The Restricted Nullcone

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Abstract. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over an algebraically closed field of characteristic $p > 0$. The restricted nullcone of $\mathfrak{g}$, denoted by $\mathcal{N}_1(\mathfrak{g})$, consists of those $x \in \mathfrak{g}$ such that $x^{[p]} = 0$. In this paper the authors provide a concrete description of this variety (via closures of Richardson orbits) when $\mathfrak{g}$ is the Lie algebra of a reductive group G and p a good prime. Various applications to representation theory and cohomology theory are provided.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field k of positive characteristic p. Assume that $\mathfrak{g}$ is restricted with restriction map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$. (See [J] for a detailed discussion of restricted Lie algebras.) The nullcone $\mathcal{N}(\mathfrak{g})$ is the closed subvariety of $\mathfrak{g}$ consisting of all elements $x \in \mathfrak{g}$ which are $[p]$-nilpotent in the sense that $x^{[p]^r} = 0$ for some $r > 0$, depending on x. More specifically, for any integer $r > 0$, let $\mathcal{N}_r(\mathfrak{g})$ be the closed subvariety consisting of those $x \in \mathfrak{g}$ satisfying $x^{[p]^r} = 0$. Thus, $\mathcal{N}_1(\mathfrak{g}) \subseteq \mathcal{N}_2(\mathfrak{g}) \subseteq \cdots$ provides a finite filtration of $\mathcal{N}(\mathfrak{g})$. In general, even for the restricted Lie algebras associated to reductive groups, little appears to be known about the structure of the varieties $\mathcal{N}_r(\mathfrak{g})$.

The restricted nullcone of $\mathfrak{g}$, defined to be $\mathcal{N}_1(\mathfrak{g})$, plays an important role in the representation theory of $\mathfrak{g}$. Let $u(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$. The category of $u(\mathfrak{g})$-modules is equivalent to the category of restricted $\mathfrak{g}$-modules. Now let $H^\bullet(u(\mathfrak{g}), k)$ be the cohomology algebra of the trivial $u(\mathfrak{g})$-module k. If p is odd, set $H = H^{2\bullet}(u(\mathfrak{g}), k)$, while if $p = 2$, put $H = H^\bullet(u(\mathfrak{g}), k)$. The algebra H is a finitely generated commutative k-algebra, and $V_{\mathfrak{g}} = \operatorname{Maxspec} H$ will be called the cohomological support variety of $\mathfrak{g}$. Given any finite dimensional restricted $\mathfrak{g}$-module M, the support in $V_{\mathfrak{g}}$ of the finitely generated H-module $\operatorname{Ext}^\bullet_{u(\mathfrak{g})}(M, M)$ is called the cohomological support variety of M; we denote it by $V_{\mathfrak{g}}(M)$.

Because $V_{\mathfrak{g}}$ and $\mathcal{N}_1(\mathfrak{g})$ are naturally homeomorphic varieties [SFB, (1.6), (5.11)], we often identify $V_{\mathfrak{g}}(M)$ with its homeomorphic image $|\mathfrak{g}|_M$ as a closed subvariety of $\mathcal{N}_1(\mathfrak{g})$. In turn, by [FP2], $|\mathfrak{g}|_M$ has a very explicit description: it is the subset of

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$\mathcal{N}_1(\mathfrak{g})$ consisting of 0, together with all $x \in \mathcal{N}_1(\mathfrak{g})$ such that M is not a projective module for the p-dimensional subalgebra $k[x]$ generated by x in $u(\mathfrak{g})$. Because M is a restricted $\mathfrak{g}$-module, $(x)^p = x^{[p]} = 0$ as operators on M. Hence, $x \in \mathcal{N}_1(\mathfrak{g})\backslash|\mathfrak{g}|_M$ if and only if x acting on M has Jordan blocks which all have size $p \times p$.

Let G be a reductive algebraic group defined over k. The Lie algebra $\mathfrak{g}$ of G carries a natural structure as a restricted Lie algebra. For $p \geq h$ (the Coxeter number), $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$. This paper aims to identify explicitly the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ for $\mathfrak{g}$ when the prime p is good for G.

While in some cases, e. g., when G has classical type A, B, C, D, $\mathcal{N}_1(\mathfrak{g})$ can be directly determined, we approach this problem generally using the theory of support varieties sketched above and, in particular, using recent results in [NPV] on the calculation of the support variety of the induced G-module $H^0(\varpi)$ for a dominant weight $\varpi \in X(T)_+$. It turns out, somewhat remarkably, that in general $\mathcal{N}_1(\mathfrak{g})$ has the form $\overline{\mathcal{O}}$ for some Richardson orbit $\mathcal{O}$ in $\mathfrak{g}$. These results are explained more fully in Section 2. In addition, this section introduces several important classes of parabolic subgroups. We call a parabolic subgroup P of G restricted provided that its associated nilpotent Richardson orbit $\mathcal{O}$ lies in $\mathcal{N}_1(\mathfrak{g})$. Then P is called strongly restricted provided that the Zariski closure $\overline{\mathcal{O}}$ is precisely the support variety of some $H^0(\varpi)$.

Section 3 proceeds to determine the restricted nullcone in case G has classical type A, B, C, D. We discuss this primarily from a support variety point of view, but also provide a description in terms of partitions. In particular, the partition approach directly relates to our determination in Theorems 3.8 and 3.9 of the restricted and strongly restricted parabolic subgroups in the classical types. In fact, using the Lusztig-Spaltenstein theory of induced nilpotent orbits, we establish that the notion of a restricted parabolic subgroup coincides with that of a strongly restricted parabolic subgroup.

Section 4 treats the exceptional cases E_6, E_7, E_8, F_4, G_2. In addition, Theorems 4.6 and 4.8 explicitly determine the restricted and strongly restricted parabolic subgroups; as in the classical cases, the two notions coincide. As a result, Theorem 4.7 determines, in all types, those orbits $\mathcal{O} \subset \mathcal{N}(\mathfrak{g})$ such that $\overline{\mathcal{O}} = |\mathfrak{g}|_{H^0(\varpi)}$ for some dominant weight ϖ.

In Section 5, we consider the restricted nullcone $\mathcal{N}_1(\mathfrak{s})$ for various restricted subalgebras $\mathfrak{s}$ of $\mathfrak{g}$. For example, when $p < h$ and $\mathfrak{s} = \mathfrak{b}$, a Borel subalgebra, $\mathcal{N}_1(\mathfrak{b})$ is not irreducible. Finally, we conclude with an application of our results to obtain divisibility information on the dimensions of restricted $\mathfrak{g}$-modules in a block.

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2. Preliminaries

2.1. Notation. Throughout we will follow standard notation. Thus, G will denote a connected reductive algebraic group defined over the algebraically closed field k of positive characteristic p. We assume that the derived group G' is a simply

connected and simple algebraic group.[1] Fix a maximal torus T of G and let Φ be the root system of G with respect to T. Let Φ^+ be a positive system for Φ and let $B \supset T$ be the Borel subgroup defined by the negative roots $-\Phi^+$. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be the set of simple roots determined by Φ^+. Any subset $J \subset \Pi$ determines a parabolic subgroup $P_J \supseteq B$, with Levi factor $L_J \supseteq T$ having root system Φ_J, the subsystem of Φ generated by J.

If $X(T)$ is the character group on T, the dominant weights $X(T)_+$ consist of those $\varpi \in X(T)$ satisfying $\langle \varpi, \alpha_i^\vee \rangle \geq 0$, $1 \leq i \leq l$. Here $\alpha^\vee$ denotes the coroot defined by the root α. For $1 \leq i \leq l$, let $\varpi_i \in X(T)_+$ satisfy $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. We call ϖ_i the fundamental dominant weight associated to α_i, even though it is uniquely determined if and only if $G = G'$. Set $\rho = \varpi_1 + \cdots + \varpi_l$. For $\alpha \in \Phi$, $\langle \rho, \alpha^\vee \rangle$ is the height of the coroot $\alpha^\vee$ with respect to the simple system $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_l^\vee\}$ of the dual root system $\Phi^\vee$.

Unless otherwise mentioned, we will assume that p is a good prime for Φ. By definition, this means that if Φ' is an integrally closed subsystem of Φ, then the abelian group $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$ has no p-torsion. This condition amounts to assuming that in classical types B_l, C_l and D_l, we have $p \neq 2$. In exceptional types E_6, E_7, F_4, G_2, the requirement is that $p > 3$; and in type E_8, we need $p > 5$. We will use the fact that if Φ' is an integrally closed subsystem of Φ which satisfies $\mathbb{Q}\Phi' \cap \Phi = \Phi'$, then there exist $J \subseteq \Phi$ and $w \in W$ (the Weyl group of G) such that $w\Phi' = \Phi_J$ [B, Prop. 24, p. 165]. Let h be the Coxeter number of Φ; thus, h is one more than the height of the highest (maximal) root in Φ^+ (or $(\Phi^\vee)^+$).

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{p}_J$, $\mathfrak{u}_J$, ... denote the Lie algebras of G, B, P_J, U_J, ..., respectively; they are all restricted Lie algebras. For any $J \subseteq \Phi$, $G \cdot \mathfrak{u}_J$ is a closed, irreducible subvariety of $\mathcal{N}(\mathfrak{g})$ of dimension equal to $2\dim \mathfrak{u}_J$. The unique open G-orbit in $G \cdot \mathfrak{u}_J$ is denoted $\mathcal{O}_J$–these are the so-called Richardson orbits in $\mathfrak{g}$.

We say that a (proper) parabolic subgroup P of G is *restricted* provided that P is conjugate to a standard parabolic P_J with the property that $\mathfrak{u}_J \subseteq \mathcal{N}_1(\mathfrak{g})$, or, equivalently, that $\mathcal{O}_J \subseteq \mathcal{N}_1(\mathfrak{g})$. For example, B is restricted if and only if $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$.

Since $X(B) = X(T)$, any weight $\varpi \in X(T)$ defines a one-dimensional rational B-module. For $\varpi \in X(T)$, let $H^0(\varpi)$ be the induced rational G-module $\mathrm{ind}_B^G \varpi$, which is non-zero if and only if $\varpi \in X(T)_+$. Moreover, $H^0(\varpi)$, $\varpi \in X(T)_+$, is a finite dimensional module with character given by Weyl's character formula.

2.2. Support varieties. For any $\varpi \in X(T)$, consider the set

$$\Phi_\varpi = \{\alpha \in \Phi : \ \langle \varpi + \rho, \alpha^\vee \rangle \in p\mathbb{Z}\}.$$

Because p is assumed to be good for Φ, this definition agrees with that given for Φ_ϖ in [NPV, §3], and so Φ_ϖ is an integrally closed subsystem of Φ such that $\mathbb{Z}\Phi/\mathbb{Z}\Phi_\varpi$ is torsion free (it can only have p-torsion by the definition of Φ_ϖ). Hence, $\mathbb{Q}\Phi_\varpi \cap \Phi = \Phi_\varpi$ and there exists a $w \in W$ such that $w\Phi_\varpi = \Phi_J$ for some $J \subseteq \Pi$. In general, we will use without mention the identity $w\Phi_\varpi = \Phi_{w\cdot\varpi + p\delta}$, for $\varpi, \delta \in X(T)$, and $w \in W$. Here $w \cdot \varpi = w(\varpi + \rho) - \rho$.

The following result, proved in [NPV, Thm. (6.2.1)], describes the support varieties of the modules $H^0(\varpi)$ for $\varpi \in X(T)_+$ in terms of closures of Richardson orbits.

[1]Occasionally, if G has type D_l, it will be convenient to work with the non-simply connected group $\mathrm{SO}_{2l}(k)$, rather than $\mathrm{Spin}_{2l}(k)$.

THEOREM . *Let $\varpi \in X(T)_+$ and choose $w \in W$ such that $w\Phi_\varpi = \Phi_J$ for some $J \subseteq \Pi$. Then $|\mathfrak{g}|_{H^0(\varpi)} = G \cdot \mathfrak{u}_J = \overline{\mathcal{O}_J}$.*

In particular, if $\varpi = 0$ then $\mathcal{N}_1(\mathfrak{g}) = |\mathfrak{g}|_k = G \cdot \mathfrak{u}_J$ where $w\Phi_0 = \Phi_J$ for some $w \in W$.[2] This shows that $\mathcal{N}_1(\mathfrak{g})$ is an irreducible variety having dimension equal to $\dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_0|$. This dimension is also equal to the Krull dimension of the k-algebra H defined in the introduction. The subset $J \subseteq \Pi$ is not unique in general. In fact, for any $w \in W$ such that $wJ \subseteq \Pi$, we have $G \cdot \mathfrak{u}_J = G \cdot \mathfrak{u}_{wJ}$.

