REPRESENTATIONS OF TAME QUIVERS AND AFFINE CANONICAL BASES

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Dedicated to Claus Michael Ringel on the occasion of his 60th birthday

Abstract. An integral PBW-basis of type $A_1^{(1)}$ has been constructed by Zhang [Z] and Chen [C] using the Auslander-Reiten quiver of the Kronecker quiver. We associate a geometric order to elements in this basis following an idea of Lusztig [L1] in the case of finite type. This leads to an algebraic realization of the canonical bases of $U_q(\hat{sl}_2)$. For any affine symmetric type, we obtain an integral PBW-basis of the generic composition algebra, by using an algebraic construction of the integral basis for a tube in [DDX], an embedding of the module category of the Kronecker quiver into the module category of the tame quiver, and a list of the root vectors of indecomposable modules according to the preprojective, regular, and preinjective components of the Auslander-Reiten quiver of the tame quiver. When the basis elements are ordered to be compatible with the geometric order given by the dimensions of the orbit varieties and the extension varieties, we can show that the transition matrix between the PBW-basis and a monomial basis is triangular with diagonal entries equal to 1. Therefore we give an algebraic way to realize the canonical bases of the quantized enveloping algebras of all symmetric affine Kac-Moody Lie algebras.

0. Introduction

0.1 Let U^+ be the positive part of the quantized enveloping algebra of U associated to a Cartan datum. In [L1], Lusztig has constructed a canonical basis of U^+ in the case of finite type by three methods. The first method can be understood in a pure combinatorial way. By applying the Lusztig's symmetries and the induced actions of the braid group on U^+, one may have a complete list of root vectors of U^+. Associated to each reduced expression of the longest element of the Weyl group, there is a PBW-basis of U^+ with a specific order and a monomial basis on the Chevalley generators such that the transition matrix between these two bases is triangular with diagonal entries equal to 1. The second method is the quiver approach. Each isomorphism class of the Dynkin quiver corresponds to a basis element E^c, $(c \in \mathbb{N}^{\Phi^+})$ of U^+. Now the representations of a fixed dimension vector of the quiver are the orbits of an algebraic groups action on an affine variety. The geometric dimension of these orbits can be applied to give an order in $\{E^c | c \in \mathbb{N}^{\Phi^+}\}$. This ordered basis relates to a monomial basis by a triangular transition matrix with diagonal entries equal to 1. By a standard linear algebra method one can

The main results in this article were reported by J. Xiao at the Fields Institute, Toronto, Canada (Workshop and Conference on Infinite Dimensional Lie Theory and its Application, July 17-25, 2003), at the Algebra seminar of the University of Sydney, Australia, October 10, 2003 and at Snowbird, Utah, USA (AMS-IMS-SIAM Summer Research Conferences, Representations of Algebraic Groups, Quantum Groups, and Lie Algebras, July 11-15, 2004).

The research was supported in part by NSF of China and by the 973 Project of the Ministry of Science and Technology of China and by NSF grant DMS-0200673.

easily obtain the canonical basis. The third method is the geometric approach by using perverse sheaves and intersection cohomology. It is also well known that an independent method to construct the crystal bases of U^+ was given in the Kashiwara's work [K]. Now it is well known that Lusztig has generalized his geometric method to construct the canonical bases of U^+ for all infinite type (see[L2] and [L3]).

0.2 A natural question is to seek an algebraic characterization of the affine canonical bases. This was attained first by Beck, Chari and Pressley in [BCP] for the quantized enveloping algebra of untwisted affine type, and then was improved and extended by Beck and Nakajima in [BN] to all twisted and untwisted affine types. Their method is to give the real root vectors by applying the Lusztig's symmetries on the generators and to construct the imaginary root vectors by using Schur functions in the Heisenberg generators. Therefore their construction depends on the Drinfeld new realization of the quantized affine algebras in [Dr].

0.3 Although most knowledge on the canonical bases in finite type can be carried out in a pure combinatorial way, it is obvious to see that the definition of the canonical bases was introduced by Lusztig in a framework of representations of quivers. Specifically, Lusztig has extended the Gabriel's theorem to build up a PBW type basis for U^+, which is ordered by the geometric properties of the corresponding orbit varieties. It is well known that the representation category of a tame quiver has been completely described by a generalization of the Gabriel's theorem and its Auslander-Reiten quiver (see[DR]). The objective of this paper is to provide a process to construct the canonical bases of U^+ of affine type by using Ringel-Hall algebra and the knowledge of the representations of tame quivers. We hope that the method we adopt in this paper is closer to the original idea of Lusztig in [L1].

In Section 1 we define the Hall algebra of a quiver in the sense of Ringel and in the sense of Lusztig respectively, and point out that the two constructions coincide essentially for the representations of a quiver over a finite field. Section 2 presents the basic geometric properties of the orbit varieties and extension varieties for the representations of quivers. In Section 3 we construct an integral PBW basis of $A_1^{(1)}$ type by using the representations of the Kronecker quiver. Most results in this section are already known for some experts (see [Z] and [C]). The category $\mathrm{mod}\,\Lambda$ of the Kronecker quiver has a strong representation-directed property [DR]. This enables us in Section 4 to arrange the positive roots in a special order. In addition, by the basic properties of the orbit varieties, we find a monomial bases whose transition matrix with the PBW basis is triangular with diagonal entries equal to 1. Section 5 is taken from [DDX], in which the integral bases and the canonical bases of $A_n^{(1)}$ type were given in terms of the nilpotent representations of the cyclic quivers. In Section 6 we consider the $\mathcal{Z}$-submodule of U^+ generated by $\langle u_M\rangle$ for M being a preprojective (resp. preinjective) module. It is a $\mathcal{Z}$-subalgebra of U^+. An integral basis for this $\mathcal{Z}$-subalgebra can be listed in an order with respect to the representation-directed property of the preprojective (resp. preinjective) component. We verify that the basis elements coincide with the products of images of Chevalley generators under the action of the sequences of Lusztig's symmetries in an admissible order. So the situation in Section 6 is essentially the same as in the finite type case. In Section 7, we show that the subalgebras corresponding to the preprojective component, preinjective component, non-homogeneous tubes, and an embedding of the module category of the Kronecker quiver can be put together, according to the representation-directed property of the tame quiver. This gives rise to an integral basis of U^+ over $\mathbb{Q}[v, v^{-1}]$. In the last section, we again find a monomial basis, which has a unipotent triangular relation with the integral PBW type basis we obtained. But this needs a little more subtle analysis of the orbit varieties and the extension varieties. Finally, the canonical basis of U^+ can be constructed in an elementary and algebraic way.

