q_r(\lambda)$$

such that $M_i/M_{i-1} \cong Z_r(\lambda_i)$ for some $\lambda_i \in X(T)$ and $i = 1, \ldots, s$, and

$$(Q_r(\lambda) : Z_r(\lambda_i)) = [Z_r(\lambda_i) : L_r(\lambda)] \quad \textit{for all } i.$$

The following facts also come from [16].

$$Soc_{G_r} Q_r(\lambda) \cong Soc_{G_rT} Q_r(\lambda) = L_r(\lambda); \tag{1}$$

$$Q_r(\lambda)/Rad_{G_r} Q_r(\lambda) \cong Q_r(\lambda)/Rad_{G_rT} Q_r(\lambda) = L_r(\lambda); \tag{2}$$

$$Q_r(\lambda + p^r\mu) \cong Q_r(\lambda) \otimes p^r\mu; \tag{3}$$

$$\text{If } p \geq 2(h-1), \text{ then } Q_r(\lambda) \text{ is a } G\text{-module for } \lambda \in X_r. \tag{4}$$

Here $h = \max\{\langle \rho, \alpha^\vee \rangle + 1 \mid \alpha \in R^+\}$ is the Coxeter number of the root system R. From (3) and (4) we have

$$Q_r(\lambda) = Q_r(\lambda_0) \otimes p^r\lambda_1 \text{ extends to a } G_rB\text{-module if } p \geq 2(h-1); \tag{5}$$

$Q_r(\lambda)$ is the injective hull and projective cover of $L_r(\lambda)$ as G_rT-module. (6)

Remark Using the proposition above and 1.3(5) one can prove the following

$$[Q_r(w_r \cdot \lambda) : L_r(w_r \cdot \mu)] = [Q_r(\lambda) : L_r(\mu)] \tag{7}$$

for all $\lambda, \mu \in X(T)$ and $w \in W$.

1.5 Socle and Radical Series Patterns Let M be a G_rT-module and $Soc^i_{G_rT}M$ and $Rad^i_{G_rT}M$ the i-th socle and radical of M respectively. They are automatically G_r-modules. In the following we define

$$S_r[M] \ = \Big\{(\mu, i, m) \ \mid [Soc^{i+1}_{G_rT}M/Soc^i_{G_rT}M : L_r(\mu)] = m\Big\}; \tag{1}$$

$$R_r[M] \ = \Big\{(\mu, i, m) \ \mid [Rad^i_{G_rT}M/Rad^{i+1}_{G_rT}M : L_r(\mu)] = m\Big\} \tag{2}$$

to be subsets of $X(T) \times \mathbf{N} \times \mathbf{N}$. We call $S_r[M]$ and $R_r[M]$ the socle and radical series patterns of M respectively. (Note the socle series pattern defined here is different from the usual one.) One can extend all the W-actions on $X(T)$ in section 1.2 to $X(T) \times \mathbf{N} \times \mathbf{N}$ by acting on the first component. Also the translation $T_\mu : X(T) \to X(T)$ by $T_\mu(\lambda) = \lambda + \mu$ can be extended to $X(T) \times \mathbf{N} \times \mathbf{N}$ in the same way. Thus following 1.3(2) and 1.4(3) one can verify

$$S_r[Z_r(\lambda + p^r\mu)] = S_r[Z_r(\lambda)] + p^r\mu; \tag{3}$$

$$R_r[Z_r(\lambda + p^r\mu)] = R_r[Z_r(\lambda)] + p^r\mu; \tag{4}$$

$$S_r[Q_r(\lambda + p^r\mu)] = S_r[Q_r(\lambda)] + p^r\mu; \tag{5}$$

$$R_r[Q_r(\lambda + p^r\mu)] = R_r[Q_r(\lambda)] + p^r\mu, \tag{6}$$

since $Soc^i_{G_rT}(Q_r(\lambda) \otimes p^r\mu) \cong (Soc^i_{G_rT}Q_r(\lambda)) \otimes p^r\mu$ (and similar results hold for the radicals and the modules $Z_r(\lambda) \otimes p^r\mu$).

From the definitions of socle and radical filtrations the following can be easily verified for a G_rT-module M:

$$Soc^i_{G_rT}(M^*) = Ker(M^* \longrightarrow (Rad^i_{G_rT}M)^*); \tag{7}$$

$$Rad^i_{G_rT}(M^*) = Ker(M^* \longrightarrow (Soc^i_{G_rT}M)^*). \tag{8}$$

This duality on the filtrations gives a duality on the filtration layers:

$$Soc^i_{G_rT}(M^*)/Soc^{i-1}_{G_rT}M^* \cong (Rad^{i-1}_{G_rT}(M)/Rad^i_{G_rT}(M))^*; \tag{9}$$
$$Rad^i_{G_rT}(M^*)/Rad^{i+1}_{G_rT}(M^*) \cong (Soc^{i+1}_{G_rT}(M)/Soc^i_{G_rT}(M))^*, \tag{10}$$

where M^* is the dual module of M.

Let T^μ_λ be the translation functor for $\lambda, \mu \in F$ (so called facet, see Jantzen [17] for definition). The exactness of T^μ_λ and the facts that $T^\mu_\lambda L_r(\lambda) = L_r(\mu)$ and $T^\mu_\lambda Z_r(\lambda) = Z_r(\mu)$ imply the invariance of the filtrations under T^μ_λ.

$$T^\mu_\lambda Rad^i_{G_rT}(Z_r(\lambda)) = Rad^i_{G_rT}T^\mu_\lambda(Z_r(\lambda));$$

$$T^\mu_\lambda Soc^i_{G_rT}(Z_r(\lambda)) = Soc^i_{G_rT}T^\mu_\lambda(Z_r(\lambda)).$$

This shows that the socle and radical series only depend on the facet type (the orbit of the facet under the translation action of $p^rX(T)$ on $X(T)$).

If M is a G-module, $S[M]$ and $R[M]$ can be defined in the same way as in (1) and (2) and the properties in (7)–(10) also hold for the socle and radical series patterns of M.

1.6 Twisted Representations Let G be a group scheme over k for a moment. If G' is another group scheme and $\varphi : G \to G'$ is an isomorphism of group schemes (I prefer to write $\varphi(G)$ for G' in this subsection.) For any $\varphi(G)$-module M, $\varphi^*(M)$ is a G-module with the same vector space and the induced action $g * m = \varphi(g)m$. Then $\varphi^* : \mathcal{M}_{\varphi(G)} \to \mathcal{M}_G$ is a category equivalence which is exact, additive and maps simple objects to simple objects. Therefore

$$\varphi^*(Soc^i_{\varphi(G)}M) = Soc^i_G\varphi^*(M); \tag{1}$$
$$\varphi^*(Rad^i_{\varphi(G)}M) = Rad^i_G\varphi^*(M). \tag{2}$$

Let $H \subseteq G$ be a closed subgroup scheme , then $\varphi(H) \subseteq \varphi(G)$ is a closed subgroup scheme. The induced isomorphism $\varphi : H \to \varphi(H)$ also induces a category equivalence $\varphi^* : \mathcal{M}_{\varphi(H)} \to \mathcal{M}_H$ which commutes with the restriction functors $\mathrm{Res}^{\varphi(G)}_{\varphi(H)}$ and Res^G_H, i.e.,

$$\mathrm{Res}^G_H \circ \varphi^* = \varphi^* \circ \mathrm{Res}^{\varphi(G)}_{\varphi(H)}. \tag{3}$$

Bearing in mind that the induction functor is the right adjoint of the restriction functor, we also have the commutativity of the functors

$$\mathrm{Ind}_H^G \circ \varphi^* = \varphi^* \circ \mathrm{Ind}_{\varphi(H)}^{\varphi(G)}. \tag{4}$$

Furthermore φ^* maps injectives to injectives. So one can check

$$R^i\mathrm{Ind}_H^G \circ \varphi^* = \varphi^* \circ R^i\mathrm{Ind}_{\varphi(H)}^{\varphi(G)}. \tag{5}$$

As $(\varphi^*)^{-1} = (\varphi^{-1})^*$, above commutativity can be reformulated as

$$R^i\mathrm{Ind}_{\varphi(H)}^{\varphi(G)} = (\varphi^{-1})^* \circ R^i\mathrm{Ind}_H^G \circ \varphi^*. \tag{6}$$

Now let us go back to the semisimple group G. Let $w \in W$ and $\dot{w} \in N(T)$ be a representative of w. Let $\varphi_w : G \to G$ be the inner automorphism defined by $\varphi_w(g) = \dot{w}^{-1}g\dot{w}$ $(\forall g \in G)$. (Though it depends on the representative $\dot{w}$ chosen, the automorphism is unique up to an inner automorphism of G by an element of T which does not affect the arguments following this, so we still denote it by φ_w without indicating the representative chosen here.) However the following facts do not depend on $\dot{w}$.

$$\varphi_w(G_rT) = G_rT; \varphi_w(B) = \dot{w}^{-1}B\dot{w} = B^w; \varphi_w(B_r) = \dot{w}^{-1}B_r\dot{w} = B_r^w.$$

Let M be a B^w-module, then $\varphi_w^*(M) = M^{w^{-1}}$(independent of the representative up to a conjugation by an element of T), with the action $b * m = \dot{w}^{-1}b\dot{w}m, \forall b \in B$, is a B-module. Similarly $M^{w^{-1}}$ is an H-module if M is an H^w-module for any subgroup scheme H of G. Let $\lambda \in X(T)$ and consider it as a T-module, then $\varphi_w^*(\lambda) = \lambda^{w^{-1}} = w(\lambda)$. In fact $t * \lambda^{w^{-1}} = \lambda(\dot{w}^{-1}t\dot{w}) = w(\lambda)(t)$.