The discussion suggests the following definition. We call a subgroup P *strongly restricted* provided that it is conjugate to a standard parabolic subgroup P_J such that there exists a weight $\varpi \in X(T)_+$ and $w \in W$ such that $w\Phi_\varpi = \Phi_J$. Equivalently, P is strongly restricted if and only if it is conjugate to some P_J for which there exists $\varpi' \in X(T)$ such that $\{\alpha \in \Phi \mid \langle \varpi', \alpha^\vee \rangle \in p\mathbb{Z}\} = \Phi_J$. In fact, if ϖ exists, we can take $\varpi' = w \cdot \varpi + \rho$; conversely, given ϖ', define $\varpi = \varpi' + (rp-1)\rho \in X(T)_+$ for some large positive integer r. If $p \geq h$, then every parabolic subgroup is evidently strongly restricted.[3]

The above theorem immediately implies that every strongly restricted parabolic subgroup P is restricted. If G has classical type B_l, C_l, D_l (l odd), then for any orbit $\mathcal{O} \subseteq \mathcal{N}_1(\mathfrak{g})$, we have $\overline{\mathcal{O}} = \mathcal{N}(\mathfrak{g}) \cap \overline{\mathcal{O}'}$ for some (necessarily Richardson) nilpotent orbit $\mathcal{O}'$ for $\mathfrak{sl}_n$ (relative to a natural embedding $\mathfrak{g} \subseteq \mathfrak{sl}_n$, $n = 2l$ or $2l+1$); see [Hum3, §7.11, 7.20]. Hence, applying the theorem, $\overline{\mathcal{O}} = |\mathfrak{g}|_M$ for some rational G-module M. The module M can be taken to have the form $H^0(\lambda)$ if and only if $\mathcal{O}$ is a Richardson class corresponding to strongly restricted parabolic subgroup P. In type D_l (l even), these remarks remain valid: If $\mathcal{O}$ is not "very even" (see, e. g., [Hum3, p. 130]) then the above statements all remain true in this case. If $\mathcal{O}$ is a very even restricted orbit, then it is a Richardson class corresponding to a restricted parabolic subgroup P. We will show in Theorem 4.8 below that if P is restricted, so $|\mathfrak{g}|_{H^0(\varpi)} = \overline{\mathcal{O}}$ for some dominant weight ϖ.

2.3. A natural question is to determine the subsets $J \subseteq \Pi$ such that the parabolic subgroup P_J is restricted. Then we will determine all J's such that P_J are strongly restricted. It follows that P_J is restricted if and only if every element in the unipotent radical U_J of P_J has order at most p, i. e., the group U_J has exponent p.[4]

For each $J \subseteq \Pi$, set $h_J = \sum_{\alpha_i \in \Pi \setminus J} m_i$ where $\alpha_0 = \sum_{i=1}^{l} m_i \alpha_i$ is the highest root in Φ^+.

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[2]Since $\Phi_0 = \emptyset$ if and only if $p \geq h$, the above theorem immediately shows that $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$ if and only if $p \geq h$. The trivial observation that, given $\alpha \in \Phi$, the root space $\mathfrak{g}_\alpha \subset \mathcal{N}_1(\mathfrak{g})$ is sometimes useful. For example, this always means that $\dim \mathcal{N}_1(\mathfrak{g}) \geq 2h^\vee - 2$, where $h^\vee$ is the dual Coxeter number of Φ [W]. The inequality is almost never an equality, however. Finally, it is sometimes helpful to observe that the "operators" $\mathcal{N}_r(-)$, $r \geq 1$, on the category of restricted Lie algebras are functorial: if $\pi : \mathfrak{k} \to \mathfrak{k}'$ is a morphism of restricted Lie algebras, then $\pi\mathcal{N}_r(\mathfrak{k}) \subseteq \mathcal{N}_r(\mathfrak{k}')$.

[3]Observe that Theorem 2.2 proves a weak version of the Johnston-Richardson theorem [JR]: suppose $P = L_P \cdot U_P$ and $Q = L_Q \cdot U_Q$ are parabolic subgroups which have G-conjugate Levi factors. If P is strongly restricted, then Q is strongly restricted and $G \cdot \mathfrak{u}_P = G \cdot \mathfrak{u}_Q$. In particular, P and Q yield the same Richardson orbit. This is established in [JR] generally, without the condition that P be strongly restricted. Thus, if $p \geq h$, Theorem 2.2 implies the full version of the Johnston-Richardson theorem.

[4]Because the Carter-Bala theory holds as long as p is good [M], we can easily reduce to the case of a distinguished parabolic subgroup and apply [McN, Thm. 1.1], for example.

PROPOSITION . *If $h_J < p$, then P_J is strongly restricted; in particular, P_J is restricted and U_J has exponent p.*

PROOF. Take $\varpi = \sum_{\alpha_i \in \Pi \setminus J} \varpi_i + (p^r - 1)\rho$. Then we have $\Phi_J = \Phi_\varpi$. Thus P_J is strongly restricted. □

2.4. Testerman in [T, Prop. 2.2, 2.3] gives a formula (called the order formula) to compute the order of unipotent elements in a unipotent class. For a *distinguished* parabolic subgroup P_J (i. e., $\dim \mathfrak{l}_J = \dim \mathfrak{u}_J/[\mathfrak{u}_J, \mathfrak{u}_J]$), the Richardson elements in U_J have order p^m if m is the smallest positive integer such that $p^m > h_J$. This formula is restated in [McN, Theorem 6.2] in terms of exponents of U_J. It is apparent that this formula is not true for general parabolic subgroups. For example, for $p = 7$, G of type E_7, and $J_1 = \{\alpha_2, \alpha_3, \alpha_5\}$, $J_2 = \{\alpha_1, \alpha_2, \alpha_7\}$, we have $h_{J_1} = 8$ and $h_{J_2} = 5$. But the Levi subgroups L_{J_1} and L_{J_2} are conjugate by a Weyl group element and thus determine the same Richardson class. Proposition 2.3 implies that P_{J_2} is strongly restricted, and therefore so is P_{J_1}. Thus, the Richardson elements in U_{J_1} have order p but $p < h_{J_1}$.

3. The computation of $\mathcal{N}_1(\mathfrak{g})$ when G has classical type

In the cases A_l, B_l, C_l, D_l, we describe $\mathcal{N}_1(\mathfrak{g})$ by explicitly identifying Φ_0 with some Φ_J, $J \subseteq \Pi$. Then $\mathcal{N}_1(\mathfrak{g})$ is identified as $G \cdot \mathfrak{u}_J$ using Theorem 2.2. A much shorter argument, however, is possible and provides the same description of the restricted nullcone, but does *not* identify Φ_0 explicitly. We discuss this briefly in Section 3.2 and after the result on B_l is proved in Section 3.5.

3.1. We begin by describing the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$ when Φ is of type A_l.

THEOREM . *Let $G = \mathrm{SL}_{l+1}(k)$ (Φ of type A_l).*

(a) *If $p \geq l+1$ then $\dim \mathcal{N}_1(\mathfrak{g}) = l(l+1)$ and $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$.*

(b) *Suppose that $p \leq l$ where $l = pm + s$ with $m > 0$ and $0 \leq s \leq p-1$. Then*

(i) *$\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ where $J \subseteq \Pi$ such that*

$$\Phi_0 \cong \Phi_J \cong \underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \ldots A_{m-1}}_{p-s-1 \text{ times}};$$

(ii) *$\dim \mathcal{N}_1(\mathfrak{g}) = l(l+1) - m(pm + 2s - p + 2)$.*

PROOF. (a) This follows immediately from remarks in (2.2) since $p \geq h = l+1$.

(b) The condition that $\langle \rho, \alpha^\vee \rangle \in p\mathbb{Z}$ is equivalent to the statement that the height of $\alpha^\vee$ is a multiple of p. For type A, we have $\Phi^\vee = \Phi$ and it is easy to see that Φ_0 has a fundamental system consisting of roots of height p. For $1 \leq j \leq p$, consider the roots of height p of the form $\beta_{i,j} = \alpha_{j+ip} + \alpha_{j+ip+1} + \cdots + \alpha_{j+(i+1)p-1}$ where $0 \leq i \leq (m-1)$. When $j + mp - 1 \leq l = pm + s$, or, in other words, $1 \leq j \leq s+1$, then $\{\beta_{0,j}, \beta_{1,j}, \ldots, \beta_{m-1,j}\}$ generates[5] a closed subroot system of type A_m. On the other hand, if $s+2 \leq j \leq p$, then $\{\beta_{0,j}, \beta_{1,j}, \cdots, \beta_{m-2,j}\}$

[5]If X is a subset of a root system Φ, then we say that X *generates* an integrally closed system Φ' of Φ if $\Phi' = \mathbb{Z}X \cap \Phi$.

generates a closed subroot system of type A_{m-1}. For distinct values of j, these subroot systems are mutually orthogonal and have Φ_0 as their union. Hence,

$$\Phi_0 \cong \underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{p-s-1 \text{ times}}.$$

In type A_l, if $K, L \subseteq \Pi$ are such that $\Phi_K \cong \Phi_L$, then there exists $w \in W$ such that $w\Phi_K = \Phi_L$. Hence, Φ_0 is W-conjugate to any Φ_J such that $\Phi_0 \cong \Phi_J$.

We can now calculate the dimension of $\mathcal{N}_1(\mathfrak{g})$:

$$\begin{aligned} \dim \mathcal{N}_1(\mathfrak{g}) &= |\Phi| - |\Phi_0| \\ &= l(l+1) - [(s+1)m(m+1) + (p-s-1)(m-1)m] \\ &= l(l+1) - m(pm + 2s - p + 2). \end{aligned}$$

□

3.2. The above proof makes use of Theorem 2.2. Here this can be easily avoided. For $G = \mathrm{SL}_{l+1}(k)$ the conjugacy classes in $\mathcal{N}(\mathfrak{g})$ are indexed by partitions $\lambda \vdash l+1$. If $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$ $(\lambda_t > 0)$ is such a partition, let x_λ be the $(l+1) \times (l+1)$ nilpotent matrix $\mathrm{diag}(N_1, N_2, \cdots, N_t)$ with lower triangular Jordan blocks N_1, $N_2, \cdots, N_t$ of sizes $\lambda_1 \times \lambda_1, \lambda_2 \times \lambda_2, \cdots, \lambda_t \times \lambda_t$. There is a uniquely determined subset $J(\lambda) \subseteq \Pi$ such that x_λ is a regular nilpotent element in the Lie algebra of $L_{J(\lambda)}$. Put $\mathcal{O}_\lambda = G \cdot x_\lambda$. For $\lambda, \mu \vdash l+1$, $\mathcal{O}_\mu \subseteq \overline{\mathcal{O}_\lambda}$ if and only if $\mu \trianglelefteq \lambda$ in the dominance order on partitions.

Note that $x_\lambda \in \mathcal{N}_1(\mathfrak{g})$ if and only if $\lambda_1 \leq p$. It follows from the definition $\Pi \setminus J(\lambda)$ has cardinality one less than the length of λ.

Also, the conjugacy classes in $\mathcal{N}(\mathfrak{g})$ are indexed by the distinct Richardson classes. In fact, the Johnston-Richardson theorem [JR] (see Footnote 2) implies that if I and J are W-conjugate subsets of Π, then $\mathcal{O}_I = \mathcal{O}_J$. Any $J \subseteq \Pi$ is W-conjugate to a unique $J(\lambda)$, $\lambda \vdash l+1$. If λ' denotes the transposed partition of λ, then a result of Kraft [K] implies that $\mathcal{O}_{\lambda'} = \mathcal{O}_{J(\lambda)} = \mathcal{O}_J$. We conclude that in type A_{l+1}, for $I, J \subseteq \Pi$, the Richardson classes $\mathcal{O}_I$ and $\mathcal{O}_J$ coincide if and only if the sets I and J are W-conjugate, i. e., if and only if the Levi factors L_I and L_J are G-conjugate. (This fact is false, in general, for other types.)

Now assume $p \leq l+1$, and write $l+1 = m'p + s'$, where $m' > 0$ and $0 \leq s' < p$. From the discussion above, it follows that $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_\tau}$, where $\tau = (p^{m'}, s') \vdash l+1$ is the partition with m' parts of size p and 1 part of size s'. We have $\tau' = ((m'+1)^{s'}, (m')^{p-s'})$. If $s' \geq 1$, then $m = m'$ and $s = s'-1$, while if $s' = 0$, then $m = m'-1$ and $s = p-1$. In both cases, $L_{J(\tau')}$ obviously has type

$$\underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{p-s-1 \text{ times}},$$

providing a conceptual explanation for the purely computational argument in the above proof. From the point of view of partitions, the dimension of $\mathcal{N}_1(\mathfrak{g})$ follows also immediately from [SS, IV, 1.8] (which anticipates and predates the result cited above in [K]).

3.3. Given $J \subseteq \Pi$, it is W-conjugate to a unique $J(\lambda)$ for $\lambda \vdash l+1$. Then $\mathcal{O}_J \subseteq \mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}_\tau}$ if and only if $\lambda' \trianglelefteq \tau$, i. e., if and only if the partition λ has length $\leq p$. Clearly, this is equivalent to requiring that $|\Pi \backslash J| = |\Pi \backslash J(\lambda)| < p$. In

addition, setting $\varpi = (p-1)\sum_{\alpha_i \in J} \varpi_i \in X(T)_+$, we have $\Phi_\varpi = \Phi_J$. Thus, we have proved the following proposition.

PROPOSITION . *In type A_l, a standard parabolic subgroup P_J is strongly restricted if and only if $|\Pi \backslash J| < p$. Furthermore, P_J is restricted if and only if it is strongly restricted. In general, a parabolic subgroup P is strongly restricted if and only if its semisimple rank is greater than or equal to* $\operatorname{rank}(G) - p$.

The discussion leads to the solution of a very special quadratic integer programming problem. Namely, consider the problem of minimizing the quadratic form $X_1^2 + \cdots + X_p^2$ subject to the conditions that the X_i be non-negative integers satisfying $X_1 + \cdots + X_p = N$ for some positive integer N. Define $l = N - 1$, and write $l = mp + s$ with $m > 0$ and $0 \le s < p$. Then $X_1 = \cdots = X_{s+1} = m$, $X_{s+2} = \cdots = X_p = m - 1$ minimizes the form. A similar result (left to the interested reader) holds for each of the cases B_l, C_l, D_l below with the quadratic form slightly modified.