Acknowledgments. (1) We are very grateful to O. Schiffmann, B. Deng and J. Du for getting our attention to the preprint [H] by A. Hubery, in which an integral PBW basis for the composition algebras of affine type are constructed according to the representations of tame quivers. (2) Our construction of the canonical bases is independent of the assumption for the existence of the Lusztig's canonical bases, also independent of the existence of the Kashiwara's crystal bases.

1. Ringel-Hall algebras

1.1 A quiver $Q = (I, H, s, t)$ consists of a vertex set I, an arrow set H, and two maps $s, t : H \to I$ such that an arrow $\rho \in H$ starts at $s(\rho)$ and terminates at $t(\rho)$.

Throughout the paper, $\mathbb{F}_q$ denotes a finite field with q elements, $Q = (I, H, s, t)$ is a fixed connected quiver, and Λ is the path algebra of Q over $\mathbb{F}_q$. By $\operatorname{mod}\Lambda$ we denote the category of all finite dimensional left Λ-modules, or equivalently finite modules. It is well-known that $\operatorname{mod}\Lambda$ is equivalent to the category of finite dimensional representations of Q over $\mathbb{F}_q$. We shall simply identify Λ-modules with representations of Q.

1.2 Ringel-Hall algebra. Given three modules L, M, N in $\operatorname{mod}\Lambda$, let g^L_{MN} denote the number of submodules W of L such that $W \simeq N$ and $L/W \simeq M$. Let $v = \sqrt{q}$ and $\mathcal{P}$ be the set of isomorphism classes of finite dimensional nilpotent Λ-modules. Then the Ringel-Hall algebra $\mathcal{H}(\Lambda)$ of Λ is by definition the $\mathbb{Q}(v)$-space with basis $\{u_{[M]} | [M] \in \mathcal{P}\}$ whose multiplication is given by

$$u_{[M]}u_{[N]} = \sum_{[L]\in\mathcal{P}} g^L_{MN} u_{[L]}.$$

Note that g^L_{MN} depends only on the isomorphism classes of M, N and L, and for fixed isomorphism classes of M, N there are only finitely many isomorphism classes $[L]$ such that $g^L_{MN} \neq 0$. It is clear that $\mathcal{H}(\Lambda)$ is associative $\mathbb{Q}(v)$-algebra with unit u_0, where 0 denotes the zero module.

The set of isomorphism classes of (nilpotent) simple Λ-modules is naturally indexed by the set I of vertices of Q. Then the Grothendieck group $G(\Lambda)$ of $\operatorname{mod}\Lambda$ is the free abelian group $\mathbb{Z}I$. The Ringel-Hall algebra $\mathcal{H}(\Lambda)$ is graded by $\mathbb{N}I$, more precisely, by dimension vectors of modules.

The Euler form $\langle -, - \rangle$ on $G(\Lambda) = \mathbb{Z}I$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i\in I} a_i b_i - \sum_{\rho\in H} a_{s(\rho)} b_{t(\rho)}$$

for $\alpha = \sum_{i\in I} a_i i$ and $\beta = \sum_{i\in I} b_i i$ in $\mathbb{Z}I$. It is well-known that for any Λ-modules M and N one has

$$\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle = \dim_{\mathbb{F}_q} \operatorname{Hom}_\Lambda(M, N) - \dim_{\mathbb{F}_q} \operatorname{Ext}_\Lambda(M, N)$$

where $\underline{\dim}\, M$ and $\underline{\dim}\, N$ denote the dimension vectors of M and N respectively. The symmetric Euler form is defined as

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle \quad \text{for} \quad \alpha, \beta \in \mathbb{Z}I.$$

This gives rise to a symmetric generalized Cartan matrix $C = (a_{ij})_{i,j\in I}$, where $a_{ij} = (i, j)$. It is easy to see that C is independent of the field $\mathbb{F}_q$ and the orientation of Q.

Throughout the paper, we concentrate on tame quivers Q. The symmetric Euler forms give rise to the Cartan matrices of all affine A, D, E type.

The twisted Ringel-Hall algebra $\mathcal{H}^*(\Lambda)$ is defined by setting $\mathcal{H}^*(\Lambda) = \mathcal{H}(\Lambda)$ as $\mathbb{Q}(v)$-vector space, but the multiplication is defined by

$$u_{[M]} * u_{[N]} = v^{\langle \underline{\dim} M, \underline{\dim} N\rangle} \sum_{[L]\in\mathcal{P}} g^L_{MN} u_L.$$

Following [R3], for any Λ-module M, we denote $\langle M\rangle = v^{-\dim M + \dim \mathrm{End}_\Lambda(M)} u_{[M]}$. Note that $\{\langle M\rangle \mid M \in \mathcal{P}\}$ a basis of $\mathcal{H}^*(\Lambda)$.

1.3 A construction by Lusztig. For any I-graded $\mathbb{F}_q$-vector space $V = \sum_{i\in I} V_i$, let $\mathbb{E}_V$ be the subset of $\oplus_{\rho:i\to j} \mathrm{Hom}(V_i, V_j)$[1] defining nilpotent representations of Q. The group $G_V = \prod_{i\in I} GL(V_i)$ acting naturally on $\mathbb{E}_V$ by

$$(g, x) \mapsto g \bullet x = x' \quad \text{where} \quad x'_\rho = g_{t(\rho)} x_\rho g^{-1}_{s(\rho)} \quad \text{for all} \quad \rho \in H.$$

Let $\mathbb{C}_G(\mathbb{E}_V)$ be the space of G_V-invariant functions $\mathbb{E}_V \to \mathbb{C}$. For $\gamma \in \mathbb{N}I$, we fix a I-graded $\mathbb{F}_q$-vector space V_γ with $\underline{\dim}\, V_\gamma = \gamma$. There is no danger of confusion if we denote by $\mathbb{E}_\gamma = \mathbb{E}_{V_\gamma}$ and $G_\gamma = G_{V_\gamma}$. Let $\alpha, \beta \in \mathbb{N}I$ such that $\alpha + \beta = \gamma$. We consider the diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_\gamma,$$

where $\mathbb{E}''$ is the set of all pairs (x, W), consisting of $x \in \mathbb{E}_\gamma$ and an x-stable I-graded subspace W of V_γ such that $\underline{\dim}\, W = \beta$; $\mathbb{E}'$ is the set of all quadruples (x, W, R', R''), where$(x, W) \in \mathbb{E}''$, R' is an invertible linear map $\mathbb{F}_q^\beta \to W$ and R'' an invertible map $\mathbb{F}_q^\alpha \to \mathbb{F}_q^\gamma/W$; $p_2(x, W, R', R'') = (x, W)$, $p_3(x, W) = x$, and $p_1(x, W, R', R'') = (x', x'')$, where $x_\rho R'_{s(\rho)} = R'_{t(\rho)} x'_\rho$ and $x_\rho R''_{s(\rho)} = R''_{t(\rho)} x''_\rho$ for all $\rho \in H$.