Denote $Z_r(w, M) = \mathrm{Ind}_{B^w}^{G_rB^w} M = \mathrm{Ind}_{B_r^w}^{G_r} M = \mathrm{Ind}_{B_r^wT}^{G_rT} M$ for a B-module M. Specially we have $Z_r(w, \lambda) = \mathrm{Ind}_{B^w}^{G_rB^w} \lambda = \mathrm{Ind}_{B_r^w}^{G_r} \lambda = \mathrm{Ind}_{B_r^wT}^{G_rT} \lambda$. If $w = 1$ we will simply denote $Z_r(1, \lambda) = Z_r(\lambda)$. So the notations in section 1.3 are the special cases of the notations defined here (the notation here is a little different from the one in [9]). Now applying (5) one gets

$$Z_r(w, \lambda) = Z_r(w(\lambda))^w. \tag{7}$$

If one denotes $H^i(G/B^w, M) = H^i(w, M)$ for any B^w-module M, then

$$H^i(w, M) = H^i(M^{w^{-1}})^w. \tag{8}$$

If M is a G-module, then the map $M \to M^{w^{-1}}$ defined by $m \mapsto \dot{w}^{-1}m$ is an isomorphism of G-modules, so

$$M \cong M^{w^{-1}}, \quad \text{if } M \text{ is a } G\text{-module.} \tag{9}$$

Note that $L_r(\lambda_0)$ is a G-module and so is $Q_r(\lambda_0)$ if $p \geq 2(h-1)$, for all $\lambda_0 \in X_r$. By applying (9) one has the following

$$
\begin{aligned}
L_r(\lambda)^w &= (L_r(\lambda_0)\otimes p^r\lambda_1)^w = L_r(\lambda_0)^w \otimes (p^r\lambda_1)^w \\
&= L_r(\lambda_0 + p^r w^{-1}(\lambda_1)) = L_r(w_r^{-1}(\lambda)); \qquad (10)\\
Q_r(\lambda)^w &= Q_r(w_r^{-1}\lambda), \quad \text{if } p \geq 2(h-1). \qquad (11)
\end{aligned}
$$

Now let us consider the socle and radical series patterns of $Z_r(w,\lambda)$. As in 1.5, W acts on $X(T)\times \mathbf{N}\times\mathbf{N}$ in several ways. So applying (1), (2), (7) and (10), one gets the following

$$
\begin{aligned}
S_r[Z_r(w,\lambda)] = S_r[Z_r(w(\lambda))^w] = w_r^{-1}(S_r[Z_r(w(\lambda))]); \qquad (12)\\
R_r[Z_r(w,\lambda)] = R_r[Z_r(w(\lambda))^w] = w_r^{-1}(R_r[Z_r(w(\lambda))]). \qquad (13)
\end{aligned}
$$

2 Socle Series of G_rT-modules and G-modules

In this section, unless indicated, we denote by $\lambda = \lambda_0 + p^r\lambda_1$ a dominant weight such that all G_rT-composition factors of $Q_r(\lambda)$ have the form $L_r(\mu_0 + p^r\mu_1)$ with $\mu_1 \in \bar{C}\cap X_+$. Furthermore, we assume $p \geq 2(h-1)$ from now on in this paper, so that $Q_r(\lambda_0)$ is a G-module.

2.1 Lemma ([16](II, ch11)) *If $\lambda = \lambda_0 + p^r\lambda_1$ with $2(h-1) \leq \langle \lambda_1 + \rho, \alpha^\vee\rangle \leq p-2(h-1)$, for all $\alpha \in R^+$, then $Q_r(\lambda)$ has all its G_rT-composition factors of the form $L_r(\mu_0 + p^r\mu_1)$ with $\mu_1 \in \bar{C}\cap X_+$.*

2.2 Proposition *If $w \in W, \mu \in X_+$ and $i \in \mathbf{N}$, then*

$$
\begin{aligned}
[H^{\ell(w)}(Q_r(w_r\cdot\lambda)) : L(\mu)] &= [Q_r(w_r\cdot\lambda) : L_r(w_r\cdot\mu)]; \qquad (1)\\
H^{\ell(w)}(Soc^i_{G_rT}Q_r(w_r\cdot\lambda)) &\subseteq Soc^i_G H^{\ell(w)}(Q_r(w_r\cdot\lambda)); \qquad (2)\\
H^{\ell(w)}(Rad^i_{G_rT}Q_r(w_r\cdot\lambda)) &\supseteq Rad^i_G H^{\ell(w)}(Q_r(w_r\cdot\lambda)). \qquad (3)
\end{aligned}
$$

Proof: Since all the composition factors of $Q_r(w_r\cdot\lambda)$ have the form $L_r(w_r\cdot\mu)$ with $\mu_1 \in \bar{C}\cap X_+$, it follows from 1.3(6) and Steinberg's tensor product theorem ([16], II 3.17) that

$$
H^i(L_r(w_r\cdot\mu)) = L(\mu_0)\otimes H^i(w\cdot\mu_1)^{(r)} = \begin{cases} L(\mu), & \text{if } i = \ell(w)\ ; \\ 0 & \text{otherwise.}\end{cases}
$$

Therefore $H^{\ell(w)}(L_r(w_r\cdot\mu)) = L(\mu)$ is simple and $H^i(Q_r(w_r\cdot\lambda)) = 0$ if $i \neq \ell(w)$. This shows that the functor $H^{\ell(w)}$ is exact on the G_rB-submodules of $Q_r(w_r \cdot \lambda)$. So (1) follows by taking a G_rB-composition series of $Q_r(w_r \cdot \lambda)$. The subquotients of the G_rT-socle or radical series of $Q_r(w_r \cdot \lambda)$ are G_rB-modules and have the form $E = \bigoplus_{\mu_0 \in X_r} L_r(\mu_0) \otimes E(\mu_0)^{(r)}$ where $E(\mu_0)$ is a B-module and all its weights are in $w \cdot \bar{C}$. Therefore $H^{\ell(w)}(E) = \bigoplus_{\mu_0 \in X_r} L(\mu_0) \otimes H^{\ell(w)}(E(\mu_0))^{(r)}$ and all the composition factors of $H^{\ell(w)}(E(\mu_0))$ have highest weight in $\bar{C}$. The strong linkage principle shows that $H^{\ell(w)}(E(\mu_0))$ is a semisimple G-module. Now (2) and (3) follow from the Steinberg tensor product theorem. ∎

Remark It follows the above proof that for all $w \in W$ and $i \geq 1$,

$$[H^{\ell(w)}(Soc^i_{G_rT} Q_r(w_r \cdot \lambda)) : L(\mu)] = [Soc^i_{G_rT} Q_r(w_r \cdot \lambda) : L_r(w_r \cdot \mu)]. \tag{4}$$

2.3 Proposition *If $i \geq 1$, then*

$$H^0(Soc^i_{G_r} Q_r(\lambda)) = Soc^i_G H^0(Q_r(\lambda)).$$

Proof: From Proposition 2.2, we only need to show

$$H^0(Soc^i_{G_rT} Q_r(\lambda)) \supseteq Soc^i_G H^0(Q_r(\lambda)). \tag{1}$$

If $i = 1$, we have $Soc_{G_r} Q_r(\lambda) = L_r(\lambda)$. So it follows from (6) in section 1.3 that $H^0(L_r(\lambda)) = L(\lambda)$. However, we have

$$H^0(Q_r(\lambda)) = H^0(Q_r(\lambda_0) \otimes p^r\lambda_1) = Q_r(\lambda_0) \otimes H^0(\lambda_1)^{(r)};$$

$$Soc_G H^0(Q_r(\lambda)) = (Soc_G Q_r(\lambda_0)) \otimes L(\lambda_1)^{(r)} = L_r(\lambda_0) \otimes L(\lambda_1)^{(r)} = L(\lambda).$$

Therefore equality holds in this case.

In general, let us consider the evaluation map

$$ev: \; H^0(Q_r(\lambda)) \longrightarrow Q_r(\lambda),$$

which is a G_rT-module homomorphism. One knows that the restriction of a semisimple G-module to G_rT is still semisimple. Therefore

$$Soc^i_G H^0(Q_r(\lambda)) \subseteq Soc^i_{G_rT} H^0(Q_r(\lambda))$$

for all $i \geq 1$ and also one has

$$ev(Soc^i_{G_rT} H^0(Q_r(\lambda))) \subseteq Soc^i_{G_rT} Q_r(\lambda).$$

Therefore $ev(Soc^i_G H^0(Q_r(\lambda)) \subseteq Soc^i_{G_rT} Q_r(\lambda)$, so we have the following commutative diagram

$$\begin{array}{ccc} H^0(Q_r(\lambda)) & \xrightarrow{ev} & Q_r(\lambda) \\ \uparrow & & \uparrow \\ Soc^i_G H^0(Q_r(\lambda)) & \xrightarrow{ev} & Soc^i_{G_rT} Q_r(\lambda) \end{array}.$$

Here the vertical maps are the inclusion maps. Applying the functor H^0 to this diagram and the naturality of the evaluation map ev, we have the commutativity of the following

$$\begin{array}{ccc} H^0(Q_r(\lambda)) & \xrightarrow{H^0(ev)=id} & H^0(Q_r(\lambda)) \\ \uparrow & & \uparrow \\ Soc^i_G H^0(Q_r(\lambda)) & \xrightarrow{H^0(ev)} & H^0(Soc^i_{G_rT} Q_r(\lambda)) \end{array}.$$

This shows (1) and we have the equality in the proposition. ∎

2.4 Lemma *Let $w \in W$,$\mu_0, \lambda_0 \in X_r$, $\mu_1 \in X(T)$ and $i \geq 1$. Then*

$$[Soc^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0 + p^r\mu_1)] = [Soc^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0 + p^r w(\mu_1))]; \quad (1)$$

$$\left[Rad^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0 + p^r\mu_1)\right] = [Rad^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0 + p^r w(\mu_1))]. \quad (2)$$

Proof: From 1.6(10)(11) we have $Q_r(\lambda_0)^w = Q_r(\lambda_0)$ and $L_r(\mu_0 + p^r\mu_1)^w = L_r(\mu_0 + p^r w(\mu_1))$. Further, 1.6(1)(2) imply

$$(Soc^i_{G_rT} Q_r(\lambda_0))^w = Soc^i_{G_rT}(Q_r(\lambda_0)^w) = Soc^i_{G_rT} Q_r(\lambda_0); \quad (3)$$

$$(Rad^i_{G_rT} Q_r(\lambda_0))^w = Rad^i_{G_rT}(Q_r(\lambda_0)^w) = Rad^i_{G_rT} Q_r(\lambda_0). \quad (4)$$

Therefore the lemma follows from:

$$\begin{aligned} & [Soc^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0 + p^r w(\mu_1))] \\ & = [(Soc^i_{G_rT} Q_r(\lambda_0))^w : L_r(\mu_0 + p^r\mu_1)^w] \\ & = [Soc^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0 + p^r\mu_1)]. \end{aligned}$$ ∎

2.5 Theorem *Suppose $\lambda = \lambda_0 + p^r\lambda_1 \in X_+$ such that all G_rT-composition factors of $Q_r(\lambda)$ have the form $L_r(\mu_0 + p^r\mu_1)$ with $\mu_1 \in X_+ \cap \bar{C}$. Then*

$$Soc^i_G H^{\ell(w)}(Q_r(w_r \cdot \lambda)) = H^{\ell(w)}(Soc^i_{G_rT} Q_r(w_r \cdot \lambda))$$

for all $w \in W$ and $i \geq 1$.