3.4. We now consider the case in which G has type B_l.

THEOREM . *Let $G = \mathrm{Spin}_{2l+1}(k)$ (Φ of type B_l) with $p \neq 2$.*

(a) *If $p \ge 2l$ then $\dim \mathcal{N}_1(\mathfrak{g}) = 2l^2$ and $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$.*

(b) *Suppose that $p \le 2l-1$ where $2l-1 = pm+s$ with $m \ge 0$ and $0 \le s \le p-1$. (Thus $m + s \equiv 1 \pmod 2$.)*

(i) *Then $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ where $J \subseteq \Pi$ such that*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ \textit{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s-1}{2}\ \textit{times}} \times B_{\frac{m+1}{2}} & \textit{if } s \textit{ is even } (m \textit{ odd}), \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s+1}{2}\ \textit{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s-2}{2}\ \textit{times}} \times B_{\frac{m}{2}} & \textit{if } s \textit{ is odd } (m \textit{ even}). \end{cases}$$

(ii) *Also,*

$$\dim \mathcal{N}_1(\mathfrak{g}) = \begin{cases} 2l^2 - \frac{m(pm+2s-p+3)+1}{2} & \textit{if } s \textit{ is even } (m \textit{ odd}), \\ 2l^2 - \frac{m(pm+2s-p+3)}{2} & \textit{if } s \textit{ is odd } (m \textit{ even}). \end{cases}$$

PROOF. (a) For Φ of type B_l, we have $h = 2l$ and $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$ for $p \ge 2l$. Furthermore, $\dim \mathcal{N}_1(\mathfrak{g}) = \dim \mathcal{N}(\mathfrak{g}) = |\Phi| = 2l^2$.

Embedding the root system of $\bar{\Phi}$ of type A_{2l-1} into the Euclidean space $\mathbb{R}^{2l}$ with standard basis $\{\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_{2l}\}$. We will use bar to indicate roots for $\bar{\Phi}$. Note that $\bar{\Phi}$ has an order 2 graph automorphism σ such that $\sigma(\bar{\alpha}_i) = \bar{\alpha}_{2l-i}$ ($\sigma(\bar{\varepsilon}_i) = -\bar{\varepsilon}_{2l+1-i}$). The root system $\Phi = \Phi_B$ of type B_l can be embedded into $\mathbb{R}^{2l}$ as follows:

$$\Phi_B = \{\bar{\alpha} + \sigma(\bar{\alpha}) \mid \bar{\alpha} \in \bar{\Phi}, \sigma(\bar{\alpha}) \neq \bar{\alpha}\} \cup \{\bar{\alpha} \mid \bar{\alpha} \in \bar{\Phi}, \sigma(\bar{\alpha}) = \bar{\alpha}\}.$$

Let $\bar{\rho}$ be the half sum of all positive roots of $\bar{\Phi}$. Then $\bar{\rho} = \rho_B$, the half sum of all positive roots of Φ_B. With both systems having the same inner product from $\mathbb{R}^{2l}$, one sees easily that if $\bar{\alpha} \neq \sigma(\bar{\alpha})$, then $\langle \bar{\alpha}, \sigma(\bar{\alpha}^\vee) \rangle = 0$ and $(\bar{\alpha} + \sigma(\bar{\alpha}))^\vee = (\bar{\alpha}^\vee + \sigma(\bar{\alpha})^\vee)/2$. Let $\Gamma : \bar{\Phi} \to \Phi_B$ be the natural map ($\Gamma(\bar{\alpha})$ is the σ-orbit sum of $\bar{\alpha}$). Thus one gets $\langle \bar{\rho}, \bar{\alpha}^\vee \rangle = \langle \rho_B, \Gamma(\bar{\alpha}) \rangle$ for all $\bar{\alpha} \in \bar{\Phi}$. In particular $\Gamma(\bar{\Phi}_0) = \Phi_0$. By Theorem 3.1, expressing $2l - 1 = mp + s$, the root system $\bar{\Phi}_0$ has type

$$\bar{\Phi}_0 \cong \underbrace{A_m \times \cdots \times A_m}_{s+1 \text{ times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{p-s-1 \text{ times}}. \tag{3.4.1}$$

The set $\{\bar\beta_{i,j} = \bar\alpha_{j+ip} + \cdots + \bar\alpha_{j+(i+1)p-1} \mid j+(i+1)p-1 \le mp+s\}$ generates $\bar\Phi_0$. The components of $\bar\Phi_0$ are generated by

$$\bar{\mathcal{B}}_j = \begin{cases} \{\bar\beta_{0,j}, \bar\beta_{1,j}, \dots, \bar\beta_{m-1,j}\} & \text{if } 1 \le j \le s+1, \\ \{\bar\beta_{0,j}, \bar\beta_{1,j}, \dots, \bar\beta_{m-2,j}\} & \text{if } s+2 \le j \le p. \end{cases}$$

Using $\bar\beta_{i,j} = \bar\varepsilon_{ip+j} - \bar\varepsilon_{(i+1)p+j}$, we easily compute

$$\sigma(\bar\beta_{i,j}) = \begin{cases} \bar\beta_{m-(i+1),s+2-j} & \text{if } 1 \le j \le s+1, \\ \bar\beta_{m-(i+2),p+s+2-j} & \text{if } s+2 \le j \le p, \end{cases}$$

$$\sigma(\bar{\mathcal{B}}_j) = \begin{cases} \bar{\mathcal{B}}_{s+2-j} & \text{if } 1 \le j \le s+1, \\ \bar{\mathcal{B}}_{p+s+2-j} & \text{if } s+2 \le j \le p. \end{cases}$$

$\Gamma(\bar{\mathcal{B}}_j)$ generates a root system of type B if and only if there are i and j such that $\sigma(\bar\beta_{i,j}) = \bar\beta_{i,j}$. In particular we have $\sigma(\bar{\mathcal{B}}_j) = \bar{\mathcal{B}}_j$. Note that $m+s \equiv 1 \pmod 2$. For s even, $i = \frac{m-1}{2}$ and $j = \frac{s}{2}+1$ and $\Gamma(\bar{\mathcal{B}}_j)$ is of type $B_{\frac{m+1}{2}}$. For s odd, $i = \frac{m-2}{2}$ and $j = \frac{p+s+2}{2}$ and $\Gamma(\bar{\mathcal{B}}_j)$ is of type $B_{\frac{m}{2}}$. For all other j, $\Gamma(\bar{\mathcal{B}}_j) = \Gamma(\sigma(\bar{\mathcal{B}}_j))$ generates a root system of type A_m or A_{m-1}. Thus the statement of (b)(i) holds. (ii) follows directly since $G \cdot \mathfrak{u}_J$ has codimension in $\mathfrak{g}$ equal to $\dim(L_J)$. □

3.5. We will describe another argument for the previous theorem. This method can also be used in proving the results for C_l and D_l below, using the suggested J. It still does, however, use Theorem 2.2.

First, it is useful to have an explicit formula for the number μ_i of roots in $\Phi^\vee$ of height i. In all cases, this result can be worked out directly, but can also be obtained easily from the following general result of Kostant (see [Hum2, Thm. p. 84]). Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be the exponents of the root system, arranged in decreasing order. Thus, $\lambda_1 = h-1, \lambda_l = 1$. Then $\mu = (\mu_1, \cdots, \mu_{h-1})$ is the partition transposed to λ by Kostant's theorem.

When Φ has type B_l, $\Phi^\vee$ has type C_l, and $\alpha \in \Phi$ belongs to Φ_0 if and only if the height of the coroot $\alpha^\vee$ is divisible by p. For type C_l, the partition $\lambda = (2l-1, 2l-3, \cdots, 3, 1)$ of exponents has dual $\mu = \lambda' = (\mu_1, \cdots, \mu_{2l-1})$, where

$$\mu_i = \begin{cases} \frac{2l-i}{2}, & i \text{ even}, \\ \frac{2l-i+1}{2}, & i \text{ odd}. \end{cases}$$

Thus, $|\Phi_0^+| = \mu_p + \mu_{2p} + \cdots + \mu_{mp}$ is given by

$$|\Phi_0^+| = \begin{cases} \frac{m(pm+2s-p+3)+1}{4} & \text{if } s \text{ is even } (m \text{ odd}), \\ \frac{m(pm+2s-p+3)}{4} & \text{if } s \text{ is odd } (m \text{ even}), \end{cases}$$

so that $\dim \mathcal{N}_1(\mathfrak{g}) = |\Phi| - |\Phi_0|$, as required.

On the other hand, let P_J be the standard parabolic subgroup of G with J as described in part (i) of Theorem 3.4. A quick calculation gives that $|\Phi_J| = |\Phi_0^+|$. Hence, $\dim G \cdot \mathfrak{u}_J = \dim \mathcal{N}_1(\mathfrak{g})$.

Finally, $|\Pi \backslash J| = \frac{p-1}{2}$. For $\alpha^\vee = \sum n_i \alpha_i^\vee \in \Phi^{\vee +}$, we have $|n_i| \le 2$. So if $\varpi = \sum_{i \in \Pi \backslash J} \varpi_i$, then $\Phi_{\varpi - \rho} = \Phi_J$. Choosing r large enough to insure that $\varpi' = \varpi + (rp-1)\rho \in X(T)_+$, it follows from Theorem 2.2 that $|\mathfrak{g}|_{H^0(\varpi')} = G \cdot \mathfrak{u}_J \subseteq \mathcal{N}_1(\mathfrak{g})$. Hence, by dimension considerations, $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$, as required.

3.6. The following theorem describes the restricted nullcone when Φ is of type D_l.

THEOREM . *Let $G = \mathrm{Spin}_{2l}(k)$ (Φ of type D_l).*

(a) *If $p \geq 2l-2$ then $\dim \mathcal{N}_1(\mathfrak{g}) = 2l^2 - 2l$ and $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$.*

(b) *Suppose that $p \leq 2l-3$ where $2l-1 = pm+s$ with $m \geq 1$ and $0 \leq s \leq p-1$. (Thus $m+s \equiv 1 \pmod 2$.)*

(i) *Then $\mathcal{N}(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ where $J \subseteq \Pi$ such that*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ times} \times \underbrace{A_{m-1} \times \ldots \times A_{m-1}}_{\frac{p-s-1}{2}\ times} \times D_{\frac{m+1}{2}} & \text{if } s \text{ is even and } m \geq 3, \\ \underbrace{A_m \times \ldots \times A_m}_{\frac{s-1}{2}\ times} \times \underbrace{A_{m-1} \times \ldots \times A_{m-1}}_{\frac{p-s}{2}\ times} \times D_{\frac{m+2}{2}} & \text{if } s \text{ is odd}, \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ times} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s-1}{2}\ times} & \text{if } s \text{ is even, } m = 1. \end{cases}$$

(ii) *Also,*

$$\dim \mathcal{N}_1(\mathfrak{g}) = \begin{cases} 2l^2 - 2l - \frac{m(pm+2s-p+1)-1}{2} & \text{if } s \text{ is even } (m \text{ odd}), \\ 2l^2 - 2l - \frac{m(pm+2s-p+1)}{2} & \text{if } s \text{ is odd } (m \text{ even}). \end{cases}$$

PROOF. (a) Let Φ be of type D_l. The Coxeter number h in this case is $h = 2l-2$ and for $p \geq 2l-2$, $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$. Thus, $\dim \mathcal{N}_1(\mathfrak{g}) = \dim \mathcal{N}(\mathfrak{g}) = |\Phi| = 2l^2 - 2l$.

(b) Set $2l-1 = pm+s$ where $m \geq 0$ and $0 \leq s \leq p-1$. Since the root system Φ is simply laced, in order to determine Φ_0 it suffices to just consider roots whose height is a multiple of p. Moreover, it is not difficult to verify that a basis of positive roots for Φ_0 will be the roots whose height is exactly p. These roots are given by

$$\begin{aligned} \eta_1 &= \alpha_{l-p} + \alpha_{l-p+1} + \cdots + \alpha_{l-2} + \alpha_{l-1}, \\ \eta_2 &= \alpha_{l-p} + \alpha_{l-p+1} + \cdots + \alpha_{l-2} + \alpha_l, \end{aligned}$$

$$\begin{aligned} \epsilon_1 &= \alpha_1 + \alpha_2 + \cdots + \alpha_{p-1} + \alpha_p, \\ \epsilon_2 &= \alpha_2 + \alpha_2 + \cdots + \alpha_{p-1} + \alpha_{p+1}, \\ &\vdots \\ \epsilon_{l-p-1} &= \alpha_{l-p-1} + \alpha_{l-p} + \cdots + \alpha_{l-3} + \alpha_{l-2}, \end{aligned}$$

$$\begin{aligned} \delta_1 &= \alpha_{l-p+1} + \alpha_{l-p+2} + \cdots + \alpha_{l-3} + \alpha_{l-2} + \alpha_{l-1} + \alpha_l, \\ \delta_2 &= \alpha_{l-p+2} + \alpha_{l-p+3} + \cdots + \alpha_{l-3} + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l, \\ &\vdots \\ \delta_{\frac{p-1}{2}} &= \alpha_{l-p+\frac{p-1}{2}} + 2\alpha_{l-p+\frac{p+1}{2}} + \cdots + 2\alpha_{l-3} + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l. \end{aligned}$$

Set $\mathcal{B}_0 = \{\eta_1, \eta_2, \epsilon_{l-2p}, \ldots, \epsilon_{l-tp}\}$. Here t is the maximal possible value such that $\mathcal{B}_0$ is in Φ_0. Observe that when $m = 1$, the set $\mathcal{B}_0$ is empty. Then $\mathcal{B}_0$ generates

a subroot system in Φ_0 of type D. We need to determine t. First we must have $l - tp \geq 1$, so $l - 1 = tp + s'$ where $t \geq 0$ and $0 \leq s' \leq p-1$. From this equation we have $2l - 1 = 2tp + 2s' + 1$. Combining the previous equation with $2l - 1 = pm + s$ yields

$$t = \frac{pm + s - 2s' - 1}{2p}. \tag{3.6.1}$$

Therefore,

$$t = \begin{cases} \frac{m}{2} + \frac{s-2s'-1}{2p} & \text{if } m \text{ is even,} \\ \frac{m-1}{2} + \frac{p+s-2s'-1}{2p} & \text{if } m \text{ is odd.} \end{cases}$$

Since $0 \leq s' \leq p-1$ and $0 \leq s \leq p-1$ and the last terms in these expressions must be integers, it follows that

$$t = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even,} \\ \frac{m-1}{2} & \text{if } m > 1 \text{ is odd.} \end{cases}$$

Hence, $\mathcal{B}_0$ generates a root system of type $D_{\frac{m+2}{2}}$ for m even and of type $D_{\frac{m+1}{2}}$ for $m > 1$ odd. When $m = 1$, the roots η and ϵ do not appear.