Given $f \in \mathbb{C}_G(\mathbb{E}_\alpha)$ and $g \in \mathbb{C}_G(\mathbb{E}_\beta)$ we define

$$f \circ g = (p_3)_!(h) \in \mathbb{C}_G(\mathbb{E}_\gamma),$$

where $h \in \mathbb{C}(\mathbb{E}'')$ is unique such that $p_2^*(h) = p_1^*(f \times g)$. Let

$$\mathbf{m}(\alpha, \beta) = \sum_{i\in I} a_i b_i + \sum_{\rho\in H} a_{s(\rho)} b_{t(\rho)}.$$

We again define the multiplication in the $\mathbb{C}$-space $\mathbf{K} = \oplus_{\alpha\in\mathbb{N}I} \mathbb{C}_G(\mathbb{E}_\alpha)$ by

$$f * g = v^{-\mathbf{m}(\alpha,\beta)} f \circ g$$

for all $f \in \mathbb{C}_G(\mathbb{E}_\alpha)$ and $g \in \mathbb{C}_G(\mathbb{E}_\beta)$. Then $(\mathbf{K}, *)$ becomes an associative $\mathbb{C}$-algebra.

Convention. *Although we are working over finite $\mathbb{F}_q$, we will regularly use G_V and $\mathbb{E}_V$ for the algebraic group and the algebraic variety which are defined over $\mathbb{F}_q$ and use the features of algebraic geometry without introducing extra notations, i. e., the set of $\mathbb{F}_q$-rational points and the algebraic variety are denoted by the same notation. This should not cause any confusion and in particular, the concept of G_V-orbits will be consistent in both cases due to Lang's theorem for this group G_V acting on $\mathbb{E}_V$. For $M \in \mathbb{E}_V$, we will use M to denote the representation of Q on V defined by M.*

For $M \in \mathbb{E}_\alpha$, let $\mathcal{O}_M \subset \mathbb{E}_\alpha$ be the G_α-orbit of M. We take $\mathbf{1}_{[M]} \in \mathbb{C}_G(V_\alpha)$ to be the characteristic function of $\mathcal{O}_M$, and set $f_{[M]} = v^{-\dim \mathcal{O}_M} \mathbf{1}_{[M]}$. We consider the subalgebra $(\mathbf{L}, *)$

[1] In fact we have $\mathbb{E}_V = \oplus_{\rho:i\to j} \mathrm{Hom}(V_i, V_j)$ except in Section 5

of $(\mathbf{K}, *)$ generated by $f_{[M]}$ over $\mathbb{Q}(v)$, for all $M \in \mathbb{E}_\alpha$ and all $\alpha \in \mathbb{N}I$. In fact $\mathbf{L}$ has a $\mathbb{Q}(v)$-basis $\{f_{[M]} | M \in \mathbb{E}_\alpha, \alpha \in \mathbb{N}I\}$, since we have the relation $\mathbf{1}_{[M]} \circ \mathbf{1}_{[N]}(W) = g^W_{MN}$ for any $W \in \mathbb{E}_\gamma$.

Proposition 1.1 *The linear map* $\varphi : (\mathbf{L}, *) \longrightarrow \mathcal{H}^*(\Lambda)$ *defined by*

$$\varphi(f_{[M]}) = \langle M \rangle, \quad \textit{for all } [M] \in \mathcal{P}$$

is an isomorphism of the associative $\mathbb{Q}(v)$*-algebras.*

Proof. Note that φ is a linear isomorphism. For $[M], [N] \in \mathcal{P}$ with $\underline{\dim}\, M = \alpha$ and $\underline{\dim}\, N = \beta$, since $\mathbf{1}_{[M]} \circ \mathbf{1}_{[N]} = \sum_{[L]} g^L_{MN} \mathbf{1}_{[L]}$ in $\mathbf{L}$, we have

$$f_{[M]} * f_{[N]} \;\; = \;\; \sum_{[L] \in \mathcal{P}} v^{-\dim \mathcal{O}_M - \dim \mathcal{O}_N - \mathbf{m}(\alpha,\beta) + \dim \mathcal{O}_L} g^L_{MN} f_{[L]}.$$

Note that $\dim \mathcal{O}_M = \dim G_\alpha - \dim \mathrm{End}_\Lambda(M)$ and $\dim G_{\alpha+\beta} - \dim G_\alpha - \dim G_\beta = \langle \alpha, \beta \rangle + \mathbf{m}(\alpha, \beta)$. In $\mathcal{H}^*(\Lambda)$ we have

$$\begin{aligned}
\langle M \rangle * \langle N \rangle &= v^{-\dim M + \dim \mathrm{End}_\Lambda(M) - \dim N + \dim \mathrm{End}_\Lambda(N) + \langle \alpha, \beta \rangle} u_{[M]} \circ u_{[N]} \\
&= \sum_L v^{\dim \mathrm{End}_\Lambda(M) + \dim \mathrm{End}_\Lambda(N) - \dim \mathrm{End}_\Lambda(L) + \langle \alpha, \beta \rangle} g^L_{MN} \langle L \rangle \\
&= \sum_L v^{\dim G_\alpha - \dim \mathcal{O}_M + \dim G_\beta - \dim \mathcal{O}_N - (\dim G_{\alpha+\beta} - \dim \mathcal{O}_L) + \langle \alpha, \beta \rangle} g^L_{MN} \langle L \rangle \\
&= \sum_L v^{-\dim \mathcal{O}_M - \dim \mathcal{O}_N + \dim \mathcal{O}_L - \mathbf{m}(\alpha,\beta)} g^L_{MN} \langle L \rangle. \;\Box
\end{aligned}$$

1.4 The free abelian group $G(\Lambda) = \mathbb{Z}I$ with the symmetric Euler form $(-,-)$ defined in 1.2 is a Cartan datum in sense of Lusztig [L5]. Associated to $(\mathbb{Z}I, (-,-))$ is the Drinfeld-Jimbo quantized enveloping algebra $U = U^- \otimes U^0 \otimes U^+$ defined over $\mathbb{Q}(t)$, where t is transcendental over $\mathbb{Q}$. It is generated by the Chevalley generators $E_i, F_i, K_i^{\pm}, i \in I$ with respect to the quantum Serre relations. Let $U^+_{\mathcal{Z}}$ be the Lusztig form of U^+ generated by $E_i^{(m)} = \frac{E_i^m}{[m]!}, i \in I$, over $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$. Let $\mathcal{Z}_v$ be the subring of $\mathbb{C}$ as the image of $\mathcal{Z}$ under the map $\mathcal{Z} \to \mathbb{C}$ with $t \mapsto v$. Let $\mathcal{C}^*(\Lambda)_{\mathcal{Z}_v}$ be the $\mathcal{Z}_v$-subalgebra of $\mathcal{H}^*(\Lambda)$ generated by $u^{(*m)}_{[S_i]} = \frac{u^{*m}_{[S_i]}}{[m]_v!}, i \in I$ where

$$[n] = \frac{t^n - t^{-n}}{t - t^{-1}}, \;\; [n]! = \Pi^n_{r=1}[r] \text{ and } \begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}$$

and $[n]_v \in \mathcal{Z}_v$ is the image of $[n]$ in $\mathcal{Z}_v$.