Proof: In the proof of Proposition 2.2 one knows the functor $H^{\ell(w)}$ is exact on the G_rB-submodules of $Q_r(w_r \cdot \lambda)$. Recall that $Soc^i_{G_rT} Q_r(w_r \cdot \lambda)$ is a G_rB-module, and $H^{\ell(w)}(L_r(\mu_0 + p^r w \cdot \mu_1)) = L(\mu_0 + p^r \mu_1)$ for all composition factors $L_r(\mu_0 + p^r w \cdot \mu_1)$ of $Q_r(w_r \cdot \lambda))$ (cf. Remark 1.4).

By Proposition 2.2 one already has

$$H^{\ell(w)}(Soc^i_{G_rT} Q_r(w_r \cdot \lambda)) \subseteq Soc^i_G H^{\ell(w)}(Q_r(w_r \cdot \lambda)).$$

So it suffices to show that every simple module has the same multiplicity on both sides. One can easily see

$$\begin{aligned} &H^{\ell(w)}(Q_r(w_r \cdot \lambda)) = H^{\ell(w)}(Q_r(\lambda_0) \otimes p^r w \cdot \lambda_1) \cong Q_r(\lambda_0) \otimes H^{\ell(w)}(w \cdot \lambda_1)^{(r)} \\ &\cong Q_r(\lambda_0) \otimes H^0(\lambda_1)^{(r)} \cong H^0(Q_r(\lambda_0 + p^r \lambda_1)) \end{aligned}$$

by the generalized tensor identity (cf. [16] I 4.8)and Theorem 1.2 since $\lambda_1 \in \bar{C}$. Now Proposition 2.3 tells us

$$Soc^i_G H^{\ell(w)}(Q_r(w_r \cdot \lambda)) \cong Soc^i_G H^0(Q_r(\lambda)) \cong H^0(Soc^i_{G_rT} Q_r(\lambda)).$$

So it is enough to show that

$$[H^0(Soc^i_{G_rT} Q_r(\lambda)) : L(\mu)] = [H^{\ell(w)}(Soc^i_{G_rT} Q_r(w_r \cdot \lambda)) : L(\mu)]$$

for all $\mu \in X_+$. It follows from 2.2(4) that this is equivalent to

$$[Soc^i_{G_rT} Q_r(\lambda) : L_r(\mu_0 + p^r \mu_1)] = [Soc^i_{G_rT} Q_r(w_r \cdot \lambda) : L_r(\mu_0 + p^r w \cdot \mu_1)].$$

Now using the following

$$Soc^i_{G_rT} Q_r(\lambda_0 + p^r w \cdot \lambda_1) = (Soc^i_{G_rT} Q_r(\lambda_0)) \otimes p^r w \cdot \lambda_1$$

for all $w \in W$, one has

$$\begin{aligned} &[Soc^i_{G_rT} Q_r(\lambda) : L_r(\mu_0 + p^r \mu_1)] = [Soc^i_{G_rT} Q_r(\lambda_0) \otimes p^r \lambda_1 : L_r(\mu_0) \otimes p^r \mu_1] \\ &= [Soc^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0) \otimes p^r(\mu_1 - \lambda_1)]. \end{aligned}$$

Similarly one gets

$$[Soc^i_{G_rT} Q_r(w_r \cdot \lambda) : L_r(\mu_0 + p^r w \cdot \mu_1)] = [Soc^i_{G_rT} Q_r(\lambda_0) : L_r(\mu_0) \otimes p^r w(\mu_1 - \lambda_1)].$$

Now the proposition follows from Lemma 2.4. ∎

2.6 For the assumption on λ as at the beginning of this section, it follows from $Z_r(\lambda) \subseteq Q_r(\lambda)$ that all composition factors of $Z_r(\lambda)$ have the form $L_r(\mu_0+p^r\mu_1)$ with $\mu_1 \in X_+\cap\bar{C}$. Therefore all composition factors of $Z_r(w\cdot\lambda)$ have the form $L_r(\mu_0+p^r w\cdot\mu_1)$ with $\mu_1 \in X_+\cap\bar{C}$ from 1.3(5). Now applying the same argument as in section 2.2 to $Z_r(w\cdot\lambda)$, together with 1.3(7) and 1.3(8), one has the following similar results.

Proposition *Let $\mu \in X_+,\ w \in W$ and $i \geq 1$. Then*

$$\begin{aligned}
[H^{\ell(w)}(Soc^i_{G_rT}Z_r(w\cdot\lambda)) : L(\mu)] &= [Soc^i_{G_rT}Z_r(w\cdot\lambda) : L_r(w_r\cdot\mu)]; & (1)\\
\left[H^{\ell(w)}(Rad^i_{G_rT}Z_r(w\cdot\lambda)) : L(\mu)\right] &= [Rad^i_{G_rT}Z_r(w\cdot\lambda) : L_r(w_r\cdot\mu)]; & (2)\\
H^{\ell(w)}(Soc^i_{G_rT}Z_r(w\cdot\lambda)) &\subseteq Soc^i_{G_rT}H^{\ell(w)}(w\cdot\lambda); & (3)\\
H^{\ell(w)}(Rad^i_{G_rT}Z_r(w\cdot\lambda)) &\supseteq Rad^i_{G_rT}H^{\ell(w)}(w\cdot\lambda). & (4)
\end{aligned}$$

Here one needs to recall $H^{\ell(w)}(Z_r(w\cdot\lambda)) = H^{\ell(w)}(w\cdot\lambda)$ for all $w \in W$.

2.7 Let $\lambda_0 \in X_r$ and $w \in W$. There is a unique weight $(\lambda_0)_w$ such that $w\cdot\lambda_0 + p^r(\lambda_0)_w \in X_r$(see [5]). If $\lambda = \lambda_0 + p^r\lambda_1$, then

$$w\cdot\lambda = w\cdot\lambda_0 + p^r w(\lambda_1) = w\cdot\lambda_0 + p^r(\lambda_0)_w + p^r(w(\lambda_1) - (\lambda_0)_w);$$

$$(w\cdot\lambda)_0 = w\cdot\lambda_0 + p^r(\lambda_0)_w \text{ and } (w\cdot\lambda)_1 = w(\lambda_1) - (\lambda_0)_w.$$

One can find in Andersen [5] that $|\langle(\lambda_0)_w, \alpha^\vee\rangle| \leq 2(h-1)$ for all $\alpha \in R^+$ and $w \in W$.

Now let us consider the composition factors of $Q_r(w\cdot\lambda) = Q_r((w\cdot\lambda)_0) \otimes p^r(w\cdot\lambda)_1$. It follows from Jantzen [16](II, ch11) that all composition factors of $Q_r((w\cdot\lambda)_0)$ have the form $L_r(\mu_0 + p^r w'(\mu_1))$ with $\mu_0 + p^r\mu_1 \leq 2(p^r-1)\rho$, therefore

$$|\langle w'(\mu_1), \alpha^\vee\rangle| \leq 2(h-1) \quad \text{for all } \alpha \in R \text{ and } w' \in W.$$

The composition factors of $Q_r(w\cdot\lambda)$ have the form

$$L_r(\mu_0) \otimes p^r w'(\mu_1) \otimes p^r(w\cdot\lambda)_1 = L_r(\mu_0 + p^r(w'(\mu_1) + (w\cdot\lambda)_1)).$$

Now we assume $w'(\mu_1)+(w\cdot\lambda)_1 \in w\cdot(\bar{C}\cap X_+)$, i.e., $w^{-1}\cdot(w'(\mu_1)+(w\cdot\lambda)_1) \in X_+\cap\bar{C}$.

Lemma *If* $5(h-1) < \langle \lambda_1 + \rho, \alpha^\vee \rangle \leq p - 5(h-1)$ *for all* $\alpha \in R^+$, *then for all* $w \in W$, *the composition factors of* $Q_r(w \cdot \lambda)$ *have the form* $L_r(\nu_0 + p^r w \cdot \nu_1)$ *with* $\nu_1 \in X_+ \cap \bar{C}$.

Proof: From above argument we only need to show that this condition does imply $\nu_1 = w^{-1} \cdot (w'(\mu_1) + (w \cdot \lambda)_1) \in X_+ \cap \bar{C}$. Note

$$\begin{aligned} w^{-1} \cdot (w'(\mu_1) + (w \cdot \lambda)_1) + \rho &= w^{-1}(w'(\mu_1) + w(\lambda_1) - (\lambda_0)_w + \rho) \\ &= \lambda_1 + w^{-1}w'(\mu_1) + w^{-1}(\rho) - w^{-1}(\lambda_0)_w. \end{aligned}$$

Let $\alpha \in R^+$. Then the lemma follows from the following calculation.

$$\begin{aligned} \langle \nu_1 + \rho, \alpha^\vee \rangle &= \langle \lambda_1, \alpha^\vee \rangle + \langle w^{-1}w'(\mu_1), \alpha^\vee \rangle + \langle w^{-1}(\rho), \alpha^\vee \rangle - \langle w^{-1}(\lambda_0)_w, \alpha^\vee \rangle \\ &\leq p - 5(h-1) + 2(h-1) + (h-1) + 2(h-1) = p; \\ \langle \nu_1 + \rho, \alpha^\vee \rangle &> 5(h-1) - 2(h-1) - (h-1) - 2(h-1) = 0. \quad \blacksquare \end{aligned}$$

Definition $\lambda \in X_+$ is called *generic* if it satisfies the condition of the lemma.