Observe that $\delta_1, \delta_2, \ldots, \delta_{\frac{p-1}{2}}$ are all mutually orthogonal roots. These roots will actually be in different irreducible components of Φ_0. Let $1 \leq j \leq \frac{p-1}{2}$ and let

$$\mathcal{B}_j = \{\epsilon_{l-tp-j}, \ldots, \epsilon_{l-2p-j}, \epsilon_{l-p-j}, \delta_j, \epsilon_{l-2p+j}, \epsilon_{l-3p+j}, \ldots, \epsilon_{l-t'p+j}\}.$$

The set $\mathcal{B}_j$ generates a root system of type A. When $m = 1$, $\mathcal{B}_j = \{\delta_j\}$ for $\frac{p-1}{2} - \frac{s}{2} < j \leq \frac{p-1}{2}$ and empty otherwise. Thus we can assume that $m > 1$. We need to determine the size of $\mathcal{B}_j$ by finding the maximal possible values of t and t'. First note that $l - tp - j \geq 1$ or $l - j = tp + s'$ where $1 \leq s' \leq p$. Combining this with the fact that $2l - 1 = pm + s$, yields

$$t = \begin{cases} \frac{m}{2} + \frac{s-2s'-2j+1}{2p} & \text{if } m \text{ is even,} \\ \frac{m-1}{2} + \frac{p+s-2s'-2j+1}{2p} & \text{if } m \text{ is odd.} \end{cases}$$

There are two cases to handle. First, assume that m is even. Then $\frac{s-2s'-2j+1}{2p}$ must be an integer. In fact the possible values because of the restrictions on s, s' and j are either 0 or -1. In the case that this quantity is zero we must have $s' + j = \frac{s+1}{2}$, otherwise if the quantity is -1, we have $s' + j = p + \frac{s+1}{2}$. Hence, for m even

$$t = \begin{cases} \frac{m}{2} & \text{if } 1 \leq j \leq \frac{s-1}{2}, \\ \frac{m-2}{2} & \text{if } \frac{s+1}{2} \leq j \leq \frac{p-1}{2}. \end{cases}$$

Observe that t' will either be t or $t+1$. By analyzing both of the cases above, we can verify that $t' = \frac{m}{2}$ in either case. Thus, $\mathcal{B}_j$ generates a root system of type A_m for $1 \leq j \leq \frac{s-1}{2}$ and a root system of type A_{m-1} for $\frac{s+1}{2} \leq j \leq \frac{p-1}{2}$.

Now assume that m is odd, then the equation above shows that $t = \frac{m-1}{2}$ for all j. We can also show that

$$t' = \begin{cases} \frac{m-1}{2} & \text{if } 1 \leq j \leq \frac{p-s-1}{2}, \\ \frac{m+1}{2} & \text{if } \frac{p-s+1}{2} \leq j \leq \frac{p-1}{2}. \end{cases}$$

Hence, for m odd, $\mathcal{B}_j$ generates a root system of type A_{m-1} for $1 \leq j \leq \frac{p-s-1}{2}$ and a root system of type A_m for $\frac{p-s+1}{2} \leq j \leq \frac{p-1}{2}$.

Part (ii) can be verified by using equation (3.4.2). □

3.7. We next consider the groups of type C_l.

THEOREM . *Let $G = \mathrm{Sp}_{2l}(k)$ (Φ of type C_l) with $p \neq 2$.*

(a) *If $p \geq 2l$ then $\dim \mathcal{N}_1(\mathfrak{g}) = 2l^2$ and $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$.*

(b) *Suppose that $p \leq 2l-1$ where $2l+1 = pm+s$ with $m \geq 1$ and $0 \leq s \leq p-1$ (thus $m+s \equiv 1 \pmod 2$).*

(i) *Then $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ where $J \subseteq \Pi$ such that*

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ \textit{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s-1}{2}\ \textit{times}} \times C_{\frac{m-1}{2}} & \textit{if } s \textit{ is even,} \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s-1}{2}\ \textit{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s}{2}\ \textit{times}} \times C_{\frac{m}{2}} & \textit{if } s \textit{ is odd.} \end{cases}$$

(ii) *Also,*

$$\dim \mathcal{N}_1(\mathfrak{g}) = \begin{cases} 2l^2 - \frac{m(pm+2s-p-1)+1}{2} & \textit{if } s \textit{ is even } (m \textit{ odd}), \\ 2l^2 - \frac{m(pm+2s-p-1)}{2} & \textit{if } s \textit{ is odd } (m \textit{ even}). \end{cases}$$

PROOF. (a) If Φ of type C_l then $h = 2l$ and $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$ for $p \geq 2l$. This implies that $\dim \mathcal{N}_1(\mathfrak{g}) = \dim \mathcal{N}(\mathfrak{g}) = |\Phi| = 2l^2$.

(b) We will make use of the result of D_{l+1} in order to handle the case for C_l. Let $\bar{\Phi}$ be the root system of type D_{l+1} with $\bar{\Pi} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{l+1}\}$ being the set of simple roots numbered as in the Appendix. Let $\sigma : \bar{\Phi} \to \bar{\Phi}$ be the order 2 graph automorphism interchanging $\bar{\alpha}_l$ and $\bar{\alpha}_{l+1}$. The roots of Φ (of type C_l) can be realized as the σ-orbit sums. Let $\Gamma(\bar{\alpha}) \in \Phi$ be the sum of all roots in the σ-orbit of $\bar{\alpha}$ for $\bar{\alpha} \in \bar{\Phi}$. Once again, let $\bar{\rho}$ be the half sum of positive roots in $\bar{\Phi}$. Similar to the argument as in 3.4, we have $\langle \bar{\rho}, \bar{\alpha}^\vee \rangle = \langle \rho, \Gamma(\alpha)^\vee \rangle$. Thus we have $\Gamma(\bar{\Phi}_0) = \Phi_0$.

By Theorem 3.6, expressing $2(l+1) - 1 = 2l + 1 = pm + s$ where $m \geq 0$ and $0 \leq s \leq p-1$, we have

$$\bar{\Phi}_0 \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ \text{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s-1}{2}\ \text{times}} \times D_{\frac{m+1}{2}} & \text{if } s \text{ is even } (m > 1 \text{ odd}), \\ \underbrace{A_{m+1} \times \cdots \times A_{m+1}}_{\frac{s-1}{2}\ \text{times}} \times \underbrace{A_m \times \cdots \times A_m}_{\frac{p-s}{2}\ \text{times}} \times D_{\frac{m+2}{2}} & \text{if } s \text{ is odd } (m \text{ even}), \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ \text{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s-1}{2}\ \text{times}} & \text{if } s \text{ is even and } m = 1. \end{cases}$$

An explicit description of $\bar{\Phi}_0$ was given in Theorem 3.6. Under the map Γ, the images of the components of type A in $\bar{\Phi}_0$ remain the same and the component of type D_{r+1} goes to a component of type C_r if $r \geq 1$. If we use the convention of C_0 as empty root system, then we have,

$$\Phi_0 \cong \Phi_J \cong \begin{cases} \underbrace{A_m \times \cdots \times A_m}_{\frac{s}{2}\ \text{times}} \times \underbrace{A_{m-1} \times \ldots A_{m-1}}_{\frac{p-s-1}{2}\ \text{times}} \times C_{\frac{m-1}{2}} & \text{if } s \text{ is even } (m \text{ odd}), \\ \underbrace{A_m \times \cdots \times A_m}_{\frac{s-1}{2}\ \text{times}} \times \underbrace{A_{m-1} \times \cdots \times A_{m-1}}_{\frac{p-s}{2}\ \text{times}} \times C_{\frac{m}{2}} & \text{if } s \text{ is odd } (m \text{ even}). \end{cases}$$

□

3.8. In this subsection, we determine the strongly restricted parabolic subgroups P_J in types B_l, C_l, D_l. In part (b) below, we say that $J \subseteq \Pi$ has a component of type D_r, $r \geq 2$, provided that (in the notation of the Appendix) $\alpha_i \in J$ for $i = l, l-1$. In case $l = 4$, we only require that J contain only one of the extremal roots $\alpha_1, \alpha_3, \alpha_4$.

THEOREM . *Let $J \subseteq \Pi$.*

(a) *Assume that G has type B_l or type C_l. Then P_J is strongly restricted if and only if $|\Pi \backslash J| \leq \frac{p-1}{2}$.*

(b) *Assume that G has type D_l. Then P_J is strongly restricted if and only if either*

  (i) *$|\Pi \backslash J| \leq \frac{p+1}{2}$ and there is $w \in W$ such that $wJ \subseteq \Pi$ and $\Pi \setminus wJ$ has a component of type D_r, $r \geq 2$, or*

  (ii) *$|\Pi \backslash J| \leq \frac{p-1}{2}$ otherwise.*

PROOF. (a) If the root system Φ has type B_l (resp., C_l), the dual root system $\Phi^\vee$ has type C_l (resp., B_l). All positive roots have coefficients at most 2 when written as linear combinations of simple roots. Let $K = \Pi \backslash J$ and $\varpi_K = \sum_{i \in K} \varpi_i + (rp-1)\rho \in X(T)_+$ for r large. Note that for $\alpha \in \Phi^+$, $\langle \varpi_K + \rho, \alpha^\vee \rangle \equiv \mathrm{ht}_J(\alpha^\vee) \pmod p$. Here $\mathrm{ht}_J(\alpha)$ is sum of the coefficients of simple roots not in J in the expression of α. If $|K| \leq \frac{p-1}{2}$, then $\langle \varpi_K + \rho, \alpha^\vee \rangle \equiv 0 \pmod p$ if and only if $\alpha \in \Phi_J$. Hence $\Phi_\varpi = \Phi_J$. So P_J is strongly restricted.

Conversely, assume $m = |K| > \frac{p-1}{2}$. In the labeling of roots in the Appendix, list $K = \{\beta_1, \cdots, \beta_m\}$ where $\beta_i = \alpha_{j_i}$ with $j_1 < j_2 < \cdots < j_m$. We will define a sequence of non-zero elements $\gamma_1^\vee, \cdots, \gamma_{2m-1}^\vee$ (not necessarily coroots) in the coroot lattice $\mathbb{Z}\Phi^\vee$ satisfying the following two conditions.

(i) For any i, $1 \leq i \leq 2m-1$, there exists a root $\delta_i \in \Phi$ such that if $\delta_i^\vee = \sum m_s \alpha_s^\vee$, then

$$\sum_{s \in K} m_s \alpha_s^\vee = \gamma_i^\vee.$$

(ii) For $i < j$, there exists $\alpha = \alpha_{i,j} \in \Phi$ such that, if $\alpha^\vee = \sum_{s=1}^l n_s \alpha_s^\vee$, then

$$\sum_{s \in K} n_s \alpha_s^\vee = \gamma_j^\vee - \gamma_i^\vee \neq 0.$$

Now suppose that P_J is strongly restricted, so there exists $\varpi \in X(T)$ such that

$$\{\alpha \in \Phi \mid \langle \varpi, \alpha^\vee \rangle \equiv 0 \pmod p\} = \Phi_J.$$

In particular, $\varpi \equiv \sum_{i \in K} d_i \varpi_i \pmod{p\mathbb{Z}\Phi_J}$ ($d_i \in \mathbb{Z}$). Because $\delta_i \notin \Phi_J$, we have $\langle \varpi, \gamma_i^\vee \rangle \equiv \langle \varpi, \delta_i^\vee \rangle \not\equiv 0 \pmod p$, $(1 \leq i \leq 2m-1)$. Since $2m-1 \geq p$, there exist $i < j$ such that $\langle \varpi, \gamma_i^\vee \rangle \equiv \langle \varpi, \gamma_j^\vee \rangle \pmod p$. Thus, $\alpha_{i,j} \in \Phi_J$, contradicting (ii). Thus P_J cannot be strongly restricted.

We now construct the sequence $\{\gamma_1, \ldots, \gamma_{2m-1}\}$.