Then we know by Ringel [R1], Green [G] and Sevenhant-Van den Bergh [SV] that $\mathcal{C}^*(\Lambda)_{\mathcal{Z}_v}$ is isomorphic to $U^+_{\mathcal{Z}} \otimes_{\mathcal{Z}} \mathcal{Z}_v$ by sending $u_i^{(*m)}$ to $E_i^{(m)}$.

We will denote $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ for $U^+_{\mathcal{Z}}$ and call it the integral generic composition algebra. In fact, as follows from Ringel's point of view, we may consider $\mathcal{Z}$ as a subring of $\prod_q \mathcal{Z}_{v_q}$ by taking $t = (v_q)$ and $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ as a $\mathcal{Z}$-subalgebra of $\prod_q \mathcal{H}^*_q(\Lambda)$ generated by $(u^{(*m)}_{[S_i \otimes \mathbb{F}_q]})$, $m \geq 1$, where $\mathcal{H}^*_q(\Lambda)$ is the Hall algebra defined using the field $\mathbb{F}_q$. In this paper, we will perform computations in each component of $\prod_q \mathcal{H}^*_q(\Lambda)$. When an expression is written as an element of $\mathbb{Z}[v, v^{-1}]$ with coefficients in $\mathbb{Z}$ are independent of the choice of the field $\mathbb{F}_q$, we say that the expression is invariant as $\mathbb{F}_q$ varies and we simply regard v as t. In this case, we simply say that v is generic in this expression.

There are bar involution $\overline{(\ )} : U^+ \to U^+$ and $U^+_{\mathcal{Z}} \to U^+_{\mathcal{Z}}$ defined by $\overline{v} = v^{-1}$, $\overline{E_i} = E_i$ and $\overline{E_i^{(m)}} = E_i^{(m)}$.

1.5 In general, if we take a special value $v = \sqrt{q}$ for the finite field $\mathbb{F}_q$, it is easy to see that

Lemma 1.2 *Given any monomial $\mathfrak{m}$ of $u_{S_i}^{(m)}$, $i \in I, m \in \mathbb{N}$ we have $\mathfrak{m} = \sum_{M\in\mathcal{P}} f_{M,q}\langle M\rangle$ in $\mathcal{H}^*(\Lambda)$ with $f_{M,q} \in \mathcal{Z}_v$. Then for each M, there is an integer b such that $v^b f_{M,q} \in \mathbb{Z}[v]$ (the subring of algebraic integers) and b is independent of $\mathbb{F}_q$.* $\square$

2. The variety of representations

We need slightly more knowledge about the geometry of representations of quivers. In this section we only consider finite quivers without oriented cycles.

2.1 Let $k = \overline{\mathbb{F}}_q$ be the algebraic closure of the finite field $\mathbb{F}_q$. For any I-graded k-vector space of dimension vector $\alpha \in \mathbb{N}I$, we have the affine variety $\mathbb{E}_\alpha$ and the action of the algebraic group G_α on $\mathbb{E}_\alpha$ over k, in a similar way as in 1.3. For any $x \in \mathbb{E}_\alpha$ we have the corresponding representation $M(x)$ of Q over k. The following properties are well-known (see[CB]).

Lemma 2.1 *For any $\alpha \in \mathbb{N}I$ and $M \in \mathbb{E}_\alpha$, we have*

(1) $\dim \mathbb{E}_\alpha - \dim \mathcal{O}_M = \dim \mathrm{End}(M) - (\alpha,\alpha)/2 = \dim \mathrm{Ext}^1(M,M)$.
(2) *$\mathcal{O}_M$ is open in $\mathbb{E}_\alpha$ if and only if M has no self-extension.*
(3) *There is at most one orbit $\mathcal{O}_M$ in $\mathbb{E}_\alpha$ such that M has no self-extension.*
(4) *If $0 \to M \to L \to N \to 0$ is a non-split exact sequence, then $\mathcal{O}_{M\oplus N} \subseteq \overline{\mathcal{O}}_L \setminus \mathcal{O}_L$.*
(5) *If $\mathcal{O}_L$ is an orbit in $\mathbb{E}_\alpha$ of maximal dimension and $L = M \oplus N$, then* $\mathrm{Ext}^1(M,N) = 0$. $\square$

Let $\mathcal{A} \subset \mathbb{E}_\alpha$ and $\mathcal{B} \subset \mathbb{E}_\beta$, we define the extension set $\mathcal{A} \star \mathcal{B}$ of $\mathcal{A}$ by $\mathcal{B}$ to be

$$\begin{aligned}\mathcal{A}\star\mathcal{B} \ = \ &\{z \in \mathbb{E}_{\alpha+\beta} |\ \text{there exists an exact sequence}\\ &\quad 0 \to M(x) \to M(z) \to M(y) \to 0 \text{ with } x \in \mathcal{B},\ y \in \mathcal{A}\}.\end{aligned}$$

Set $\mathrm{codim}\,\mathcal{A} = \dim \mathbb{E}_\alpha - \dim \mathcal{A}$. It follows from [Re] that

Lemma 2.2 *Given any $\alpha, \beta \in \mathbb{N}I$, we consider the affine varieties $\mathbb{E}_\alpha$ and $\mathbb{E}_\beta$ over k. Let $\mathcal{A} \subset \mathbb{E}_\alpha$ and $\mathcal{B} \subset \mathbb{E}_\beta$. If the both $\mathcal{A}, \mathcal{B}$ are irreducible algebraic varieties and are stable under the action of G_α and G_β respectively, then $\mathcal{A}\star\mathcal{B}$ is irreducible and stable under the action of $G_{\alpha+\beta}$, too. Moreover,*

$$\mathrm{codim}\,\mathcal{A}\star\mathcal{B} = \mathrm{codim}\,\mathcal{A} + \mathrm{codim}\,\mathcal{B} - \langle\beta,\alpha\rangle + r,$$

where $0 \le r \le \min\{\dim_k \mathrm{Hom}(M(y), M(x)) | y \in \mathcal{B}, x \in \mathcal{A}\}$. $\square$