Remark Using the remark in 1.4 one can show, for generic λ, that the composition factors of $Q_r(w_r^{-1} \cdot (w \cdot \lambda)) = Q_r((w \cdot \lambda)_0 + p^r w^{-1} \cdot (w \cdot \lambda)_1)$ are of the form $L_r(\mu_0 + p^r \mu_1)$ with $\mu_1 \in \bar{C} \cap X_+$. So Theorem 2.5 can be applied to get the equality

$$Soc_G^i H^{\ell(w)}(Q_r(w \cdot \lambda)) = H^{\ell(w)}(Soc_{G_rT}^i Q_r(w \cdot \lambda)).$$

2.8 Let $\lambda \in X_+$ be generic. Then $H^i(Q_r(w \cdot \lambda)) = 0$ unless $\ell(w) = i$. Further, the functor $H^{\ell(w)}$ is exact on the subcategory of G_rB-modules whose composition factors are also composition factors of $Q_r(w \cdot \lambda)$. In this paragraph we denote, for simplicity, the following:

$$Z = Z_r(w \cdot \lambda),\ S^i H^{\ell(w)}(Z) = Soc_G^i H^{\ell(w)}(Z_r(w \cdot \lambda)),\ S^i Z = Soc_{G_rT}^i Z_r(w \cdot \lambda);$$

$$Q = Q_r(w \cdot \lambda),\ S^i H^{\ell(w)}(Q) = Soc_G^i H^{\ell(w)}(Q_r(w \cdot \lambda)),\ S^i Q = Soc_{G_rT}^i Q_r(w \cdot \lambda).$$

Since $Z \subseteq Q$ then $H^{\ell(w)}(Z) \subseteq H^{\ell(w)}(Q)$. Recall that $S^i Z = (S^i Q) \cap Z$ and $S^i H(Z) = (S^i H(Q)) \cap H^{\ell(w)}(Z)$.

Theorem *If* $\lambda = \lambda_0 + p^r\lambda_1 \in X_+$ *such that* $5(h-1) < \langle\lambda_1 = \rho, \alpha^\vee\rangle \leq p - 5(h-1)$ *for all* $\alpha \in R^+$*, then for all* $w \in W,\ \mu \in X_+$ *and* $i \geq 1$*,*

$$\begin{aligned} Soc^i_G H^{\ell(w)}(w\cdot\lambda) &= H^{\ell(w)}(Soc^i_{G_r} Z_r(w\cdot\lambda)); && (1)\\ \left[Soc^i_G H^{\ell(w)}(w\cdot\lambda) : L(\mu)\right] &= [Soc^i_{G_r} Z_r(w\cdot\lambda) : L_r(w_r\cdot\mu)], && (2)\end{aligned}$$

Proof: (2) follows from (1) and 2.6(1). To prove (1) it is enough to prove $H^{\ell(w)}(S^iZ) = H^{\ell(w)}(Z) \cap H^{\ell(w)}(S^iQ)$ (cf. Theorem 2.5). Applying the exact functor $H^{\ell(w)}$ to the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} 0 & \to & Z & \to & Q & \to & Q/Z \\ & & \uparrow & & \uparrow & & \| \\ 0 & \to & S^iZ & \to & S^iQ & \to & Q/Z \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}, \qquad (3)$$

we can get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} 0 & \to & H^{\ell(w)}(Z) & \to & H^{\ell(w)}(Q) & \to & H^{\ell(w)}(Q/Z) \\ & & \uparrow & & \uparrow & & \| \\ 0 & \to & H^{\ell(w)}(S^iZ) & \to & H^{\ell(w)}(S^iQ) & \to & H^{\ell(w)}(Q/Z) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}. \qquad (4)$$

Diagram (4) yields $H^{\ell(w)}(S^iZ) = H^{\ell(w)}(Z) \cap H^{\ell(w)}(S^iQ)$. So the theorem follows. ∎

3 Duality and Radical Series

3.1 Recall from 1.5 that for a G-module M, $S[M]$ and $R[M]$ as subsets of $X(T) \times \mathbf{N} \times \mathbf{N}$ are called the socle and radical series patterns of M while $S_r[M]$ and $R_r[M]$ are defined for a G_rT-module M.

Lemma *If* $\lambda \in X_+$ *is generic as in* 2.7*, then*

$$\begin{aligned} S[H^{\ell(w)}(w\cdot\lambda)] &= w_r^{-1}\cdot S_r[Z_r(w\cdot\lambda)]; && (1)\\ S[H^{\ell(w)}(Q_r(w\cdot\lambda))] &= w_r^{-1}\cdot S_r[Q_r(w\cdot\lambda)]. && (2)\end{aligned}$$

Proof: Recall from 1.2 the action $w_r^{-1} \cdot \mu = \mu_0 + p^r w^{-1} \cdot \mu_1$. Now the Lemma follows from Theorem 2.5, Theorem 2.8 and 2.2(4). $\blacksquare$

3.2 Let $\lambda_0 \in X_r$ and $(\lambda_0)_w \in X(T)$ be defined in 2.7 such that $w \cdot (\lambda_0) + p^r(\lambda_0)_w \in X_r$. If $\lambda = \lambda_0 + p^r\lambda_1$, let $\nu = p^r\lambda_1 - \rho$ be the special point which is the lower vertex of the r-box containing λ (an r-box is a fundamental domain in $X(T) \otimes \mathbf{R}$ under the translation action of $p^r X(T)$, see Doty and Sullivan [11] for details).

Proposition *Suppose $\lambda = \lambda_0 + p^r\lambda_1 \in X_+$ is generic. Then*

$$S[H^{\ell(w)}(w \cdot \lambda)] = (w_r^{-1})_\nu \cdot S[H^0((w \cdot \lambda)_0 + p^r\lambda_1)] - p^r w^{-1}((\lambda_0)_w). \qquad (1)$$

Proof: Since $(w \cdot \lambda)_1 = w(\lambda_1) - (\lambda_0)_w$ and $w \cdot \lambda = (w \cdot \lambda)_0 + p^r(w(\lambda_1) - (\lambda_0)_w)$ (cf. 2.7), $(w \cdot \lambda)_0 + p^r\lambda_1$ is also dominant and generic. Applying 3.1 we have

$$S[H^0((w \cdot \lambda)_0 + p^r\lambda_1)] = S_r[Z_r((w \cdot \lambda)_0 + p^r\lambda_1)]. \qquad (2)$$

The following calculation is based on 1.5(3).

$$\begin{aligned} &S[H^{\ell(w)}(Z_r(w \cdot \lambda))] = w_r^{-1} \cdot S_r[Z_r(w \cdot \lambda)] \\ &= w_r^{-1} \cdot (S_r[Z_r((w \cdot \lambda)_0 + p^r\lambda_1)] - p^r(\lambda_1 - w(\lambda_1)) - p^r((\lambda_0)_w)) \\ &= w_r^{-1} \cdot (S_r[Z_r((w \cdot \lambda)_0 + p^r\lambda_1)] - p^r\lambda_1) + p^r\lambda_1 - p^r w^{-1}((\lambda_0)_w) \\ &= (w_r^{-1})_\nu \cdot S[H^0((w \cdot \lambda)_0 + p^r\lambda_1)] - p^r w^{-1}((\lambda_0)_w). \quad \blacksquare \end{aligned}$$

Remark Using exactly the same argument as in 2.3, one can show that

$$Soc^i_G(H^0(\lambda)) = H^0(Soc^i_{G_rT} Z_r(\lambda))$$

provided that all composition factors of $Z_r(\lambda)$ have the form $L_r(\mu_0 + p^r\mu_1)$ with $\mu_1 \in \bar{C} \cap X_+$. This was proved by Andersen [5] for λ satisfying

$$2(h-1) \leq \langle \lambda_1, \alpha^\vee \rangle \leq p - 2(h-1), \qquad \text{for all } \alpha \in R^+. \qquad (3)$$

However, if λ is generic, then

$$w_\nu \cdot \lambda = (w \cdot \lambda)_0 + p^r((w \cdot \lambda)_1 + \lambda_1 - w(\lambda_1)) = (w \cdot \lambda)_0 + p^r(\lambda_1 - (\lambda_0)_w);$$

$$\begin{aligned} &\langle \lambda_1 - (\lambda_0)_w, \alpha^\vee \rangle \leq \langle \lambda_1, \alpha^\vee \rangle + |\langle (\lambda_0)_w, \alpha^\vee \rangle| \leq p - 5(h-1) + 2(h-1) \leq p - 2(h-1); \\ &\langle \lambda_1 - (\lambda_0)_w, \alpha^\vee \rangle \geq 5(h-1) - 2(h-1) \geq 2(h-1). \end{aligned}$$

Therefore, if we take $w_\nu \cdot \lambda$ instead of $(w \cdot \lambda)_0 + p^r\lambda_1$ in (2), then the same proof in this proposition will show

$$S[H^{\ell(w)}(w \cdot \lambda)] = (w_r^{-1})_\nu \cdot S[H^0(w_\nu \cdot \lambda)]. \tag{4}$$

This gives us an algorithm to calculate the socle series pattern of $H^{\ell(w)}(w \cdot \lambda)$ out of the socle series pattern of $H^0(w_\nu \cdot \lambda)$. For λ generic, take the lower vertex ν of the box containing λ, apply w_ν "dot" action (take ν as center instead of $-\rho$ and apply the "dot" action of W in this situation) to λ to get $w_\nu \cdot \lambda$. Now take the socle series pattern of $H^0(w_\nu \cdot \lambda)$ as known. For each box containing a highest weight of a composition factor of $H^0(w_\nu \cdot \lambda)$, take its upper vertex and move the upper vertex back by w_ν^{-1} to get an upper vertex of a box and translate everything (including socle level and multiplicity in each level) from the former box to this new one. Then one can get the socle series pattern of $H^{\ell(w)}(w \cdot \lambda)$. The action of w_ν^{-1} is exactly the same as the alcove identification operator defined in Doty and Sullivan [11].