Type B_l: If $j_m = l$ set

$$\gamma_i^\vee = \begin{cases} \beta_1^\vee + \cdots + \beta_i^\vee & 0 < i \leq m, \\ \beta_1^\vee + \cdots + \beta_{2m-i-1}^\vee + 2\beta_{2m-i}^\vee + \cdots + 2\beta_{m-1}^\vee + \beta_m^\vee & m < i \leq 2m-1. \end{cases}$$

If $j_m < l$, we change the definition of $\gamma_i^\vee$ for $i > m$ to be $\gamma_i^\vee = \beta_1^\vee + \cdots + \beta_{2m-i-1}^\vee + 2\beta_{2m-i}^\vee + \cdots + 2\beta_m^\vee$. Now the existences of α_{ij} and δ_i follows by inspection of the root system $\Phi^\vee$ which is of type C_l

Type C_l: Set

$$\gamma_i^\vee = \begin{cases} \beta_1^\vee + \cdots + \beta_i^\vee & 1 \le i \le m, \\ \beta_1^\vee + \cdots + \beta_{2m-i}^\vee + 2\beta_{2m-i+1}^\vee + \cdots + 2\beta_m^\vee & m+1 \le i \le 2m-1. \end{cases}$$

The existences of α_{ij} and δ_i follows by inspecting the root system $\Phi^\vee$ which is of type B_l.

(b) Let Φ be of type D_l. Suppose first that condition (i) holds. Replacing J with $wJ \subseteq \Pi$, we can assume that $K = \Pi \backslash J$ has a component of type D_r, $r \ge 2$. List $K = \{\beta_1, \cdots, \beta_m\}$, where $\beta_s = \alpha_{i_s}$ with $i_1 < \cdots < i_{m-1} = l-1 < i_m = l$. (We leave the special case of D_4 to the reader.) If $m \le \frac{p+1}{2}$, let $\varpi = \sum_{s=1}^m \varpi_{i_s}$. Then $\Phi_J = \{\alpha \in \Phi \,|\, \langle \varpi, \alpha^\vee \rangle \equiv 0 \pmod p\}$; in fact, for any $\alpha \in \Phi^+ \backslash \Phi_J^+$, $0 < \langle \varpi, \alpha^\vee \rangle \le 2m-2 \le p-1$. Thus, P_J is strongly restricted in this case. Note that $0 \le \langle \varpi_i, \alpha^\vee \rangle \le 2$ for all $\alpha \in \Phi^+$ for all classical root systems. If (ii) holds, i. e., $m \le \frac{p-1}{2}$, then, for any $\alpha \in \Phi^+ \backslash \Phi_J^+$, $0 < \langle \varpi, \alpha^\vee \rangle \le 2m \le p-1$. Thus $\Phi_J = \{\alpha \in \Phi \,|\, \langle \varpi, \alpha^\vee \rangle \equiv 0 \pmod p\}$.

Conversely, suppose that P_J is strongly restricted, so there exists $\varpi = \sum a_i \varpi_i \in X(T)$ such that $\Phi_J = \{\alpha \in \Phi \mid \langle \varpi, \alpha^\vee \rangle \equiv 0 \pmod p\}$. We can assume that $a_i = 0$ for all $i \in J$ and $1 \le a_i \le p-1$ for $i \in K$. We use the embedding $C_{l-1} \subseteq \mathbb{Z}D_l$ described in the proof of 3.7(b). If α' are roots of C_{l-1} and ϖ' are weights of C_{l-1}, then $\alpha_i' = \alpha_i$ for all $i \le l-2$ and $\alpha_{l-1}' = \alpha_{l-1} + \alpha_l$. Similarly, $\varpi_i' = \varpi_i$ for $i \le l-2$ and $\varpi_{l-1}' = \varpi_{l-1} + \varpi_l$. Let σ and Γ be defined in 3.7. If J does not contain α_{l-1} and α_l (i. e., K contains a component of type D_r for $r \ge 2$), we can replace ϖ by $\varpi + \sigma(\varpi)$ since p is odd ($a_{l-1} + a_l \not\equiv 0 \pmod p$). Thus we can assume that $a_l = a_{l-1}$ and $\sigma(\varpi) = \varpi$. Set $\varpi' = \Gamma(\varpi)$ as a weight for type C_{l-1}. Then $P'_{\Gamma(J)}$ is strongly restricted in type C_{l-1} since

$$\Phi'_{\Gamma(J)} = \Gamma(\Phi_J) = \{\alpha' \in \Phi' \mid \langle \varpi', \alpha' \rangle \equiv 0 \pmod p\}.$$

We have $|K| = l - |J| = l - |\Gamma(J)| \le \frac{p-1}{2} + 1 = \frac{p+1}{2}$.

If J contains both α_{l-1} and α_l, then

$$\Phi'_{\Gamma(J)} = \Gamma(\Phi_J) = \{\alpha' \in \Phi' \mid \langle \Gamma(\varpi), {\alpha'}^\vee \rangle \equiv 0 \pmod p\}$$

since $\sigma(\Phi_J) = \Phi_J$ and $\Gamma(\varpi) = \varpi$. Thus $P'_{\Gamma(J)}$ is strongly restricted in type C_{l-1} and $|K| = l - |J| = l - |\Gamma(J)| - 1 \le \frac{p-1}{2}$ by the result we proved for type C_{l-1}.

Assume that J contains exactly one of α_{l-1} and α_l. Consider $\varpi' = \varpi + \sigma(\varpi)$ and $J' = J \cap \{\alpha_1, \ldots, \alpha_{l-2}\}$. It is straightforward to check that $\Phi_{\varpi'} = \Phi_{J'}$ by using the fact that $\langle \varpi', \alpha^\vee \rangle = \langle \varpi, \alpha^\vee + \sigma(\alpha)^\vee \rangle$ and p is odd. Applying the above proved case to J', we have $l - |J'| \le \frac{p+1}{2}$, we have $|K| = l - |J| = l - |J'| - 1 \le \frac{p-1}{2}$. □

3.9. In this subsection, we will show that, when G has classical type, all restricted parabolic subgroups are strongly restricted. Thus, let G have root system Φ of type B_l, C_l or D_l. In type D_l, we will replace the simply connected group $G = \mathrm{Spin}_{2l}(k)$ by $\mathrm{SO}_{2l}(k)$. Thus, there exists an embedding via the standard representation of G into $\mathrm{GL}_N(k)$, where $N = 2l+1$ (resp. $N = 2l$) for type B_l (resp. C_l and D_l). In this way, we can regard $\mathfrak{g}$ as a restricted subalgebra of $\mathfrak{gl}_N(k)$. With this identification, the restriction map $x \mapsto x^{[p]}$ identifies with the ordinary pth power map $x \mapsto x^p$. Let $\mathcal{P}(N)$ be the set of partitions $\lambda = (\lambda_1, \cdots, \lambda_N)$ of N. (Notation: $\lambda \in \mathcal{P}(N) \iff \lambda \vdash N$.) As discussed in (3.2), the set $\mathcal{P}(N)$ indexes the nilpotent orbits in $\mathfrak{gl}_N$. (Note that nilpotent orbits in $\mathfrak{gl}_N$ are the same as those in

$\mathfrak{sl}_N$ by either GL_N or SL_N.) For $X = B, C, D$, let $\mathcal{P}_X(N) \subseteq \mathcal{P}(N)$ be the subset indexing the nilpotent classes in type B_l, C_l, D_l, respectively. For $\lambda \in \mathcal{P}_X(N)$, let $\mathcal{O}_\lambda$ be the corresponding nilpotent orbit. (In case, $\mathfrak{g}$ needs to be mentioned, we write $\mathcal{O}(\mathfrak{g})_\lambda$.) These sets are described in [CM, Thm. 5.1.2–5.1.4]; see also [M, Chapter 3].[6] In type D_l, $\lambda \in \mathcal{P}_D(N)$ indexes two nilpotent orbits, denoted $\mathcal{O}^I_\lambda$ and $\mathcal{O}^{II}_\lambda$ in case λ is very even (i. e., all its parts λ_i are even integers). However, $\mathcal{O}^I_\lambda$ is restricted if and only if $\mathcal{O}^{II}_\lambda$. We also remark that given $\lambda, \mu \in \mathcal{P}_X(N)$, $\mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\mu$ if and only if $\lambda \trianglelefteq \mu$.

In the proof below, we will make use of the Lusztig-Spaltenstein theory of induced nilpotent orbits. For a recent treatment, see [M, Chp. 3] (also, [CM]). Thus, if $\mathfrak{l}$ is a Levi factor (of some parabolic subalgebra $\mathfrak{p}$), induction associates to any nilpotent orbit $\mathcal{O}$ in $\mathfrak{l}$ a nilpotent orbit $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}\mathcal{O}$ in $\mathfrak{g}$. Induction is transitive in the sense that if $\mathfrak{l}_1 \subseteq \mathfrak{l}_2$ are Levi subalgebras, then $\mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}} \circ \mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{l}_2}\mathcal{O}$. If $\mathcal{O}$ is a Richardson orbit associated to a parabolic subgroup P with Levi subalgebra $\mathfrak{l}$, then $\mathcal{O} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}0$, where 0 denotes the trivial nilpotent orbit in $\mathfrak{l}$. Finally, for $\lambda \in \mathcal{P}_X(N)$, let $\lambda_X \in \mathcal{P}_X(N)$ denote the collapse of λ [CM, Lem. 6.3.3].

THEOREM . *Let G the simple of type B_l, C_l or D_l with p a good prime. Then every restricted parabolic subgroup is strongly restricted.*

PROOF. For $J \subseteq \Pi$, let $\mathfrak{l}_J \subset \mathfrak{p}_J$ be the Levi factor of the parabolic subalgebra $\mathfrak{p}_J = \mathrm{Lie}(P_J)$. We can assume that $J \neq \Pi$. There exist positive integers $m_1, \ldots, m_t$ such that $\Pi \setminus J = \{\alpha_{m_1}, \alpha_{m_1+m_2}, \ldots, \alpha_{m_1+\cdots+m_t}\}$. Then $|\Pi \setminus J| = t$. Set $s = m_1 + \cdots + m_t$ and $r = l - s$. Then $[\mathfrak{p}_J, \mathfrak{p}_J]$ has the root system

$$\Phi_J = A_{m_1-1} \times A_{m_2-1} \times \cdots \times A_{m_t-1} \times X_r.$$

(If $m_i = 1$, interpret $A_{m_i-1} = \emptyset$.) When X is of type D and $s = l - 1$, then we can replace J by its image under the graph automorphism of the root system and assume that $s = l$. Thus we can assume that $r \neq 1$ in the case of type D.

Let $P = P_K$ be the maximal parabolic subgroup with $K = \Pi \setminus \{\alpha_s\}$. The (standard) Levi subalgebra $\mathfrak{l}$ of $\mathrm{Lie}(P) = \mathfrak{p}$ can be decomposed as $\mathfrak{l} = \mathfrak{gl}_s \oplus \mathfrak{g}' \subseteq \mathfrak{l}$, with $\mathfrak{g}'$ having the Cartan type X_r.

Thus, we can decompose the (standard) Levi subalgebra $\mathfrak{l}_J$ as $\mathfrak{l}_J = \mathfrak{l}'_J \oplus \mathfrak{g}'$, where $\mathfrak{l}'_J$ is a Levi subalgebra of $\mathfrak{gl}_s$. Note that in type D_l, $r \neq 1$: when $X_r = \emptyset$, $\mathfrak{l} = \mathfrak{gl}_l$.

Let $\mathcal{O}_\mu = \mathrm{Ind}_{\mathfrak{l}_J}^{\mathfrak{g}} 0$ be the Richardson orbit obtained by inducing the trivial nilpotent orbit 0 in $\mathfrak{l}_J$ to $\mathfrak{g}$. Then $\mathcal{O}_\mu = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{l}}$ with $\mathcal{O}_{\mathfrak{l}} = \mathrm{Ind}_{\mathfrak{l}_J}^{\mathfrak{l}} 0$. The decompositions of $\mathfrak{l}$ and $\mathfrak{l}'_J$ give $\mathcal{O}_{\mathfrak{l}} = \mathcal{O}_\sigma \oplus \mathcal{O}_\delta$ where $\mathcal{O}_\sigma = \mathrm{Ind}_{\mathfrak{l}'_J}^{\mathfrak{gl}_s} 0$ is a Richardson orbit in $\mathfrak{gl}_s$ corresponding to a partition $\sigma \vdash s$. By (3.2), the transpose partition σ' equals to the partition obtained from the composition $(m_1, \cdots, m_t)$ of s [Sp, II.7.1]. Thus, $\sigma_1 = t$. Also, $\mathcal{O}_\delta$ is the zero orbit in $\mathfrak{g}'$, which corresponds to the partition

$$\delta = \begin{cases} (1^{N(r)}) & \text{if } X_r = B_r,\ C_r \text{ for } r \geq 1 \text{ and } D_r \text{ for } r \geq 2, \\ (1) & \text{if } X_r = B_r \text{ and } r = 0, \\ (0) & \text{if } X_r = C_r,\ D_r \text{ and } r = 0. \end{cases}$$

[6] We often refer to [CM] in the text below. Although the results there are stated entirely for characteristic 0, they remain valid in positive characteristic p as long as p is good. See [M], [Sp, I.2.5], etc. In some cases, results are proved for unipotent classes in arbitrary characteristic, but good characteristic is needed to pass to nilpotent classes via the Springer isomorphism.

(The case $X_r = D_1$ does not appear here.)