2.2 For any $\alpha, \beta \in \mathbb{N}I$, we consider the diagram

$$\mathbb{E}_\alpha \times \mathbb{E}_\beta \xleftarrow{p_1} \mathbb{E}' \xrightarrow{p_2} \mathbb{E}'' \xrightarrow{p_3} \mathbb{E}_{\alpha+\beta},$$

which is defined over k, by a similar way as in 1.3. It follows from the definition that $\mathcal{A}\star\mathcal{B} = p_3p_2(p_1^{-1}(\mathcal{A}\times\mathcal{B}))$. Thus we have $\overline{\mathcal{A}}\star\overline{\mathcal{B}} \subseteq \overline{\mathcal{A}\star\mathcal{B}}$ since p_1 is a locally trivial fibration (see Lemma 2.3). For any $M \in \mathbb{E}_\alpha, N \in \mathbb{E}_\beta$ and $L \in \mathbb{E}_{\alpha+\beta}$ we define

$$\mathbf{Z} = p_2p_1^{-1}(\mathcal{O}_M \times \mathcal{O}_N),\ \mathbf{Z}_{L,M,N} = \mathbf{Z} \cap p_3^{-1}(L).$$

Then it follows from [L1] that

Lemma 2.3 *For the diagram above and $M \in \mathbb{E}_\alpha, N \in \mathbb{E}_\beta$ and $L \in \mathbb{E}_{\alpha+\beta}$, we have the following properties.*

(1) *The map p_2 is a principal $G_\alpha \times G_\beta$ fibration.*
(2) *The map p_1 is a locally trivial fibration with smooth connected fibres of dimension*

$$\sum_{i\in I} a_i^2 + \sum_{i\in I} b_i^2 + \mathbf{m}(\alpha,\beta).$$

(3) *The map p_3 is proper.*
(4) *The variety $\mathbf{Z}$ is smooth and irreducible of dimension*

$$\dim \mathbf{Z} = \dim(\mathcal{O}_M) + \dim(\mathcal{O}_N) + \mathbf{m}(\alpha,\beta).$$

(5) *If L is an extension of M by N, then*

$$\dim(\mathcal{O}_L) \le \dim(\mathcal{O}_M) + \dim(\mathcal{O}_N) + \mathbf{m}(\alpha,\beta).$$

(6) *If $\mathcal{O}_L$ is dense in $p_3\mathbf{Z}$, then*

$$\dim(\mathcal{O}_L) = \dim(\mathcal{O}_M) + \dim(\mathcal{O}_N) + \mathbf{m}(\alpha,\beta) - \dim \mathbf{Z}_{L,M,N}.$$

(7) *Assume that $\mathrm{Ext}(M,N) = 0$ and $\mathrm{Hom}(N,M) = 0$. If $M' \in \overline{\mathcal{O}}_M$ and $N' \in \overline{\mathcal{O}}_N$ such that either $M' \in \overline{\mathcal{O}}_M \backslash \mathcal{O}_M$ or $N' \in \overline{\mathcal{O}}_N \backslash \mathcal{O}_N$, then $X \in \overline{\mathcal{O}}_{M\oplus N} \backslash \mathcal{O}_{M\oplus N}$ for all $X \in \overline{\mathcal{O}}_{M'} \star \overline{\mathcal{O}}_{N'}$. In particular, $\dim \mathcal{O}_X < \dim \mathcal{O}_{M\oplus N}$.*

As a consequence of Lemma 2.2 we have

Lemma 2.4 *Given any two representations M and N of Q over k, if $\mathrm{Ext}(M,N) = 0$, then $\overline{\mathcal{O}}_M \star \overline{\mathcal{O}}_N = \overline{\mathcal{O}}_{M\oplus N}$, i.e., $\mathcal{O}_{M\oplus N}$ is open and dense in $\overline{\mathcal{O}}_M \star \overline{\mathcal{O}}_N$.* □

Lemma 2.5 *Let $M, N, X \in \mathrm{mod}\,\Lambda$. Then $\mathcal{O}_X$ is open in $\mathcal{O}_M \star \mathcal{O}_N$ if and only if $\mathcal{O}_X$ is open in $\overline{\mathcal{O}}_M \star \overline{\mathcal{O}}_N$. In that case for any $Y \in \overline{\mathcal{O}}_M \star \overline{\mathcal{O}}_N$ we have $\dim \mathcal{O}_Y \le \dim \mathcal{O}_X$.*

Proof. This follows from $\mathcal{O}_X \subseteq \mathcal{O}_M \star \mathcal{O}_N \subseteq \overline{\mathcal{O}}_M \star \overline{\mathcal{O}}_N \subseteq \overline{\mathcal{O}_M \star \mathcal{O}_N}$ and Lemma 2.2. □

3. The integral bases from the Kronecker quiver

The most results in this section can be found in [Z] and [C], and some others maybe in [BK]. For a completeness, we give some proofs here.

3.1 Representations of the Kronecker quiver over finite fields. Let $\mathbb{F}_q$ be the finite field with q elements and Q the Kronecker quiver. Let $\Lambda = \mathbb{F}_q Q$ be the path algebra. It is known that the structure of the preprojective and preinjective components of $\mathrm{mod}\,\Lambda$ is the same as those of $\mathrm{mod}\,kQ$ for k being an algebraically closed field. However the regular components of $\mathrm{mod}\,\Lambda$ is different with that of $\mathrm{mod}\,kQ$.

The set of dimension vectors of indecomposable representations

$$\Phi^+ = \{(l+1,l),(m,m),(n,n+1) | l \ge 0, m \ge 1, n \ge 0\}.$$

The dimension vector $(n+1,n)$ and $(n,n+1)$ correspond to preprojective and preinjective indecomposable representations respectively. For $\alpha = (n+1,n)$ or $(n,n+1)$ (real roots), there is only one isoclass of indecomposable representation with dimension vector α which will be denoted by V_α. Define an order $\prec$ on Φ^+ by

$$(1,0) \prec \cdots \prec (m+1,m) \prec (m+2,m+1) \prec \cdots \prec (k,k) \prec (k+1,k+1)$$
$$\prec \cdots \prec (n+1,n+2) \prec (n,n+1) \prec \cdots \prec (0,1).$$

Any Λ-module is given by the date $(V_1, V_2; \sigma, \tau)$, where V_1 and V_2 are finite dimensional vector space over $\mathbb{F}_q$, σ and τ are $\mathbb{F}_q$-linear maps from V_2 to V_1.