3.3 Recall that $L(\mu)^* = L(\mu^*)$ and $H^i(\mu)^* = H^{N-i}(w_0 \cdot \mu^*)$ by Serre duality. Now using 1.5(9) and 1.5(10) for a G-module M one has

$$S[M]^* = R[M^*] \text{ and } R[M]^* = S[M^*], \tag{1}$$

where the $*$ acts on $X(T) \times \mathbf{N} \times \mathbf{N}$ by acting on the first component. Therefore Serre duality implies

$$S[H^i(\mu)]^* = R[H^{N-i}(w_0 \cdot \mu^*)]. \tag{2}$$

Lemma *Let $\lambda \in X_+$ be generic. Then*

$$\begin{aligned} &R[H^{\ell(ww_0)}(ww_0 \cdot \lambda^*)] = \\ &\quad ((w_0 w^{-1} w_0)_r)_{\nu^*} \cdot R[H^N(w_0 \cdot ((w \cdot \lambda)_0^* + p^r \lambda_1^*))] - p^r (w^{-1}(\lambda_0)_w)^*. \end{aligned} \tag{3}$$

Proof: It follows from (2) that

$$S[H^{\ell(w)}(w \cdot \lambda)]^* = R[H^{\ell(ww_0)}(w_0 \cdot (w \cdot \lambda)^*)].$$

Note that $(w \cdot \lambda)^* = w_0 w w_0 \cdot \lambda^*$. Thus we have

$$S[H^{\ell(w)}(w \cdot \lambda)]^* = R[H^{\ell(ww_0)}(ww_0 \cdot \lambda^*)].$$

On the other hand,

$$\begin{aligned}
&((w_r^{-1})_\nu \cdot S[H^0((w\cdot\lambda)_0 + p^r\lambda_1)] - p^r w^{-1}(\lambda_0)_w)^* \\
&= ((w_0 w^{-1} w_0)_r)_{\nu^*} \cdot S[H^0((w\cdot\lambda)_0 + p^r\lambda_1)]^* - p^r(w^{-1}(\lambda_0)_w)^* \\
&= ((w_0 w^{-1} w_0)_r)_{\nu^*} \cdot R[H^N(w_0\cdot((w\cdot\lambda)_0^* + p^r\lambda_1^*))] - p^r(w^{-1}(\lambda_0)_w)^*.
\end{aligned}$$

Therefore the lemma follows from Proposition 3.2. ∎

3.4 Lemma *Let $\mu \in X_+$ be generic. Then*

$$R[H^N(w_0\cdot\mu)] = ((w_0^{-1})_r)_\eta \cdot R[H^0((w_0\cdot\mu)_0 + p^r\mu_1)] - p^r w_0^{-1}(\mu_0)_{w_0}, \quad (1)$$

where $\eta = p^r\mu_1 - \rho$ is a special point.

Proof: Let $w = w_0$ in Lemma 3.3. Then

$$R[H^0(\lambda^*)] = ((w_0^{-1})_r)_{\nu^*}\cdot R[H^N(w_0\cdot((w_0\cdot\lambda)_0^* + p^r\lambda_1^*))] - p^r(w_0^{-1}(\lambda_0)_{w_0})^*. \quad (2)$$

Recall the properties of the $*$ action in 1.2 we have

$$\begin{aligned}
&(w_0^{-1}(\lambda_0)_{w_0})^* = -w_0(w_0^{-1}(\lambda_0)_{w_0}) = -(\lambda_0)_{w_0} = w_0^{-1}\cdot(\lambda_0)_{w_0}^*; \\
&(w_0\cdot\lambda)_0^* = (w_0\cdot\lambda_0)^* + p^r(\lambda_0)_{w_0}^* = w_0\cdot\lambda_0^* + p^r(\lambda_0)_{w_0}^*; \\
&w_0\cdot((w_0\cdot\lambda)_0^* + p^r\lambda_1^*) = w_0\cdot(w_0\cdot\lambda)_0^* + p^r w_0(\lambda_1^*) = \lambda_0^* + p^r w_0(\lambda_0)_{w_0}^* + p^r w_0(\lambda_1^*); \\
&(\lambda_0)_{w_0}^* = (\lambda_0^*)_{w_0}.
\end{aligned}$$

Therefore we have

$$R[H^0(\lambda^*)] = ((w_0^{-1})_r)_{\nu^*}\cdot R[H^N(\lambda_0^* + p^r w_0(\lambda_0^*)_{w_0} + p^r w_0(\lambda_1^*))] - p^r w_0^{-1}(\lambda_0^*)_{w_0}. \quad (3)$$

Let $\mu = (w_0\cdot\lambda)_0^* + p^r\lambda_1^*$. Then $\mu_1 = \lambda_1^*$. Therefore μ is generic if and only if λ^* is generic. Furthermore we have

$$\begin{aligned}
\mu_0 &= (w_0\cdot\lambda)_0^* = (w_0\cdot\lambda^*)_0 = w_0\cdot\lambda_0^* + p^r(\lambda_0^*)_{w_0}; \\
w_0\cdot\lambda^* &= (w_0\cdot\lambda^*)_0 + p^r(w_0\cdot\lambda^*)_1 \\
&= \mu_0 + p^r(w_0(\lambda_1^*) - (\lambda_0^*)_{w_0}) = \mu_0 + p^r w_0(\mu_1) - p^r(\lambda_0^*)_{w_0}; \\
\lambda^* &= w_0\cdot\mu_0 + p^r\mu_1 - p^r w_0^{-1}(\lambda_0^*)_{w_0}.
\end{aligned}$$

Note that

$$w_0\cdot\mu_0 + p^r(-w_0(\lambda_0^*)_{w_0}) = w_0\cdot(w_0\cdot\lambda_0^* + p^r(\lambda_0^*)_{w_0} - p^r(\lambda_0^*)_{w_0}) = \lambda_0^* \in X_r.$$

Thus $-w_0^{-1}(\lambda_0^*)_{w_0} = (\mu_0)_{w_0}$ and

$$\lambda^* = w_0 \cdot \mu_0 + p^r \mu_1 + p^r (\mu_0)_{w_0} = (w_0 \cdot \mu)_0 + p^r \mu_1.$$

Therefore, for any μ as in the Lemma, take $\lambda = ((w_0 \cdot \mu)_0 + p^r \mu_1)^*$. So μ is generic if and only if λ is generic. Now by substituting the arguments in (3) by those in terms of μ and solving out the H^N term, one has

$$\begin{aligned} R[H^N(w_0 \cdot \mu)] &= ((w_0^{-1})_r)_{\nu^*} \cdot (R[H^0((w_0 \cdot \mu)_0 + p^r \mu_1)] - p^r (\mu_0)_{w_0}) \\ &= ((w_0^{-1})_r)_{\nu^*} \cdot R[H^0((w_0 \cdot \mu)_0 + p^r \mu_1)] - p^r w_0 (\mu_0)_{w_0}, \end{aligned}$$

where $\nu^* = \eta$ and the Lemma follows. ∎

3.5 Theorem *Suppose $\lambda \in X_+$ is generic. Then*

$$R[H^{\ell(w)}(w \cdot \lambda)] = (w_r^{-1})_\nu \cdot R[H^0((w \cdot \lambda)_0 + p^r \lambda_1)] - p^r w^{-1}(\lambda_0)_w. \quad (1)$$

Proof: Let $\mu = (w \cdot \lambda)_0^* + p^r \lambda_1^*$ in Lemma 3.3. The following follows from Lemma 3.4.

$$\begin{aligned} R[H^{\ell(ww_0)}(ww_0 \cdot \lambda^*)] &= ((w_0 w^{-1} w_0)_r)_{\nu^*} \cdot R[H^N(w_0 \cdot \mu)] - p^r (w^{-1}(\lambda_0)_w)^* \\ &= ((w_0 w^{-1} w_0)_r)_{\nu^*} \cdot (((w_0^{-1})_r)_{\nu^*} \cdot R[H^0((w_0 \cdot \mu)_0 + p^r \mu_1)] - p^r w_0^{-1} (\mu_0)_{w_0}) - p^r (w^{-1}(\lambda_0)_w)^* \\ &= ((w_0 w^{-1})_r)_{\nu^*} \cdot R[H^0((w_0 \cdot \mu)_0 + p^r \mu_1)] - p^r (w_0 w^{-1}((\mu_0)_{w_0}) + (w^{-1}(\lambda_0)_w)^*), \end{aligned} \quad (2)$$

where $\mu_1 = \lambda_1^*$, so μ and λ^* are in the same box with the lower vertex ν^*.

Note that $(w \cdot \lambda)_0^* = (w_0 w w_0 \cdot \lambda^*)_0$. Then

$$\mu = (w \cdot \lambda)_0^* + p^r \lambda_1^* = w_0 w w_0 \cdot \lambda^* + p^r (\lambda_1^* - (w_0 w w_0 \cdot \lambda^*)_1). \quad (3)$$

Therefore $w_0 \cdot \mu = w w_0 \cdot \lambda^* + p^r w_0 (\lambda_1^* - (w_0 w w_0 \cdot \lambda^*)_1)$ and

$$(w_0 \cdot \mu)_0 + p^r \mu_1 = (w w_0 \cdot \lambda^*)_0 + p^r \lambda_1^*. \quad (4)$$

Also from (3) one has $\mu_0 = (w_0 w w_0 \cdot \lambda^*)_0 = w_0 w w_0 \cdot \lambda_0^* + p^r (\lambda_0^*)_{w_0 w w_0}$ and

$$\begin{aligned} w_0 \cdot \mu_0 &= w w_0 \cdot \lambda_0^* + p^r w_0 ((\lambda_0^*)_{w_0 w w_0}) \\ &= w w_0 \cdot \lambda_0^* + p^r (\lambda_0^*)_{w w_0} + p^r (w_0 ((\lambda_0^*)_{w_0 w w_0}) - (\lambda_0^*)_{w w_0}). \end{aligned} \quad (5)$$

From (5) one also has $(\mu_0)_{w_0} = (\lambda_0^*)_{w w_0} - w_0 ((\lambda_0^*)_{w_0 w w_0})$. Therefore

$$w_0 w^{-1} ((\mu_0)_{w_0}) = w_0 w^{-1} ((\lambda_0^*)_{w w_0}) - w_0 w^{-1} w_0 ((\lambda_0^*)_{w_0 w w_0}). \quad (6)$$

Since $(w\cdot\lambda_0+p^r(\lambda_0)_w)^* = w_0ww_0\cdot\lambda_0^*+p^r(\lambda_0)_w^* \in X_r$, then $(\lambda_0)_w^* = (\lambda_0^*)_{w_0ww_0}$. Therefore, by applying (6) we have

$$\begin{aligned}
&w_0w^{-1}((\mu_0)_{w_0}) + (w^{-1}(\lambda_0)_w)^* \\
&\quad = w_0w^{-1}((\lambda_0^*)_{ww_0}) - w_0w^{-1}w_0((\lambda_0^*)_{w_0ww_0}) + w_0w^{-1}w_0((\lambda_0)_w^*) \\
&\quad = w_0w^{-1}((\lambda_0^*)_{ww_0}) - w_0w^{-1}w_0((\lambda_0^*)_{w_0ww_0}) + w_0w^{-1}w_0((\lambda_0^*)_{w_0ww_0}) \\
&\quad = w_0w^{-1}((\lambda_0^*)_{ww_0}). \qquad (7)
\end{aligned}$$