The procedure outlined in [CM, p. 115-116] provides a procedure to determine μ from Φ_J. Also see [Sp, II.7.4]. Set $\lambda = 2\sigma + \delta$. Then $\mathcal{O}_\mu = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{l}}$, where $\mu = \lambda_X$ [CM, Thm. 7.3.3]. By [CM, Lemma 6.3.3], we have $\lambda_1 = 2\sigma_1 + 1 \leq \mu_1 + 1$ in all types of $r \geq 1$ and in type B_l if $r = 0$. The parabolic subgroup P_J is therefore restricted if and only if $\mu_1 \leq p$. Thus, except possibly when $r = 0$ in types C_l and D_l, if P_J is restricted, then $|\Pi \backslash J| = t \leq \frac{p-1}{2}$ since p is odd. Thus, P_J is strongly restricted by Theorem 3.8.

It remains to consider the case $r = 0$ in types C and D. Suppose first that G has type D_l. Since $r = 0$, $s = l$. By the Johnston-Richardson theorem, we can replace J by any W-conjugate wJ which is contained in Π. There are then two cases to consider:

<u>Case 1:</u> $\Pi \backslash J$ has a component of type D_a, $a \geq 2$. (See remarks immediately before statement of Theorem 3.8.) In this case, $m_t = 1$ in the composition $(m_1, \cdots, m_t)$ and $\sigma_1 > \sigma_2$. Thus, $\lambda = 2\sigma$ has a unique maximal even part $\lambda_1 = 2\sigma_1$. Since $\mu = \lambda_D$, we have $\mu_1 = 2\sigma_1 - 1$ by the collapse procedure [CM, Lem. 6.3.3]. Therefore, $\mathcal{O}_\mu$ is restricted if and only if $\mu_1 = 2\sigma_1 - 1 \leq p$, i. e., $\sigma_1 \leq (p+1)/2$. Since $\sigma_1 = |\Pi \backslash J|$, we finally obtain that P_J is restricted if and only if $|\Pi \backslash J| \leq \frac{p+1}{2}$. By Theorem 3.8(b), such P_J are strongly restricted.

<u>Case 2:</u> For any $w \in W$ with $wJ \subseteq \Pi$, wJ contains at least one of α_{l-1}, α_l. If wJ contains both α_{l-1} and α_l, we are in the case of $r \geq 2$ when J is replaced by wJ to get that P_J is restricted if and only if $|\Pi \setminus J| \leq (p-1)/2$. So we assume that wJ contains only one of α_{l-1} and α_l for all $w \in W$ with $wJ \subseteq \Pi$. Clearly, $m_t \geq 2$. If $m_i = 1$, for some $i < t$, then $\{\alpha_1, \ldots, \alpha_{l-1}\} \setminus J$ has a component of type A_a with $a \geq 2$. Now in a root system of type A_{l-1}, there is an element $w \in W_{\{\alpha_1, \ldots, \alpha_{l-1}\}}$ such that $wJ \subseteq \{\alpha_1, \ldots, \alpha_{l-1}\}$ and wJ does not contain α_{l-1}, which contradicts the assumption. Thus, in the composition $(m_1, \cdots, m_t)$, $m_i > 1$ for all i and $\lambda_1 = \lambda_2 = 2\sigma_1$. By the collapse procedure again, $\mu_1 = \lambda_1 = 2|\Pi \backslash J|$. It follows that $\mathcal{O}_\mu$ is restricted if and only if $|\Pi \backslash J| \leq \frac{p-1}{2}$ (p is odd). By Theorem 3.8(b) again, P_J is restricted if and only if it is strongly restricted.

Finally, suppose that G has type C_l and $r = 0$. Because $\lambda = 2\sigma$, the leading part λ_1 of λ is even. Therefore, the leading part of λ_X equals λ_1. It follows that P_J is restricted if and only if $|\Pi \backslash J| \leq \frac{p-1}{2}$. Now apply Theorem 3.8(a). □

Although we chose to prove Theorem 3.8 directly, we remark that the methods in the above proof could be used to shorten somewhat the arguments for that theorem.

3.10. Recall the closed subvarieties $\mathcal{N}_r(\mathfrak{g})$, $r = 1, 2, \ldots$, defined in the Introduction:

$$\mathcal{N}_r(\mathfrak{g}) = \{x \in \mathcal{N}(\mathfrak{g}) \mid x^{[p]^r} = 0\}.$$

We wish to point out that the partition point of view yields the following result.

THEOREM . *Let G be simple of classical type A, B, C, or D and p be good. For any $r \geq 1$, the closed subvariety $\mathcal{N}_r(\mathfrak{g})$ of $\mathcal{N}(\mathfrak{g})$ is irreducible.*

PROOF. First suppose that G has type A_l. Write $l + 1 = qp^r + d$, with $0 \leq d < p^r$. Let $\lambda \vdash l + 1$ be the partition with q parts of size p^r and 1 part of size d. Then λ is maximal with respect to the dominance partial order among all partitions of

$l+1$ which have parts of size at most p^r. Therefore, $\mathcal{N}_r(\mathfrak{g}) = \overline{\mathcal{O}}_\lambda$ is irreducible in this case.

Now assume that G has type B, C or D. Let $\mathfrak{g} \hookrightarrow \mathfrak{gl}_N(k)$ be the embedding via the standard representation where $N := N(l) = 2l+1$ (resp. $2l$) for $\Phi = B_l$ (resp. C_l and D_l). Let $\lambda_X \in \mathcal{P}_\epsilon(N)$ be the X-collapse of λ for $X = B, C, D$. By [CM, 6.3.3], λ_X is the unique maximal partition in $\mathcal{P}_X(N)$ dominated by λ. Thus, $\mathcal{N}_r(\mathfrak{g}) = \overline{\mathcal{O}}_{\lambda_X}$ is irreducible. □

4. The computation of $\mathcal{N}_1(\mathfrak{g})$: exceptional groups

4.1. We now turn our attention to the exceptional groups E_6, E_7, E_8, F_4 and G_2. The computation of Φ_0 and $\dim \mathcal{N}_1(\mathfrak{g})$ is fairly straightforward in these cases. For E_6, E_7, E_8, the roots in Φ_0 will consist of the ones of heights which are divisible by p. For example, for E_8, $p = 11$, the roots in Φ_0 are given by:

$$\{\pm\beta_1, \pm\beta_2, \pm\beta_3, \pm\beta_4, \pm\beta_5, \pm\beta_6, \pm\beta_7, \pm\beta_8\}$$

where

$$\begin{aligned}
\beta_1 &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 \\
\beta_2 &= \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \\
\beta_3 &= 2\alpha_2 + \alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \\
\beta_4 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 \\
\beta_5 &= \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \\
\beta_6 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \\
\beta_7 &= \alpha_1 + 3\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 3\alpha_7 + \alpha_8 \\
\beta_8 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 3\alpha_7 + 2\alpha_8.
\end{aligned}$$

The roots $\beta_1, \cdots, \beta_6$ of height 11 form a base for Φ_0: notice that $\beta_3 + \beta_6 = \beta_7$ and $\beta_2 + \beta_5 = \beta_8$. Therefore, $\Phi_0 \cong A_2 \times A_2 \times A_1 \times A_1$. This procedure can be used to calculate Φ_0 for all exceptional algebras. After Φ_0 is identified with some subroot system of Φ, it follows that $\dim \mathcal{N}_1(\mathfrak{g}) = |\Phi| - |\Phi_0|$ [NPV, (6.2.2) Cor.].

4.2. A more complicated task involves finding a set $J \subseteq \Pi$ such that $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$. For E_6 and F_4 when p is good, E_7 when $p = 13, 17$ and E_8 when $p = 19, 23, 29$ there is only one G-orbit whose dimension equals $\dim \mathcal{N}_1(\mathfrak{g}) = |\Phi| - |\Phi_0|$ [CM, §8.4]. In these cases J can be chosen such that Φ_J is of the same type as Φ_0. The remaining cases are E_7 ($p = 5, 7, 11$) and E_8 ($p = 7, 11, 13, 17$). In all these cases there are two G-orbits with the same dimension as $\mathcal{N}_1(\mathfrak{g})$. However, for each of the listed cases except E_7 and $p = 11$, we can apply the following lemma to show that Φ_0 is W-conjugate to Φ_J when Φ_0 is of the same type as Φ_J.

Lemma . *Let Π denote the simple roots of Φ. Suppose that*

(a) *J and J_2 are subsets of Π;*
(b) *$J_1 = J \cap J_2$;*
(c) *each root in $J \setminus J_1$ is orthogonal to all roots in J_2.*

If $w \in W_{J_2}$ such that $wJ_1 \subseteq J_2$ then $wJ = (J \setminus J_1) \cup wJ_1$ and $w\Phi_J = \Phi_{wJ}$.

The proof of the lemma is straightforward because every element in $J \setminus J_1$ is fixed by W_{J_2}. This lemma is used several times in the cases E_7 ($p = 5, 7$) and E_8 ($p = 7, 11, 13, 17$) to check that J for $G \cdot \mathfrak{u}_J$ only depends on the type of Φ_0. In

application, one chooses J_2 such that Φ_{J_2} is irreducible of type A. Note that for a type A root system, two subsets of simple roots are W-conjugate if and only if they generate isomorphic subroot systems (cf. 3.2).

4.3. Now let us consider the case when the Lie algebra is E_7 and $p = 11$. In this situation, $\Phi_0^+ = \{\alpha, \beta, \gamma\}$ with

$$\begin{aligned} \alpha &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\ \beta &= \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7, \\ \gamma &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \end{aligned}$$

where the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ are ordered as in the conventional notion given in the Appendix. The root system Φ_0 is of type $A_1 \times A_1 \times A_1$ with α, β, γ as base. Since p is good there is a subset $J \subseteq \Pi$ such that $w\Phi_0 = \Phi_J$ for some $w \in W$. There are two possibilities for the set J corresponding to nilpotent orbits of the same dimension $\dim G \cdot \mathfrak{u}_J$. We have $J = \{\alpha_2, \alpha_5, \alpha_7\}$ or $J = \{\alpha_2, \alpha_3, \alpha_5\}$. We will show that J has to be the second case. First observe that there is no positive root $\sigma = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 + n_5\alpha_5 + n_6\alpha_6 + n_7\alpha_7$ in E_7 such that $\langle \alpha_2, \sigma^\vee \rangle = \langle \alpha_5, \sigma^\vee \rangle = \langle \alpha_7, \sigma^\vee \rangle = -1$ because the system of equations

$$\begin{aligned} \langle \alpha_2, \sigma^\vee \rangle &= 2n_2 - n_4 = -1 \\ \langle \alpha_5, \sigma^\vee \rangle &= -n_4 + 2n_5 - n_6 = -1, \\ \langle \alpha_6, \sigma^\vee \rangle &= -n_6 + 2n_7 = -1, \end{aligned}$$

has no integer solution (n_4 and n_6 have to be odd). On the other hand, by taking σ as the highest root, we see that $\langle \alpha, \sigma^\vee \rangle = \langle \beta, \sigma^\vee \rangle = \langle \gamma, \sigma^\vee \rangle = -1$. Hence, J has to be conjugate to the set $\{\alpha_2, \alpha_3, \alpha_5\}$ under W.

4.4. From this analysis a set of representatives $J \subseteq \Pi$ can be chosen such that $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$. The summary of results for the exceptional Lie algebras (when p is good) is provided in the following tables. We also indicate the closure of the orbit equal to $\mathcal{N}_1(\mathfrak{g})$ via the standard labels given in [CM].

Type E_6:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	62	$A_2 \times A_1 \times A_1$	$\{1, 2, 4, 6\}$	$A_4 + A_1$
7	66	$A_1 \times A_1 \times A_1$	$\{2, 3, 5\}$	$E_6(a_3)$
11	70	A_1	$\{4\}$	$E_6(a_1)$
≥ 13	72	$\emptyset$	$\emptyset$	E_6

Type E_7:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	106	$A_3 \times A_2 \times A_1$	$\{1, 2, 3, 5, 6, 7\}$	$A_4 + A_2$
7	114	$A_2 \times A_1 \times A_1 \times A_1$	$\{1, 2, 3, 5, 7\}$	A_6
11	120	$A_1 \times A_1 \times A_1$	$\{2, 3, 5\}$	$E_7(a_3)$
13	122	$A_1 \times A_1$	$\{4, 6\}$	$E_7(a_2)$
17	124	A_1	$\{4\}$	$E_7(a_1)$
≥ 19	126	$\emptyset$	$\emptyset$	E_7

Type E_8:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
7	212	$A_4 \times A_2 \times A_1$	$\{1,2,3,5,6,7,8\}$	$A_6 + A_1$
11	224	$A_2 \times A_2 \times A_1 \times A_1$	$\{1,2,3,5,6,8\}$	$E_8(a_6)$
13	228	$A_2 \times A_1 \times A_1 \times A_1$	$\{2,3,5,6,8\}$	$E_8(a_5)$
17	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{2,3,5,7\}$	$E_8(a_4)$
19	234	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_8(a_3)$
23	236	$A_1 \times A_1$	$\{4,6\}$	$E_8(a_2)$
29	238	A_1	$\{4\}$	$E_8(a_1)$
≥ 31	240	$\emptyset$	$\emptyset$	E_8

Type F_4:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	40	$A_2 \times A_1$	$\{1,3,4\}$	$F_4(a_3)$
7	44	$A_1 \times A_1$	$\{1,3\}$	$F_4(a_2)$
11	46	A_1	$\{3\}$	$F_4(a_1)$
≥ 13	48	$\emptyset$	$\emptyset$	F_4

Type G_2:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	10	A_1	$\{2\}$	$G_2(a_1)$
≥ 7	12	$\emptyset$	$\emptyset$	G_2

4.5. This subsection will sketch a determination of the standard restricted parabolic subgroups P_J, $J \subseteq \Pi$, of the exceptional groups G of types E_6, E_7, E_8, F_4 and G_2. As always, we will assume that the characteristic p is good. Recall from (2.1) that P_J is restricted provided that $\mathfrak{u}_J \subseteq \mathcal{N}_1(\mathfrak{g})$, or equivalently, that $\mathcal{O}_J \subseteq \mathcal{N}_1(\mathfrak{g})$. As in (2.2), P_J is strongly restricted provided that there exists $\varpi \in X(T)_+$ such that $\Phi_J = w\Phi_\varpi$ for some $w \in W$. In this case, $|\mathfrak{g}|_{H^0(\varpi)} = \overline{\mathcal{O}_J} \subseteq \mathcal{N}_1(\mathfrak{g})$, so that P_J is restricted as well. It will come out in the next subsection that P_J is restricted if and only if it is strongly restricted.