Proposition 3.1. *The isomorphism classes of the regular quasi-simple modules in* $\operatorname{mod}\Lambda$ *are indexed by spec*$(\mathbb{F}_q[x])$. *That is, each regular quasi-simple module is isomorphic to the representation* $(V_1, V_2; \sigma, \tau)$, *where* $V_1 = V_2 = \mathbb{F}_q[x]/(p(x))$ *for an irreducible polynomial* $p(x)$ *in* $\mathbb{F}_q[x]$, σ *is the identity map and* τ *is given by the multiplication by* x, *except* $(\mathbb{F}_q, \mathbb{F}_q; 0, 1)$.

3.2 The integral bases from the representations. In this section, let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional Λ-modules, $\mathcal{H}$ be the Ringel-Hall algebra of Λ over $\mathbb{Q}(v)$, where $v^2 = q$, and $\mathcal{H}^*$ be the twisted form of $\mathcal{H}$. If $\mathbf{d} \in \mathbb{N}I$ be a dimension vector, we set in $\mathcal{H}$

$$R_{\mathbf{d}} = \sum_{\substack{[M]\in\mathcal{P}, M \text{ regular} \\ \underline{\dim}\, M = \mathbf{d}}} u_{[M]}.$$

For an element $x = \sum_{[M]\in\mathcal{P}} c_{[M]} u_{[M]} \in \mathcal{H}$, we call $u_{[M]}$ to be a (non-zero) term of x if $c_{[M]} \neq 0$. Furthermore,

$$R(x) = \sum_{[M]\in\mathcal{P}, M \text{ regular}} c_{[M]} u_{[M]}$$

is called the regular part of x. According to our notation, we denote $u_\alpha = u_{[V_\alpha]}$ for $\alpha = (n-1, n)$ or $(n, n+1)$ being real roots.

Let $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$. be the simple root vectors. The orientation of Q implies $\langle \alpha_1, \alpha_2 \rangle = 0$ and $\langle \alpha_2, \alpha_1 \rangle = -2$. Thus for $\delta = (1, 1)$ we have $\langle \delta, \alpha_1 \rangle = -1$, $\langle \alpha_1, \delta \rangle = 1$, $\langle \delta, \alpha_2 \rangle = 1$ and $\langle \alpha_2, \delta \rangle = -1$.

3.3 In this section, the multiplication in $\mathcal{H}$ will be simply written as xy instead of $x \circ y$. The following can computed easily as in [Z].

Lemma 3.2. *Let i and j be two positive integers. Then*

$$u_{(j-1,j)} u_{(i,i-1)} = R(u_{(j-1,j)} u_{(i,i-1)}) + q^{i+j-2} u_{(i,i-1)} u_{(j-1,j)}.$$

Lemma 3.3

$$\begin{aligned} R_\delta &= u_{(0,1)} u_{(1,0)} - u_{(1,0)} u_{(0,1)}, \\ u_{(n+1,n)} &= \frac{1}{q+1}(R_\delta u_{(n,n-1)} - q u_{(n,n-1)} R_\delta), \\ u_{(n,n+1)} &= \frac{1}{q+1}(u_{(n-1,n)} R_\delta - q R_\delta u_{(n-1,n)}). \end{aligned}$$

Lemma 3.4 *Let i and j be two positive integers and $n = i + j - 1$. Then*

$$R(u_{(j-1,j)} u_{(i,i-1)}) = R(u_{(n-1,n)} u_{(1,0)}) = R(u_{(0,1)} u_{(n,n-1)}).$$

Lemma 3.5 *Let $m, n \geq 1$. Then*

$$\begin{aligned} u_{(m-1,m)}R_{n\delta} &= \sum_{0\leq i\leq n} \frac{q^i - q^{n+1}}{1-q} R_{i\delta} u_{(m+n-i-1,m+n-i)}, \\ R_{n\delta}u_{(m,m-1)} &= \sum_{0\leq i\leq n} \frac{q^i - q^{n+1}}{1-q} u_{(m+n-i,m+n-i-1)} R_{i\delta}. \end{aligned}$$

3.4 We give here some quantum commutative relations in $\mathcal{H}$ and in $\mathcal{H}^*$. We define

$$\begin{aligned} E_{(n+1,n)} &= \langle u_{(n+1,n)}\rangle = v^{-\dim V_{(n+1,n)}+\dim \operatorname{End} V_{(n+1,n)}} u_{(n+1,n)} = v^{-2n} u_{(n+1,n)}, \\ E_{(n,n+1)} &= \langle u_{(n,n+1)}\rangle = v^{-\dim V_{(n,n+1)}+\dim \operatorname{End} V_{(n,n+1)}} u_{(n,n+1)} = v^{-2n} u_{(n,n+1)}. \end{aligned}$$

Set b $E_1 = E_{(1,0)}, E_2 = E_{(0,1)}$ for the Chevalley generators and, for $n \geq 1$,

$$\tilde{E}_{n\delta} = E_{(n-1,n)} * E_1 - v^{-2} E_1 * E_{(n-1,n)}.$$

In the following we give a sequence of computations we will need. Most of them are known.

Lemma 3.6 $\tilde{E}_{n\delta} = v^{-3n+1} R(u_{(n-1,n)}u_{(1,0)})$.
Proof. By taking $u_1 = u_{(1,0)}$ we have

$$\begin{aligned} \tilde{E}_{n\delta} &= v^{-2(n-1)}\big(v^{\langle (n-1)\delta+\alpha_2,\alpha_1\rangle} u_{(n-1,n)}u_1 - v^{-2} v^{\langle \alpha_1,(n-1)\delta+\alpha_2\rangle} u_1 u_{(n-1,n)}\big) \\ &= v^{-3n+1}\big(u_{(n-1,n)}u_1 - v^{2(n-1)} u_1 u_{(n-1,n)}\big) \\ &= v^{-3n+1} R(u_{(n-1,n)}u_1) \quad \text{by Lemma 3.2.} \quad \square \end{aligned}$$

Lemma 3.7 *In $\mathcal{H}^*$ we have*

$$\begin{aligned} [\tilde{E}_\delta, E_{(n+1,n)}] &= [2]_v E_{(n+2,n+1)}, \\ [E_{(n,n+1)}, \tilde{E}_\delta] &= [2]_v E_{(n+1,n+2)}. \end{aligned}$$