Now substituting (4) and (7) into (2) we have

$$R[H^{\ell(ww_0)}(ww_0\cdot\lambda^*)] = ((w_0w^{-1})_r)_{\nu^*}\cdot R[H^0((ww_0\cdot\lambda^*)_0+p^r\lambda_1^*)] - p^rw_0w^{-1}((\lambda_0^*)_{ww_0}).$$

Since $\lambda = \lambda_0 + p^r\lambda_1$ is generic if and only if $\lambda^* = \lambda_0^* + p^r\lambda_1^*$ is generic, we can replace λ^* by λ and ww_0 by w and the theorem follows. ∎

3.6 Let $L_r(\mu)$ be a simple G_rT-module. Then

$$\begin{aligned}
L_r(\mu)^* &= (L_r(\mu_0)\otimes p^r\mu_1)^* \\
&= L_r(\mu_0)^*\otimes p^r(-\mu_1) = L_r(\mu_0^* + p^r(-\mu_1)). \qquad (1)
\end{aligned}$$

Let $\bar{\cdot}: X(T)\to X(T)$ be the involution defined by

$$\overline{\mu_0 + p^r\mu_1} = \mu_0^* + p^r(-\mu_1) \qquad (2)$$

for all $\mu\in X(T)$. It follows from the definition of $\bar{\cdot}$ and $*$ that

$$\mu^* = (w_0)_r(\overline{\mu}). \qquad (3)$$

Let $\lambda\in X(T)$. Using 1.5(9), 1.5(10) and (1) one can show for a G_rT-module M of finite length that

$$\overline{S_r[M]} = R_r[M^*]; \qquad (4)$$
$$\overline{R_r[M]} = S_r[M^*]. \qquad (5)$$

Now let $M = Z_r(w\cdot\lambda)$, we have the following

$$\begin{aligned}
&R_r[Z_r(w\cdot\lambda)]^* = (w_0)_r(\overline{R_r[Z_r(w\cdot\lambda)]}) \\
&\quad = (w_0)_r(S_r[Z_r(w\cdot\lambda)^*]) = (w_0)_r(S_r[Z_r(2(p^r-1)\rho - w\cdot\lambda)]) \\
&\quad = (w_0)_r(S_r[Z_r(-2\rho - w\cdot\lambda)] + p^r\rho) - p\rho = (w_0)_r\cdot(S_r[Z_r(w_0\cdot(w\cdot\lambda)^*)]) \\
&\quad = (w_0)_r\cdot(S_r[Z_r(ww_0\cdot\lambda^*)]). \qquad (6)
\end{aligned}$$

Now we are ready to prove

Theorem *Let $\lambda = \lambda_0 + p^r\lambda_1$ be generic. Then*

$$R[H^{\ell(w)}(w \cdot \lambda)] = w_r^{-1} \cdot R_r[Z_r(w \cdot \lambda)]; \tag{7}$$
$$Rad_G^i H^{\ell(w)}(w \cdot \lambda) = H^{\ell(w)}(Rad^i_{G_rT} Z_r(w \cdot \lambda)). \tag{8}$$

Proof: Since $Rad_G^i H^{\ell(w)}(w \cdot \lambda) \subseteq H^{\ell(w)}(Rad^i_{G_rT} Z_r(w \cdot \lambda))$ as in 2.6(4), (8) follows easily from (7) by counting composition factors. To prove (7), consider from Serre duality,

$$R[H^{\ell(w)}(w \cdot \lambda)]^* = S[H^{\ell(w)}(w \cdot \lambda)^*] = S[H^{\ell(ww_0)}(ww_0 \cdot \lambda^*)]. \tag{9}$$

On the other hand, by using (6) one has

$$\begin{aligned}(w_r^{-1} \cdot R_r[Z_r(w \cdot \lambda)])^* &= (w_0 w^{-1} w_0)_r \cdot (R_r[Z_r(w \cdot \lambda)]^*) \\ &= (w_0 w^{-1} w_0)_r \cdot ((w_0)_r \cdot S_r[Z_r(ww_0 \cdot \lambda^*)]) = (w_0 w^{-1})_r \cdot S_r[Z_r(ww_0 \cdot \lambda^*)]. \end{aligned} \tag{10}$$

Since λ is generic if and only if λ^* is generic, one can apply Lemma 3.1 to (9) and (10) to get

$$R[H^{\ell(w)}(w \cdot \lambda)]^* = (w_r^{-1} \cdot R_r[Z_r(w \cdot \lambda)])^*.$$

Now the theorem follows by applying the involution $*$ once more. ∎

3.7 Let $\lambda \in X(T)$, then $L_r(\lambda)^* = L_r(\overline{\lambda})$. Recall that $Q_r(\lambda)$ is the projective cover and the injective hull of $L_r(\lambda)$. So $Q_r(\lambda)^*$ is the projective cover and injective hull of $L_r(\overline{\lambda})$. Therefore,

$$Q_r(\lambda)^* \cong Q_r(\overline{\lambda}). \tag{1}$$

Recall that if M is a G-module, the r-th Frobenius twist $M^{(r)}$ is a G-module and

$$(M^{(r)})^* = (M^*)^{(r)}. \tag{2}$$

Now we consider the dual of $H^i(Q_r(\lambda))$ for $\lambda_1 \in W \cdot \bar{C}$. By using (1), (2) and Serre duality we get

$$\begin{aligned}(H^i(Q_r(\lambda)))^* &= (H^i(Q_r(\lambda_0) \otimes p^r\lambda_1))^* \\ &= (Q_r(\lambda_0) \otimes H^i(\lambda_1)^{(r)})^* = Q_r(\lambda_0)^* \otimes (H^i(\lambda_1)^{(r)})^* \\ &= Q_r(\lambda_0^*) \otimes (H^i(\lambda_1)^*)^{(r)} = Q_r(\lambda_0^*) \otimes H^{N-i}(w_0 \cdot \lambda_1^*)^{(r)} \\ &= H^{N-i}(Q_r(\lambda_0^* + p^r(w_0 \cdot \lambda_1^*))) = H^{N-i}(Q_r((w_0)_r \cdot \lambda^*)). \end{aligned} \tag{3}$$

Theorem *Let $\lambda = \lambda_0 + p^r\lambda_1 \in X_+$ such that*

$$2(h-1) \leq \langle \lambda_1 + \rho, \alpha^\vee \rangle \leq p - 2(h-1), \qquad \forall \alpha \in R^+.$$

Then, for all $w \in W$,

$$\begin{aligned} S[H^{\ell(w)}(Q_r(w_r \cdot \lambda))] &= w_r^{-1} \cdot S_r[Q_r(w_r \cdot \lambda)]; & (4) \\ S[H^{\ell(w)}(Q_r(w_r \cdot \lambda))] &= (w_r^{-1})_\nu \cdot S[H^0(Q_r(\lambda))]; & (5) \\ R[H^{\ell(w)}(Q_r(w_r \cdot \lambda))] &= w_r^{-1} \cdot R_r[Q_r(w_r \cdot \lambda)]; & (6) \\ R[H^{\ell(w)}(Q_r(w_r \cdot \lambda))] &= (w_r^{-1})_\nu \cdot R[H^0(Q_r(\lambda))]. & (7) \end{aligned}$$

Here $\nu = p^r\lambda_1 - \rho$ is the lower vertex of the box containing λ.

Proof: (4) follows from Theorem 2.5 and definitions of the patterns in 1.5. Since

$$\begin{aligned} & w_r^{-1} \cdot S_r[Q_r(w_r \cdot \lambda)] = w_r^{-1} \cdot S_r[Q_r(\lambda_0) \otimes p^r w \cdot \lambda_1] \\ & = w_r^{-1} \cdot (S_r[Q_r(\lambda_0) \otimes p^r\lambda_1] + p^r(w \cdot \lambda_1 - \lambda_1)) \\ & = (w_r^{-1})_\nu \cdot S_r[Q_r(\lambda)], \end{aligned}$$

(5) follows by applying (4) for $w = 1$.

To prove (6), we use the duality in (3) and get

$$R[H^{\ell(w)}(Q_r(w_r \cdot \lambda))]^* = S[H^{\ell(w)}(Q_r(w_r \cdot \lambda))^*] = S[H^{\ell(ww_0)}(Q_r((w_0)_r \cdot (w_r \cdot \lambda)^*))].$$

Note that $(w_r \cdot \lambda)^* = (w_0 w w_0)_r \cdot \lambda^*$ and

$$S[H^{\ell(ww_0)}(Q_r((ww_0)_r \cdot \lambda^*))] = (ww_0)_r^{-1} \cdot S_r[Q_r((ww_0)_r \cdot \lambda^*)].$$

Since λ^* also satisfies the condition of the theorem, so combining (4) we have

$$R[H^{\ell(w)}(Q_r(w \cdot \lambda))]^* = (ww_0)_r^{-1} \cdot S_r[Q_r((ww_0)_r \cdot \lambda^*)]. \tag{8}$$

On the other hand,

$$\begin{aligned} & R_r[Q_r(w_r \cdot \lambda)]^* = (w_0)_r(\overline{R_r[Q_r(w_r \cdot \lambda)]}) \\ & = (w_0)_r(S_r[Q_r(w_r \cdot \lambda)^*]) = (w_0)_r(S_r[Q_r(\overline{w_r \cdot \lambda})]). \end{aligned}$$

From

$$\overline{w_r \cdot \lambda} = \overline{(\lambda_0 + p^r w \cdot \lambda)} = \lambda_0^* + p^r(-w \cdot \lambda_1) = \lambda_0^* + p^r w_0 \cdot (w \cdot \lambda_1)^* + p^r 2\rho$$
$$= \lambda_0^* + p^r w w_0 \cdot \lambda_1^* + p^r 2\rho = (ww_0)_r \cdot \lambda^* + p^r 2\rho,$$

we have

$$\begin{aligned} R_r[Q_r(w_r \cdot \lambda)]^* &= (w_0)_r(S_r[Q_r((ww_0)_r \cdot \lambda^*)] + p^r 2\rho) \\ &= (w_0)_r \cdot S_r[Q_r((ww_0)_r \cdot \lambda^*)]. \end{aligned}$$

Therefore, by (8)

$$(w_r^{-1} \cdot R_r[Q_r(w_r \cdot \lambda)])^* = (w_0 w^{-1} w_0)_r \cdot R_r[Q_r(w_r \cdot \lambda)]^*$$
$$= (w_0 w^{-1})_r \cdot S_r[Q_r((ww_0)_r \cdot \lambda^*)] = R[H^{\ell(w)}(Q_r(w \cdot \lambda))]^*$$

and (6) follows. (7) follows from (6) by the similar argument as in (5). ∎

4 Structural Correspondence

4.1 Recall in 1.6 we defined $Z_r(w, \lambda) = \mathrm{Ind}_{B_r^w T}^{G_r T} \lambda$ which is a G_rT-module and extends to a G_rB^w-module, with $Z_r(1, \lambda) = Z_r(\lambda)$. Further, $Z_r(w, \lambda) = Z_r(w(\lambda))^{w^{-1}}$.