For any reductive group, an even orbit is always a Richardson orbit. For exceptional type, the remaining Richardson orbits can be read off from the list given in [Hi, p. 370] (which identifies them both by label and as the Richardson class associated to a specific parabolic P_J; e. g., in type E_6, there is an odd Richardson class labeled $2A_1$ which corresponds to $J = \Pi \backslash \{\alpha_1\}$). However, a Richardson class need not correspond uniquely to a standard parabolic P_J; if J, K are W-conjugate subsets of Π, then P_J and P_K determine the same Richardson orbit [JR], while the converse statement need not necessarily hold–see [Hi].

Suppose the label of the Richardson class determined by P_J is known. Then, using the results given in (4.4), together with the Hasse diagrams in [C, 13.4] (see also [Sp]), it can be determined whether P_J is restricted.

We can now summarize some simple facts that we will use below:

C1. If a standard parabolic subgroup P_J is strongly restricted then $|\Phi_J^+| \geq |\Phi_0^+|$.

C2. If $h_J < p$, then P_J is strongly restricted.

C3. If P_J is strongly restricted, so is P_{wJ} for any $w \in W$ such that $wJ \subseteq \Pi$.

To provide a list of all $J \subseteq \Pi$ such that P_J is strongly restricted, we proceed in three steps:

(1) Determine all $J \subseteq \Pi$ such that $|\Phi_J^+| \geq |\Phi_0^+|$.

(2) Exclude those J in (1) for which P_J is not restricted. As mentioned above, for a general P_J, there is no available description of the label of the associated Richardson orbit in the Hasse diagram. But because there exist only a few dimensions $|\Phi| - |\Phi_J|$ where there is a non-restricted Richardson orbit, there is in practice only a small number of possible subsets J to consider for exclusion. In most of these remaining cases, the label can be readily determined directly, or, using Lemma 4.2, there exists $w \in W$ such that the Richardson orbit associated to P_{wJ} has a known label. In summary, we can list the subsets J appearing in (1), for which P_J is not restricted:

$$\begin{array}{ll} E_6,\ p=5: & \Phi_J \cong A_2 \times A_2.(J = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}). \\ E_7,\ p=5: & \Phi_J \cong D_4, J = \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}. \\ E_7,\ p=7: & \Phi_J \cong A_3, A_2 \times A_2. \\ E_7,\ p=11: & J = \{\alpha_2, \alpha_5, \alpha_7\}. \\ E_8,\ p=7: & \Phi_J \cong D_5, A_5. \\ E_8,\ p=11: & \Phi_J \cong D_4. \\ E_8,\ p=13: & \Phi_J \cong A_3. \end{array} \tag{4.5.1}$$

(3) For the remaining J in (1), either **C2** applies to determine that P_J is strongly restricted, or we use **C3**: apply Lemma 4.2 (choosing J_2 of type A) to find $w \in W$ such that $wJ \subseteq \Pi$ satisfies **C2**. There are only a small number of cases for which the above method fails. Finally, by using a small computer program (Maple or LiE), we can find $\varpi' = \sum_{\alpha_i \in (\Pi \setminus J)} a_i \varpi_i$ with $0 \leq a_i < p$ and $\Phi_J = \Phi_{\varpi' - \rho}$.

Thus we summarize the result in the following

Theorem . *Assume that G has type E_6, E_7, E_8, F_4, G_2 and that p is a good prime. If $p \geq h$, then all standard parabolic subgroups P_J are strongly restricted, and hence restricted. Otherwise, P_J is a restricted parabolic subgroup if and only if J satisfies one of the conditions listed below. This listing is given by prime and rank.*

(a) *G has type E_6.*
  (i) *$p = 11$: $\Phi_J \neq \emptyset$.*
  (ii) *$p = 7$: J has type A_2 or $|J| \geq 3$.*
  (iii) *$p = 5$:* $\begin{cases} \Phi_J \cong A_3; \\ \Phi_J \cong A_4,\ D_4,\ A_3 \times A_1,\ A_2 \times A_1 \times A_1; \\ |J| \geq 5. \end{cases}$

(b) *G has type E_7.*
  (i) *$p = 17$: $\Phi_J \neq \emptyset$.*
  (ii) *$p = 13$: $|J| \geq 2$.*
  (iii) *$p = 11$: $\Phi_J \cong A_2$, $|J| \geq 3$ with $J \neq \{\alpha_2, \alpha_5, \alpha_7\}$.*
  (iv) *$p = 7$:* $\begin{cases} \Phi_J \cong A_4, D_4, A_2 \times A_1 \times A_1 \times A_1; \\ \Phi_J \cong A_3 \times A_1 \text{ with } J \neq \{\alpha_2, \alpha_5, \alpha_6, \alpha_7\}, \{\alpha_2, \alpha_4, \alpha_5, \alpha_7\}; \\ |J| \geq 5. \end{cases}$

(v) $p = 5$: $\begin{cases} \Phi_J \cong D_4 \times A_1, D_5, A_4 \times A_1, A_5 (J \neq \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}); \\ |J| \geq 6. \end{cases}$

(c) *G has type E_8.*

(i) $p = 29$: $J \neq \emptyset$.

(ii) $p = 23$: $|J| \geq 2$.

(iii) $p = 19$: $|J| \geq 3$ *or* $\Phi_J \cong A_2$.

(iii) $p = 17$: $|J| \geq 4$ *or* $\Phi_J \cong A_3$, $A_2 \times A_1$.

(iv) $p = 13$: $\begin{cases} \Phi_J \cong D_4, A_3 \times A_1, A_4, A_2 \times A_2; \\ |J| \geq 5. \end{cases}$

(v) $p = 11$: $\begin{cases} \Phi_J \cong A_4; \\ \Phi_J \cong D_5, D_4 \times A_1, A_5, A_4 \times A_1, A_3 \times A_2, A_3 \times A_1 \times A_1; \\ |J| \geq 6. \end{cases}$

(vi) $p = 7$: $\begin{cases} \Phi_J \cong A_6, A_5 \times A_1, E_6, D_6, D_5 \times A_1, D_4 \times A_2; \\ |J| \geq 7. \end{cases}$

(d) *G has type F_4.*

(i) $p = 11$: $\Phi_J \neq \emptyset$.

(ii) $p = 7$: $|J| \geq 2$.

(iii) $p = 5$: $\Phi_J \cong B_2$ *or* $|J| \geq 3$.

(e) *G has type G_2.*

(i) $p = 5$: $|J| \geq 1$.

4.6. In (3.3), we listed all strongly restricted parabolic subgroups for type A_l groups. As a consequence, all restricted parabolic subgroups are strongly restricted in these cases. For the exceptional groups, in the process of establishing Theorem 4.5, we see that for any given J, either P_J is not restricted (i. e., $|\Phi_J| < |\Phi_0|$ or J is in the list (4.5.1)), or P_J is strongly restricted. Thus, we conclude:

THEOREM . *Assume that G is of type E_6, E_7, E_8, G_2, F_4 and that p is a good prime. Then every restricted parabolic subgroup is strongly restricted.*

4.7. Combining the results of the previous two subsections with those in (3.3) and (3.9), together with the discussion in (2.2), we obtain the following result.

THEOREM . *Let G be a simple algebraic group over an algebraically closed field of good characteristic. Let $\mathcal{O} \subseteq \mathcal{N}(\mathfrak{g})$ be any nilpotent G-orbit. Then there exists a dominant weight ϖ such that $|\mathfrak{g}|_{H^0(\varpi)} = \overline{\mathcal{O}}$ if and only if $\mathcal{O}$ is a Richardson orbit and is contained in the restricted nullcone $\mathcal{N}_1(\mathfrak{g})$. (The possible orbits $\mathcal{O}$ are explicitly listed in Theorems 3.8 and 4.8.)*

4.8. We now summarize the properties of parabolic subgroups.

THEOREM . *Let G be a simple algebraic group over an algebraically closed field k of good characteristic p. Let P be a parabolic subgroup with unipotent radical P_u, and, by abuse of notation, let $\mathfrak{u}_P$ denote the Lie algebra of P_u. Then the following are equivalent.*

(i) *P is restricted, i. e., $\mathfrak{u}_P \subset \mathcal{N}_1(\mathfrak{g})$.*

(ii) *$G \cdot \mathfrak{u}_P = |\mathfrak{g}|_{H^0(\varpi)}$ for some dominant weight ϖ.*

(iii) *The semisimple group $[P/P_u, P/P_u]$ (the commutator group) has the root system isomorphic to $\Phi_\varpi = \{\alpha \in \Phi \mid \langle \varpi + \rho, \alpha^\vee \rangle \in p\mathbb{Z}\}$ for some integral weight ϖ.*

Proof. Note that (iii) implies (ii) by Theorem 2.2 and (ii) implies (i) since all support varieties are contained in $\mathcal{N}_1(\mathfrak{g})$. By Theorem 3.3, 3.9, and 4.6. (i) implies that P is strongly restricted, i. e., P is conjugate to a standard parabolic subgroup which is strongly restricted. Thus, its Levi subgroup has the root system isomorphic to Φ_ϖ for some ϖ. □

5. Applications

5.1. We conclude this paper with some applications of our results to the restricted nullcone $\mathcal{N}_1(\mathfrak{p})$ where P is a parabolic subgroup of our reductive group G and $\mathfrak{p}$ is the Lie algebra of P. We will first consider the case when $P = B$ where B is a Borel subgroup. Let $x \in \mathcal{N}(\mathfrak{g})$ and $\mathcal{B}_x$ be the set of Borel subalgebras in $\mathfrak{g}$ containing x. Also, let $C_G(x)$ be the centralizer x in G, and $C_G(x)^0$ be its connected component of the identity. Set $A(x) = C_G(x)/C_G(x)^0$. The group $A(x)$ acts naturally on the set of the irreducible components of $\mathcal{B}_x$.

Theorem . *Let G be a reductive algebraic group with p good and let B be a Borel subgroup of G.*

(a) *If $p \geq h$ then $\mathcal{N}_1(\mathfrak{b}) = \mathfrak{u}$ where $\mathfrak{u}$ is the unipotent radical of $\mathfrak{b}$.*
(b) *If $p < h$ then $\mathcal{N}_1(\mathfrak{b})$ is not irreducible. In this case let J be such that $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ where $J \subseteq \Pi$. Moreover, let $x \in \mathfrak{g}$ with $\overline{G \cdot x} = G \cdot \mathfrak{u}_J$.*
   (i) *The irreducible components of $\mathcal{N}_1(\mathfrak{b})$ all have dimensions which are less than or equal to* $\dim \mathfrak{u}_J$.
   (ii) *The number of components of $\mathcal{N}_1(\mathfrak{b})$ of maximal dimension* $\dim \mathfrak{u}_J$ *equals the number of $A(x)$-orbits on the set of irreducible components of $\mathcal{B}_x$.*
   (iii) *In particular, assume that $G \cong SL_{l+1}$, and take $x = x_\lambda$ for some $\lambda \vdash (l+1)$, then the number components of maximal dimension in $\mathcal{N}_1(\mathfrak{b})$ equals the dimension of the irreducible complex representation of $W = S_{l+1}$ corresponding to the partition λ.*

Proof. (a) For $p \geq h$, $\mathcal{N}_1(\mathfrak{b}) = \mathcal{N}(\mathfrak{b}) = \mathfrak{u}$. (b) Now assume $p < h$, so that, using Theorem 2.2, $\mathcal{N}_1(\mathfrak{g}) = G \cdot \mathfrak{u}_J$ for some *nonempty* subset J of Π. Observe that $\mathcal{N}_1(\mathfrak{b}) = \mathcal{N}_1(\mathfrak{g}) \cap \mathfrak{u}$. The irreducible components of $\mathcal{N}_1(\mathfrak{b})$ all have dimension less than or equal to $\dim \mathfrak{u}_J$. In particular, $\mathfrak{u}_J$ must be an irreducible component of $\mathcal{N}_1(\mathfrak{b})$. Since $J \neq \emptyset$, any nonzero root vector $x \in \mathfrak{g}_\alpha$, $\alpha \in J$, satisfies $x \in \mathcal{N}_1(\mathfrak{b}) \backslash \mathfrak{u}_J$. Hence, $\mathcal{N}_1(\mathfrak{b})$ is not irreducible.