Proof. We only check the first equation. By definition and Lemma 3.3, we

$$\begin{aligned} [\tilde{E}_\delta, E_{(n+1,n)}] &= v^{-2(n+1)} v^{\langle \delta, n\delta+\alpha_1\rangle} R_\delta u_{(n+1,n)} - v^{-2(n+1)} v^{\langle n\delta+\alpha_1,\delta\rangle} u_{(n+1,n)} R_\delta \\ &= v^{-2(n+1)} v^{-1}((q+1)u_{(n+2,n+1)} + q u_{(n+1,n)} R_\delta) - v^{-2(n+1)} v u_{(n+1,n)} R_\delta \\ &= v^{-2(n+1)}(v+v^{-1}) u_{(n+2,n+1)} = [2]_v E_{(n+2,n+1)}. \square \end{aligned}$$

Lemma 3.8 $E_{(2,1)} * E_1 = v^2 E_1 * E_{(2,1)}$ *and* $E_2 * E_{(1,2)} = v^2 E_{(1,2)} * E_2$.
Proof. Let $M = V_{(1,0)} \oplus V_{(2,1)}$. Then $E_{(2,1)} * E_1 = v^{-2} v^{\langle \delta+\alpha_1,\alpha_1\rangle} u_{(2,1)} u_1 = v^2 u_\beta$ and $E_1 * E_{(2,1)} = v^{-2} v^{\langle \alpha_1, \delta+\alpha_1\rangle} u_1 u_{(2,1)} = u_\beta$. This proves the first equality. The second equality follows from a similar computation. $\square$

Lemma 3.9 *For any non-negative integers r and s, we have in $\mathcal{H}^*$*

$$\tilde{E}_{(r+s+1)\delta} = E_{(r,r+1)} * E_{(s+1,s)} - v^{-2} E_{(s+1,s)} * E_{(r,r+1)}.$$

Proof. Using Lemma 3.2, we have

$$\begin{aligned}
&E_{(r,r+1)} * E_{(s+1,s)} - v^{-2}E_{(s+1,s)} * E_{(r,r+1)} \\
&= v^{-3(r+s)-2}u_{(r,r+1)}u_{(s+1,s)} - v^{-(r+s)-2}u_{(s+1,s)}u_{(r,r+1)} \\
&= v^{-3(r+s)-2}(R(u_{(r,r+1)}u_{(s+1,s)}) + q^{r+s}u_{(s+1,s)}u_{(r,r+1)}) - v^{-(r+s)-2}u_{(s+1,s)}u_{(r,r+1)} \\
&= v^{-3(r+s)-2}R(u_{(r,r+1)}u_{(s+1,s)}) = v^{-3(r+s)-2}R(u_{(r+s,r+s)}u_1) = \tilde{E}_{(r+s+1)\delta}.\Box
\end{aligned}$$

Lemma 3.10 *There exist $a_h^{(r)} \in \mathbb{Z}[v, v^{-1}]$ for all $r \in \mathbb{N} \setminus \{0\}$ and $h \in \{0, 1, \cdots, \lfloor \frac{r}{2} \rfloor\}$ such that for all $n > m$ in $\mathbb{N}$,*

$$\begin{aligned}
E_{(n+1,n)} * E_{(m+1,m)} &= \sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} a_h^{(n-m)} E_{(m+h+1,m+h)} * E_{(n-h+1,n-h)} \\
E_{(m,m+1)} * E_{(n,n+1)} &= \sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} a_h^{(n-m)} E_{(n-h+1,n-h)} * E_{(m+h,m+h+1)}
\end{aligned}$$

Proof. We verify the first equation only.

$$\begin{aligned}
E_{(n+1,n)} * E_{(m+1,m)} &= v^{-2(n+m)}v^{\langle n\delta+\alpha_1, m\delta+\alpha_1\rangle}u_{(n+1,n)}u_{(m+1,m)} \\
&= v^{-2(n+m)}v^{-n+m+1}\sum_{h=0}^{\lfloor \frac{n-m}{2} \rfloor} g^{M_h}_{V_{(n+1,n)}V_{(m+1,m)}}u_{[M_h]}.
\end{aligned}$$

where $M_h = V_{(n-h+1,n-h)} \oplus V_{(n+h+1,n+h)}$. Since $n-h \geq m+h$, the strong representation-directed property implies

$$E_{(m+h+1,m+h)} * E_{(n-h+1,n-h)} = v^{-2(n+m)}v^{-m+n-2h+1}g^{M_h}_{V_{(m+h+1,m+h)}V_{(n-h+1,n-h)}}u_{[M_h]}.$$

Therefore $E_{(m+h+1,m+h)} * E_{(n-h+1,n-h)} = v^{-3m-n-2h+1}u_{\lambda(h)}$ if $m+h \neq n-h$ or $E_{(m+h+1,m+h)} * E_{(n-h+1,n-h)} = v^{-3m-n-2h+1}(q+1)u_{\lambda(h)}$ if $m+h = n-h$. By comparing the coefficients of the two sides, we may find the precise numbers $a_h^{(r)} \in \mathbb{Z}[v, v^{-1}]$ for the equation. $\Box$

For $k \geq 0$, we inductively define

$$E_{0\delta} = 1, \quad E_{k\delta} = \frac{1}{[k]}\sum_{s=1}^{k} v^{s-k}\tilde{E}_{s\delta} * E_{(k-s)\delta}.$$

Lemma 3.11 *We have $E_{k\delta} = v^{-2k}R_{k\delta}$.*

Proof. If $k = 1$, $E_\delta = \tilde{E}_\delta = v^{-2}R_\delta$. We assume that the assertion is true for all numbers $t < k$. Then using Lemma 3.6, and [Z] (Lem 3.7, Thm 4.1, Lem 4.7) , we have

$$\begin{aligned}
E_{k\delta} &= \frac{1}{[k]}\sum_{s=1}^{k} v^{s-k}v^{-3s+1}R(u_{(s-1,s)}u_1)*v^{-2(k-s)}R_{(k-s)\delta}\\
&= \frac{1}{[k]}\sum_{s=1}^{k} v^{-3k+1}R(u_{(s-1,s)}u_1)*R_{(k-s)\delta}\\
&= \frac{1}{[k]}\sum_{s=1}^{k} v^{-3k+1}a_s(R_\delta,R_{2\delta},\cdots,R_{s\delta})R_{(k-s)\delta}\\
&= \frac{v^{-3k+1}}{[k]}\frac{1-q^k}{1-q}R_{k\delta} = v^{-2k}R_{k\delta}. \quad \square
\end{aligned}$$

Lemma 3.12 *For $m,n\in\mathbb{N}$ we have in $\mathcal{H}^*$*

$$\begin{aligned}
E_{n\delta}*E_{(m+1,m)} &= \sum_{k=0}^{n}[n+1-k]E_{(m+n+1-k,m+n-k)}*E_{k\delta};\\
E_{(m,m+1)}*E_{n\delta} &= \sum_{k=0}^{n}[n+1-k]E_{k\delta}*E_{(m+n-k,m+n-k+1)}.
\end{aligned}$$