Let $St_r = L_r((p^r - 1)\rho) = Z_r((p^r - 1)\rho) = H^0((p^r - 1)\rho)$ be the Steinberg module. A weight in the orbit $W(p^r - 1)\rho$ of the Weyl group W is called an extremal weight of the Steinberg module St_r. Generally for the G_rT-module $Z_r(\lambda)$ the *extremal weights* are defined to be the weights in $\lambda - (p^r - 1)\rho + W(p^r - 1)\rho = \lambda + (p^r - 1)W \cdot 0$

Humphreys in [14] conjectured that there should be a structural correspondence between the G-module $H^{\ell(w)}(w \cdot \lambda)$ and the G_rT-module of the kind $Z_r(w, \mu)$ for some extremal weight μ of $Z_r(\lambda)$, at least for λ in *general position* in the lowest p^2-alcove. The following theorem gives a correspondence on the socle and radical structures between the conjectured modules.

Theorem *Suppose $\lambda = \lambda_0 + p^r \lambda_1 \in X_+$ is generic. Then*

$$S[H^{\ell(w)}(w \cdot \lambda)] = S_r[Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0)]; \quad (1)$$
$$R[H^{\ell(w)}(w \cdot \lambda)] = R_r[Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0)]. \quad (2)$$

Proof: From 1.6(12) we have

$$S_r[Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0)] = w_r^{-1}(S_r[Z_r(w(\lambda + (p^r - 1)w^{-1} \cdot 0))]).$$

Since

$$w(\lambda + (p^r - 1)w^{-1} \cdot 0) = w \cdot \lambda + p^r(\rho - w(\rho)),$$

thus

$$S_r[Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0)] = w_r^{-1}(S_r[Z_r(w \cdot \lambda)] + p^r(\rho - w(\rho)))$$

$$= w_r^{-1}(S_r[Z_r(w \cdot \lambda)] + p^r\rho) - p^r\rho = w_r^{-1} \cdot S_r[Z_r(w \cdot \lambda)].$$

Now (1) follows from Lemma 3.1.

By using Theorem 3.6 and the same argument, one can get (2). ∎

4.2 Let $\sigma : G \to G$ be an automorphism of the semisimple algebraic group G such that $\sigma(U_\alpha) = U_{-w_0(\alpha)}$. Under this automorphism we have $\sigma(G_rT) = G_rT$, $\sigma(B_rT) = B_rT$ and $\sigma(B) = B$ etc. Since σ induces an automorphism of the root system which maps positive roots to positive roots, then $\sigma(U) = U$. Note that $\sigma(t) = w_0t^{-1}w_0^{-1}, \forall t \in T$. So from (5) in 1.6, we have

$$R^i\mathrm{Ind}_B^G E^\sigma = (R^i\mathrm{Ind}_B^G E)^\sigma \text{ and } \mathrm{Ind}_{B_rT}^{G_rT} M^\sigma = (\mathrm{Ind}_{B_rT}^{G_rT} M)^\sigma,$$

where E and M are B-module and B_rT-module respectively, and $\sigma^*(E) = E^\sigma$.

For the one dimensional T-module λ we have $\lambda^\sigma = \lambda^*$. Hence

$$\begin{aligned} H^i(\lambda^*) &= H^i(\lambda)^\sigma; && (1)\\ L(\lambda^*) &= L(\lambda)^\sigma; && (2)\\ Z_r(\lambda^*) &= Z_r(\lambda)^\sigma; && (3)\\ L_r(\lambda^*) &= L_r(\lambda)^\sigma. && (4) \end{aligned}$$

Therefore by 1.6(1) and 1.6(2) we have

$$S[H^i(\lambda^*)] = S[H^i(\lambda)^\sigma] = S[H^i(\lambda)]^*;$$

$$R[H^i(\lambda^*)] = R[H^i(\lambda)^\sigma] = R[H^i(\lambda)]^*.$$

Lemma *Let $\lambda \in X_+$ be generic. Then*

$$R[H^{\ell(w)}(w \cdot \lambda)] = S[H^{\ell(ww_0)}(w_0 w \cdot \lambda)]$$

for all $w \in W$.

Proof: By Serre duality we have

$$S[H^{\ell(w)}(w \cdot \lambda)]^* = R[H^{\ell(w)}(w \cdot \lambda)^*] = R[H^{\ell(ww_0)}(ww_0 \cdot \lambda^*)].$$

On the other hand

$$S[H^{\ell(w)}(w \cdot \lambda)]^* = S[H^{\ell(w)}(w \cdot \lambda)^\sigma] = S[H^{\ell(w)}((w \cdot \lambda)^*)] = S[H^{\ell(w)}(w_0 w w_0 \cdot \lambda^*)].$$

Now the lemma follows by replacing λ^* and w with λ and ww_0 . ∎

4.3 Theorem *Let $\lambda \in X_+$ be generic and $\nu = p^r \lambda_1 - \rho$ be the lower vertex of the box containing λ. Then*

$$S[H^0((w_0)_\nu \cdot \lambda)] = ((w_0)_r)_\nu \cdot R[H^0(\lambda)].$$

Proof: Let $w = 1$ in Lemma 4.2, we have

$$R[H^0(\lambda)] = S[H^{\ell(w_0)}(w_0 \cdot \lambda)].$$

Now by (4)in 3.2, $S[H^{\ell(w_0)}(w_0 \cdot \lambda)] = ((w_0^{-1})_r)_\nu \cdot S[H^0((w_0)_\nu \cdot \lambda)]$ and now the theorem follows. ∎

Remark From this theorem and the theorems in section 3 one can see that generically the radical and socle structures of all cohomology modules are determined by the structures of Weyl modules. When $r = 1$, moreover, for a p-regular weight λ the structures only depend on the alcove type within a box. This theorem tells us that there is a duality between the alcove types such that the socle series of the Weyl module with highest weight in an alcove comes from the radical series of the Weyl module with highest weight in the dual alcove. In other words, generically, the socle and radical structures of all cohomology modules are determined by only half of the Weyl module structures in a box.

4.4 Let $\lambda \in X_+$. Define $D(\lambda) = \{(\mu, m) \mid [H^0(\lambda) : L(\mu)] = m\}$ to be the *decomposition pattern.* It follows from 3.2(4) that

$$(w_r)_\nu \cdot D(\lambda) = D(w_\nu \cdot \lambda) \tag{1}$$

for λ generic and $\nu = p^r\lambda_1 - \rho$ and $w \in W$. Combining this with the translation invariance of patterns, one can conclude that the decomposition patterns of all Weyl modules in the generic case are determined by patterns in only one p^r-alcove. Furthermore, the total number of composition factors of a Weyl module in the generic case only depends on the facet type containing the highest weight.

Corollary ([16], II 9.14) *If $\lambda \in p^r X_+ - \rho$ is generic, then $H^{\ell(w)}(w \cdot \lambda)$ is simple for all $w \in W$.*

Proof: Since $S[H^{\ell(w)}(w \cdot \lambda)] = w_r^{-1} \cdot S_r[Z_r(w \cdot \lambda)]$, let us consider $S_r[Z_r(w \cdot \lambda)]$. As $\lambda = p^r\nu - \rho$ and $w \cdot \lambda = p^r w(\nu) - \rho$, we have $S_r[Z_r(w \cdot \lambda)] = S_r[Z((p^r - 1)\rho)] + p^r(w(\nu) - \rho)$. Now the corollary follows from the simplicity of the Steinberg modules. ∎

4.5 Another question is when $L(\mu)$ is a composition factor of $H^0(\lambda)$, i.e., find $D(\lambda)_+ = \{\mu \mid [H^0(\lambda) : L(\mu)] \neq 0\}$. However, for λ generic and $\nu = p^r\lambda_1 - \rho$, $\mu \in D(\lambda)_+$ implies $(w_r)_\nu \cdot \mu \in D(w_\nu \cdot \lambda)_+$ for all $w \in W$. So the strong linkage principle implies $(w_r)_\nu \cdot \mu \uparrow w_\nu \cdot \lambda$ for all $w \in W$. This turns out to be a sufficient condition when $r = 1$. However, for $r \geq 2$ this is not sufficient, for details see [13]. The condition here determines exactly the strong linkage class of λ when $r = 1$ as in [13].

4.6 In this section we assume $r = 1$. Andersen in [6] has proved that in generic cases the socle filtration and the radical filtration of $H^0(\lambda)$ coincide by assuming the Jantzen conjecture. Recall that a module of finite length is called *rigid* if its radical filtration and socle filtration coincide. The following theorem extends Andersen's results to all cohomology modules.

Theorem *Suppose $r = 1$. Let $\lambda \in X_+$ be generic and $w \in W$. Then $H^{\ell(w)}(w \cdot \lambda)$ is rigid if Lusztig's conjecture is true.*

Proof: Andersen and Kaneda in [7] have proved by assuming Lusztig's conjecture that the G_1T-module $Z_1(w, \lambda + (p^r - 1)w^{-1} \cdot 0)$ is rigid and has Loewy length $N + 1$. Now the theorem follows from Theorem 4.1. ∎

Remark For the groups of rank 2, the Lusztig conjecture has been proved. In this case the cohomology modules generically are rigid for the weights in the lowest p^2-alcove. The radical filtration and Andersen's filtration [4] coincide generically for the group of type B_2 since the socle filtration of $H^0(\lambda)$ for λ dominant has been proved to be the dual filtration of the Jantzen filtration of the Weyl module $V(\lambda)$ in this case (see [8] , [11] for example).