To see (b)(ii), observe that, by results of Spaltenstein (see, e. g., [M, §7.4]), if $\mathcal{O}$ is a G-orbit in $\mathcal{N}(\mathfrak{g})$, then the irreducible components of $\mathcal{O} \cap \mathfrak{b}$ all have the same dimension $\frac{1}{2} \dim \mathcal{O}$. Write $\mathcal{N}_1(\mathfrak{b})$ as a finite union $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_m$ of G-orbits with $\mathcal{O}_1 = G \cdot x$ being the unique orbit of maximal dimension (which is $2 \dim \mathfrak{u}_J$. Then $\mathcal{N}_1(\mathfrak{b})$ is the union of the various $\mathcal{O}_i \cap \mathfrak{b}$, $i = 1, \cdots, m$. It follows that the irreducible components of $\mathcal{N}_1(\mathfrak{b})$ of maximal dimension (which is $\dim \mathfrak{u}_J$) are precisely the closures in $\mathfrak{b}$ of the irreducible components of $\mathfrak{b} \cap \mathcal{O}_1$. Since the number of irreducible components of $\mathfrak{b} \cap O$ equals the number of orbits of $A(x)$ on $\mathcal{B}_x$ [M, p. 219], it remains only to check that if C, C' are two distinct irreducible components of $\mathfrak{b} \cap \mathcal{O}$, then $\overline{C} \neq \overline{C}'$. For $w \in W$, put $\mathfrak{u}_w = \mathfrak{u} \cap w(\mathfrak{u})$. By Joseph's theorem (see [M, Thm. 7.7]), there exist $x, y \in W$ such that $C = \overline{B \cdot (\mathfrak{u}_x)} \cap \mathcal{O}$ and $C' = \overline{B \cdot (\mathfrak{u}_y)} \cap \mathcal{O}$, where $\mathfrak{u}_x \cap \mathcal{O}$ is dense in $\mathfrak{u}_x$ and $\mathfrak{u}_y \cap \mathcal{O}$ is dense in $\mathfrak{u}_y$. It follows

easily that $\overline{B \cdot \mathfrak{u}_x} = \overline{C}$ and $\overline{B \cdot \mathfrak{u}_y} = \overline{C'}$. Intersecting with $\mathcal{O}$ gives that $C = C'$, as required.

Finally, (b)(iii) follows from known facts involving orbital varieties (i. e., the components of $\mathfrak{b} \cap \mathcal{O}$) and Springer representations; see [M, §7.3]. □

The results in Sections 2 and 3 can be used to explicitly compute the dimension of $\mathcal{N}_1(\mathfrak{b})$. For example, consider the case when $\mathcal{N}_1(\mathfrak{g})$ is the closure $\overline{\mathcal{O}}_{\text{sub-reg}}$ of the sub-regular class in $\mathcal{N}(\mathfrak{g})$. This situation occurs in types $(A_l, p = l), \ldots, (E_8, p = 29)$. Since $\sum_{\beta \in \Phi^+} x_\beta$, $x_\beta \in \mathfrak{g}_\beta$, is then contained in $\mathcal{N}_1(\mathfrak{g})$ if and only if some $x_\alpha = 0$, $\alpha \in \Pi$, we see directly that $\mathcal{N}_1(\mathfrak{b})$ is the union of the *linear* subvarieties $\mathfrak{u}_{J_\alpha}$, $\alpha \in \Pi$, where $J_\alpha = \{\alpha\}$. Thus, $\mathcal{N}_1(\mathfrak{g})$ has $l = \operatorname{rank} G$ irreducible components. On the other hand, the corresponding W-module is the complex, irreducible reflection module (which has dimension l).

5.2. We have the following result concerning the structure of $\mathcal{N}_1(\mathfrak{p})$ for a parabolic subgroup P of G.

PROPOSITION . *Let $P = L \cdot U_P$ be a parabolic subgroup of G.*

(a) *For any prime p (possibly not good), we have*

$$\mathcal{N}_1(\mathfrak{p}) \subseteq \mathcal{N}_1(\mathfrak{l}) \times \mathfrak{u}_P \subseteq \mathcal{N}(\mathfrak{p}).$$

(b) *If $p \geq h$, then*
   (i) $\mathcal{N}_1(\mathfrak{p}) = \mathcal{N}_1(\mathfrak{l}) \times \mathfrak{u}_P = \mathcal{N}(\mathfrak{l}) \times \mathfrak{u}_P = \mathcal{N}(\mathfrak{p})$.
   (ii) $\mathcal{N}_1(\mathfrak{p})$ *is an irreducible variety.*
   (iii) $\dim \mathcal{N}_1(\mathfrak{p}) = \dim P - \operatorname{rank} G$.

PROOF. We first prove (a). Given $\ell \in \mathcal{N}_1(\mathfrak{l})$, $x \in \mathfrak{u}_P$,

$$(\ell + x)^{[p]} = \ell^{[p]} + x^{[p]} + z = x^{[p]} + z \in \mathfrak{u}_P,$$

for some $z \in [\mathfrak{l}, \mathfrak{u}_P] \subseteq \mathfrak{u}_P$. Hence, $\mathcal{N}_1(\mathfrak{l}) \times \mathfrak{u}_P \subseteq \mathcal{N}(\mathfrak{p})$. Also, if $y = \ell + x \in \mathcal{N}_1(\mathfrak{p})$, $\ell \in \mathfrak{l}, x \in \mathfrak{u}_P$, the same calculation shows that $\ell^{[p]} = 0$, so $\mathcal{N}_1(\mathfrak{p}) \subseteq \mathcal{N}_1(\mathfrak{l}) \times \mathfrak{u}_P$.

Finally, if $p \geq h$, $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$, so (b) follows trivially. □

For $p < h$, the precise determination of the variety $\mathcal{N}_1(\mathfrak{p})$ remains open, e. g., when is $\mathcal{N}_1(\mathfrak{p})$ irreducible? In a particular case, suppose $p = 2$ and that $\mathfrak{u}_P^{[p]} = 0$. Then $\mathcal{N}_1(\mathfrak{p}) = \{l + x \in \mathcal{N}_1(\mathfrak{l}) \times \mathfrak{u}_P \mid [l, x] = 0\}$ which is a type of commuting variety.

5.3. We now assume again that the prime p is good for the reductive group G. We conclude with a result showing that certain prime powers p^d must divide the dimension of every module in a block for $u(\mathfrak{g})$. It is well-known that the irreducible $u(\mathfrak{g})$ modules $L_1(\varpi)$ correspond bijectively to the set $X_1(T)_+$ of restricted dominant weights consisting of all $\varpi \in X(T)_+$ such that $\langle \varpi, \alpha^\vee \rangle < p$ for all $\alpha \in \Pi$. In fact, $L_1(\varpi)$ can be taken to be the restriction to $u(\mathfrak{g})$ of the irreducible G-module $L(\varpi)$ of highest weight ϖ. If $\lambda \in X(T)$, let $\lambda = \lambda^0 + p\lambda^1$ for some $\lambda^0 \in X_1(T)_+$ and $\lambda^1 \in X(T)$ and set $L_1(\lambda) = L(\lambda^0)$. Fix $\lambda \in X(T)_+$ and let $\mathcal{B}_\lambda$ be the $u(\mathfrak{g})$-block containing the irreducible $u(\mathfrak{g})$-module $L_1(\lambda)$. The irreducible modules in $\mathcal{B}_\lambda$ are precisely the $L_1(\mu)$ for $\mu \in W \cdot \lambda + pX(T)$ and $L_1(\mu) \cong L_1(\nu)$ if and only if $\mu \equiv \nu \pmod p$ (we can take G simply connected). See [Jan1].

Now let $J(\lambda) \subseteq \Pi$ satisfy $w\Phi_\lambda = \Phi_{J(\lambda)}$ for some $w \in W$. Define

$$b_\lambda = \frac{1}{2}(\dim \mathcal{N}_1(\mathfrak{g}) - \dim G \cdot \mathfrak{u}_{J(\lambda)}).$$

The definition of b_λ is well-defined by Footnote 3.

THEOREM . *For any $M \in \mathcal{B}_\lambda$, we have $p^{b_\lambda} | \dim M$. Furthermore, for some $\lambda \in X(T)_+$ with $H^0(\lambda) \in \mathcal{B}_\lambda$, one has p^{b_λ} as the maximal power of p dividing* $\dim\ H^0(\lambda)$.

PROOF. For any $\mu \in X(T)_+$, we have, by the Steinberg Tensor Product Theorem, $L(\mu) \cong L_1(\mu) \otimes L(\mu^1)^{[1]}$, where $L(\mu^1)^{[1]}$ is a trivial $u(\mathfrak{g})$-module. Thus, $|\mathfrak{g}|_{L(\mu)} = |\mathfrak{g}|_{L_1(\mu)}$. Suppose that $S = L_1(\varpi) \in \mathcal{B}_\lambda$ for some $\varpi \in X_1(T)_+$. Then $L_1(\varpi)$ is $u(\mathfrak{g})$-isomorphic to a G-submodule of $H^0(\varpi)$. All the other G-composition factors $L(\mu)$ of $H^0(\varpi)$ have highest weights μ in $(W \cdot \lambda + pX(T)) \cap X(T)_+$ and strictly less than ϖ in the usual partial ordering on $X(T)_+$ (though they need not be restricted). By Theorem 2.2, we have $|\mathfrak{g}|_{H^0(\mu)} = |\mathfrak{g}|_{H^0(\varpi)} = |\mathfrak{g}|_{H^0(\lambda)}$. By using the lemma below, a straightforward induction on the set of dominant weights in $W \cdot \lambda + pX(T)$ shows that $|\mathfrak{g}|_{L(\mu)} \subseteq |\mathfrak{g}|_{H^0(\lambda)}$. Also, $|\mathfrak{g}|_S \subseteq |\mathfrak{g}|_{H^0(\lambda)}$. Of course, $\dim |\mathfrak{g}|_{H^0(\lambda)} = 2 \dim \mathfrak{u}_{J(\lambda)} = |\Phi| - |\Phi_\lambda|$. [7]

Since S extends to a G-module, we can apply the divisibility result [NPV, Theorem 3.5.1(b)] to conclude that $p^b | \dim S$, where $b = \frac{1}{2}(|\Phi| - |\Phi_0| - \dim |\mathfrak{g}|_S)$. Clearly, $b \geq b_\lambda$. Thus, p^{b_λ} divides the dimension of all simple modules in $\mathcal{B}_\lambda$, and so divides the dimensions of all $M \in \mathcal{B}_\lambda$.

The final assertion follows by using Weyl's dimension formula. □

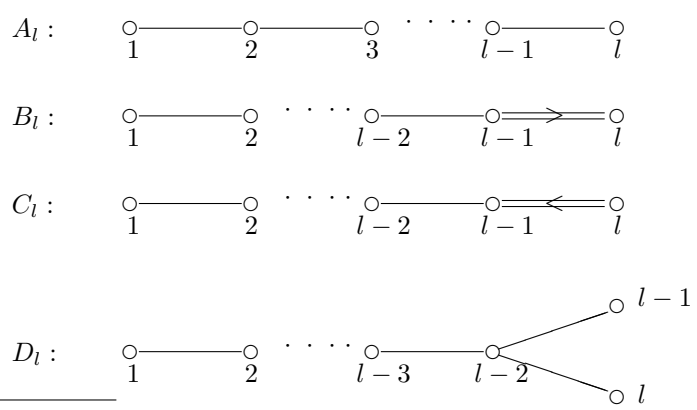
LEMMA . *Let $0 \to N \to E \to M \to 0$ be a short exact sequence of $u(\mathfrak{g})$-modules. Then*

(i) $|\mathfrak{g}|_E \subseteq |\mathfrak{g}|_N \cup |\mathfrak{g}|_M$.
(ii) *If either $|\mathfrak{g}|_M \subseteq |\mathfrak{g}|_E$ or $|\mathfrak{g}|_N \subseteq |\mathfrak{g}|_E$ then $|\mathfrak{g}|_E = |\mathfrak{g}|_N \cup |\mathfrak{g}|_M$.*

PROOF. The proof is straightforward using the characterization of the support variety in [FP2] and in (ii) the fact that $u(\mathfrak{g})$ is a Frobenius algebra. □

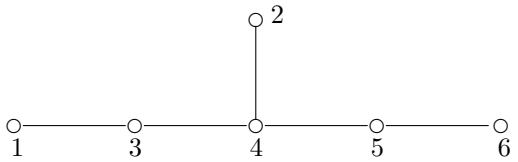
We close by mentioning that Humphreys, in a paper written over 30 years ago, established the above result for the case $p > h$. See [Hum1, Thm. 6.3].

6. Appendix: Labeling of Dynkin diagrams

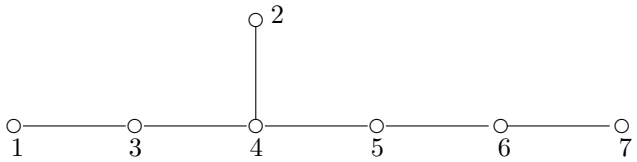


[7] Our notation here differs somewhat from that in [NPV]. In our case, $d(\Phi, p) = |\Phi_0|$ since p is good.

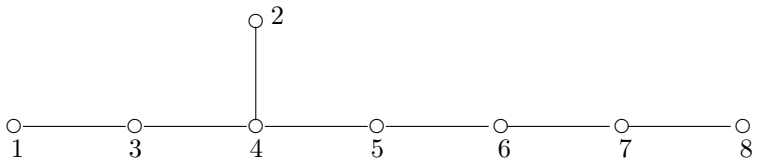
E_6 :



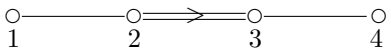
E_7 :



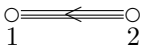
E_8 :



F_4 :



G_2 :



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