4.7 For any $r \geq 1$ and $\lambda \in X_+$ generic, the rigidity of $H^{\ell(w)}(w \cdot \lambda)$ depends on the rigidity of Weyl modules.

Theorem *Suppose $\lambda \in X_+$ is generic. Then the following are equivalent*

(i) $H^0(\lambda)$ *is rigid;*

(ii) $H^{\ell(w)}(w \cdot \lambda)$ *is rigid for all* $w \in W$*;*

(iii) $Z_r(\lambda)$ *is rigid as* G_rT*-module;*

(iv) $Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0)$ *is rigid for all* $w \in W$*.*

Proof: The equivalence of (i) and (ii) can be derived from Proposition 3.2 and Theorem 3.5. Lemma 3.1 and Theorem 3.6 give the equivalence of (i) and (iii). On the other hand, Theorem 4.1 gives the equivalence of (ii) and (iv). So the Theorem follows. ∎

5 Nongeneric Structure

Using the structure of $Z_r(\lambda)$ and Kempf's vanishing theorem, Bai–Wang–Wen in [8] derived the socle structure of $H^0(\lambda)$ for λ dominant and close to a chamber wall. In this section we derive from $Z_r(\lambda)$ and $Q_r(\lambda)$ that $H^i(\lambda)$ has simple socle for certain i and λ. This extends Andersen's work on the structure of H^1 in [1].

5.1 Recall that the Weyl group W as a reflection group has a "dot" action on $X(T) \otimes \mathbf{R}$. The closure of each connected component (so called chamber

) of the complement of the union of all the hyperplanes through $-\rho$ is a fundamental domain of this action. So there is a correspondence between the set of chambers and the Weyl group such that for each chamber there is a unique $w \in W$ such that $w \cdot X_+$ is contained in this chamber (which we will call the w-chamber). The chamber containing X_+ is called the dominant chamber. We define an order relation in the set of chambers by reversing the Bruhat order on W, i.e., we say the w-chamber is higher than the w'-chamber if $w < w'$. So the dominant chamber is the unique highest chamber and the w_0-chamber is the unique lowest chamber. In this sense one can see that the w-chamber is higher than the w'-chamber if and only if there is a sequence of positive roots $\alpha_1, \alpha_2, \ldots, \alpha_t$ such that $w' = s_{\alpha_t} \ldots s_{\alpha_1} w$ and the $s_{\alpha_{i-1}} \ldots s_{\alpha_1} w$-chamber is on the positive side of the α_i-wall for each i.

We have used the term " p^r-box" (we simply say "box" if r is understood) in section 3. If $\lambda \in X(T)$, the upper vertex of the box containing λ is $\lambda^v = p^r\lambda_1 + (p^r-1)\rho$ and the lower vertex is $\lambda_v = p^r\lambda_1 - \rho$. If $w \in W$, one can easily verify

$$w_r \cdot \lambda = \lambda_0 - (p^r-1)\rho + w \cdot \lambda^v. \tag{1}$$

This is the reason that the algorithm in 3.2 is so stated.

5.2 Lemma *Let $\lambda_1 \in W \cdot \bar{C}$. Then $H^i(L_r(\lambda)) \neq 0$ if and only if there exists a (unique) $w \in W$ such that $w \cdot \lambda^v \in X_+$ and $i = \ell(w)$.*

Proof: Suppose there is a $w \in W$ such that $w \cdot \lambda^v \in X_+$. One can verify that

$$w \cdot \lambda^v = w \cdot (p^r\lambda_1 + (p^r-1)\rho) = p^r w \cdot \lambda_1 + (p^r-1)\rho \in X_+$$

if and only if $w \cdot \lambda_1 \in X_+$. Since $\lambda_1 \in W \cdot \bar{C}$, we have $w \cdot \lambda_1 \in \bar{C}$, so $H^i(\lambda_1) \neq 0$ if and only if $i = \ell(w)$. Therefore $H^i(L_r(\lambda)) = L_r(\lambda_0) \otimes H^i(\lambda_1)^{(r)} \neq 0$ if and only if $i = \ell(w)$.

Conversely, suppose $w \cdot \lambda^v \notin X_+$ for all $w \in W$. Then λ^v has to be on a chamber wall, i.e., $\langle \lambda^v + \rho, \alpha^\vee \rangle = 0$ for some $\alpha \in R^+$. Let $w \in W$ such that $w(\alpha) \in S$ is simple, then $\langle w \cdot \lambda^v + \rho, w(\alpha)^\vee \rangle = \langle \lambda^v + \rho, \alpha^\vee \rangle = 0$ and $\langle w \cdot \lambda_1, w(\alpha)^\vee \rangle = -1$. Now since $\lambda_1 \in W \cdot \bar{C}$ we have $H^i(\lambda_1) = 0$ for all $i \geq 0$. The lemma now follows from the generalized tensor identity. ∎

5.3 We have defined in 4.5 $D_r(\lambda)_+$ which is the set of highest weights of composition factors of $Z_r(\lambda)$. $D_r(\lambda)_+$ is computable from Doty and Sullivan [13].

Theorem *Suppose $\lambda \in X(T)$ such that $\lambda_1 \in W \cdot \bar{C}$ and λ^v lies in the w-chamber. If none of the weights in $D_r(\lambda)_+$ has its upper vertex in a higher chamber, then $H^i(\lambda) = 0$ for all $i = 0, \ldots, \ell(w) - 1$, and $H^{\ell(w)}(\lambda) \neq 0$.*

Proof: From Lemma 5.2 we have $H^i(L_r(\mu)) = 0$ for all $\mu \in D_r(\lambda)_+$ and $i = 0, \ldots, \ell(w) - 1$ since $w > w'$ implies $\ell(w) > \ell(w')$. However, $H^{\ell(w)}(L_r(\lambda)) \neq 0$ by Lemma 5.2. Now by induction one can easily show $H^i(M) = 0$ for all G_rB-subquotients M of $Z_r(\lambda)$ and $0 \leq i < \ell(w)$. So we have the exact sequence

$$0 \to H^{\ell(w)}(L_r(\lambda)) \to H^{\ell(w)}(Z_r(\lambda)) = H^{\ell(w)}(\lambda).$$

This proves the theorem. ∎

Remark In [1], Andersen proved that for certain weights close to chamber walls the $H^1 \neq 0$. Using above theorem one can see that if λ has its upper vertex in the w-chamber (with λ itself in a lower chamber) and far from any upper walls of the w-chamber then λ has a lower nonvanishing cohomology besides its standard nonvanishing one. Combining this theorem with Serre duality, one can see that λ has two nonvanishing cohomology groups if the box containing λ crosses a chamber wall and is far from any other chamber walls. These two i's can be calculated which are two successive numbers and λ has only these two nonvanishing cohomology groups. This situation is called the semi-standard vanishing as predicted by Humphreys [15]. It is conjectured that if λ is close to a chamber wall but far away from any other chamber walls, extra nonvanishing cohomology appears only when the box containing λ crosses the chamber wall. This can be seen for the groups of rank 2 if λ is far from $-\rho$. However, when λ is close to $-\rho$, the situation is different. In the case when G has type B_2 and $r = 1$, one can use the above theorem to see that each alcove listed in Andersen's picture [3] Fig.1 with extra nonvanishing cohomology is contained in a box which crosses a chamber wall. Further one can determine the composition factors of some extra cohomology. In the $s_\beta s_\alpha s_\beta$-chamber one finds that there is a "2" missing from an alcove in the first box on the top. However, if we look at the pattern $D_1(\lambda)_+$ for λ in that alcove, we will find $H^2(L_1(\mu)) = 0$ for all $\mu \in D_1(\lambda)_+$ by Lemma 5.2. $H^2(L_1(\lambda)) = 0$ because the upper vertex $p\lambda_1 + (p-1)\rho$ of the box is on the $\alpha + \beta$ wall. A similar situation happens for the group of type G_2.

5.4 Let $\lambda \in X(T)$. Define

$$\begin{aligned} D_r^{-1}(\lambda)_+ &= \{\mu \in X(T) \mid [Z_r(\mu) : L_r(\lambda)] \neq 0\} \\ &= \{\mu \in X(T) \mid [Q_r(\lambda) : Z_r(\mu)] \neq 0\}. \end{aligned} \quad (1)$$

It follows from Doty and Sullivan [13] that $D_r^{-1}(\lambda)_+$ is computable, so is $D_r(D_r^{-1}(\lambda)_+)_+$.

Theorem *Let $\lambda \in X(T)$ and $w \in W$ such that λ^v is in the w-chamber and $\lambda_1 \in W \cdot \bar{C}$. If none of the weights in $D_r(D_r^{-1}(\lambda)_+)_+$ lies in a higher chamber, then*

$$Soc_G H^{\ell(w)}(\lambda) = L(w_r^{-1} \cdot \lambda).$$

Proof: Since $D_r(\lambda)_+ \subset D_r(D_r^{-1}(\lambda)_+)_+$ we have $H^i(M) = 0$ for all subquotients M of $Q_r(\lambda)$ and $0 \leq i < \ell(w)$ by Lemma 5.2. Therefore the exact sequence $0 \to Z_r(\lambda) \to Q_r(\lambda)$ induces an exact sequence

$$0 \to H^{\ell(w)}(\lambda) \to H^{\ell(w)}(Q_r(\lambda)) = Q_r(\lambda_0) \otimes H^{\ell(w)}(\lambda_1)^{(r)}.$$

We know from Theorem 5.3 that $H^{\ell(w)}(\lambda) \neq 0$. On the other hand, $H^{\ell(w)}(\lambda_1)$ is simple and $Q_r(\lambda_0)$ has a simple socle $L(\lambda_0)$. So $H^{\ell(w)}(Q_r(\lambda))$ has a simple socle $L(w_r^{-1} \cdot \lambda)$ and the theorem follows. ∎

Comparing with Andersen's work on the socle of the first cohomology one can see that $H^{\ell(w)}(\lambda)$ has simple socle when λ is in a box with upper vertex in the w-chamber but λ itself is in a lower chamber. So the socle for H^1 is a special case. Some application of this theorem to the group of type B_2 has been made. In that case the socle series of the $H^0(\lambda)$ are known. Calculations on diagrams enable us to find out at least one of socle and radical series of all weights in the lowest p^2-alcove for all nonvanishing cohomology. (Except a few alcoves close to $-\rho$. For details see [18].)

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