Proof. Again it only needs to verify the first equation. By Lemma 3.11, we have

$$\begin{aligned}
E_{n\delta}*E_{(m+1,m)} &= v^{-2n}R_{n\delta}*v^{-2m}u_{(m+1,m)} = v^{-2(n+m)}v^{-n}R_{n\delta}u_{(m+1,m)}\\
&= v^{-3n-2m}\sum_{k=0}^{n}\frac{q^k-q^{n+1}}{1-q}u_{(m+n-k+1,m+n-k)}R_{k\delta} \quad \text{(by Lemma 3.5)}\\
&= \sum_{k=0}^{n}\frac{v^{-3n-2m}}{1-v^2}(v^{2k}-v^{2n+2})v^{2k}v^{-k}v^{2(n+m-k)}E_{(m+n-k+1,m+n-k)}*E_{k\delta}\\
&= \sum_{k=0}^{n}[n+1-k]E_{(m+n-k+1,m+n-k)}*E_{k\delta}.\square
\end{aligned}$$

3.5 Let $\mathcal{L}$ be the $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$-subalgebra of $\mathcal{H}^*$ generated by the set

$$\{E^{(s)}_{(m+1,m)},E_{k\delta},E^{(t)}_{(n,n+1)}|m\geq 0,n\geq 0,s\geq 1,t\geq 1,k\geq 1\}.$$

It contains the divided powers $E_1^{(s)}, E_2^{(t)}$, $s,t\in\mathbb{N}$, of the Chevalley generators. We have obtained an integral basis of $\mathcal{L}$ over $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$ consisting of the monomials on the vectors

$$\{E^{(s)}_{(m+1,m)},E_{k\delta},E^{(t)}_{(n,n+1)}|m\geq 0,n\geq 0,s\geq 1,t\geq 1,k\geq 1\}$$

with respect to the order given in 3.1. This conclusion can be easily seen from the facts: (1) the commutation relations in the above lemmas imply that any element in the algebra $\mathcal{L}$ is a $\mathcal{Z}$-linear combination of the monomials we defined above; (2) those monomials are linearly independent over $\mathcal{Z}$ (even over $\mathbb{Q}(v)$) by the definition of Ringel-Hall algebras.

Notice that our formulas in the lemmas are unchanged when we take $v=\sqrt{q}$ for different finite fields $\mathbb{F}_q$. This means that the vectors and the formulas are obtained in their generic forms. We may regard the parameter v transcendental over $\mathbb{Q}$. As previously we have defined that $\mathcal{C}^*_{\mathcal{Z}}$ is the integral form of the generic composition algebra. By the well-known Ringel-Green theorem (see [G],[R1]), we have $\mathcal{C}^*_{\mathcal{Z}} \simeq U^+_{\mathcal{Z}}$, where $U^+_{\mathcal{Z}}$ is the Lusztig integral form of the positive part of $U_q(\hat{sl}_2)$, and E_1, E_2 are chosen as the Chevalley generators.

For any $n > m \geq 0$, let $P_{(n,m)}$ (resp. $I_{(m,n)}$) be an isomorphism class of preprojective (resp. preinjective) representation with $\underline{\dim}\, P_{(n,m)} = (n,m)$ (resp. $\underline{\dim}\, I_{(m,n)} = (m,n)$). In the following formulas, the summation is taken over all nonzero preprojective and preinjective representations of indicated dimension vectors.

Lemma 3.13 *In the following formulas all P and I are non-zero.*

$$(1)\quad E_2^{(*n)} * E_1^{(*n+1)} = E_{(n+1,n)} + \sum_{1\leq l\leq n} v^{-l-1} E_{(n-l+1,n-l)} * E_{l\delta}$$
$$+ \sum_{\substack{0\leq l\leq n-1\\ p\geq 1, s\geq 0, t\geq 0\\ s+t+l+(p-1)=n}} v^{-\dim \mathrm{End}(P) - dim\, \mathrm{End}(I)} v^{-p(l+t)-(s+l)(p-1)} \langle P_{(s+p,s)}\rangle * E_{l\delta} * \langle I_{(t,t+p-1)}\rangle;$$

$$(2)\quad E_2^{(*n+1)} * E_1^{(*n)} = E_{(n,n+1)} + \sum_{1\leq l\leq n} v^{-l-1} E_{l\delta} * E_{(n-l,n-l+1)}$$
$$+ \sum_{\substack{0\leq l\leq n-1\\ p\geq 1, s\geq 0, t\geq 0\\ s+t+l+(p-1)=n}} v^{-\dim \mathrm{End}(P) - dim\, \mathrm{End}(I)} v^{-p(l+s)+(t+l)(p-1)} \langle P_{(s+p-1,s)}\rangle * E_{l\delta} * \langle I_{(t,t+p)}\rangle;$$

$$(3)\quad E_2^{(*n)} * E_1^{(*n)} = E_{n\delta}$$
$$+ \sum_{\substack{0\leq l\leq n-1, p\geq 1\\ s\geq 0, t\geq 0, s+t+l+p=n}} v^{-\dim \mathrm{End}(P) - \dim \mathrm{End}(I)} v^{-p(s+2l+t)} \langle P_{(s+,s)}\rangle * E_{l\delta} * \langle I_{(t,t+p)}\rangle.$$

Proof. We only verify the first relation in (1), the others can be verified in a similar way. We have the following relation in $\mathcal{H}$ (see [R3]).

$$u_2^n u_1^{n+1} = \psi_n(q)\psi_{n+1}(q)(u_{(n+1,n)} + \sum_{1\leq l\leq n} u_{(n-l+1,n-l)} R_{l\delta} + \sum_{\substack{0\leq l\leq n-1, p\geq 1\\ s\geq 0, t\geq 0, s+t+l+(p-1)=n}} u_{[P]} R_{l\delta} u_{[I]})$$

where P is a non-zero preprojective representation with $\underline{\dim}\, P = (s+p, s)$ and I is a non-zero preinjective representation with $\underline{\dim}\, I = (t, t+p-1)$ and

$$\psi_n(q) = \frac{(1-q)\cdots(1-q^n)}{(1-q)^n}.$$

Then by a routing calculation according to the relation in 3.2, we have the relation in (1). □

3.6 Note that the dimensions of $P, I, \mathrm{End}\, I$ and $\mathrm{End}\, P$ over $\mathbb{F}_q$ are invariant as $\mathbb{F}_q$ varies. By induction using Lemma 3.13, the set

$$\{E_{(m+1,m)}, E_{k\delta}, E_{(n,n+1)} | m\geq 0, n\geq 0, k\geq 1\}$$

lie in $\mathcal{C}^*_{\mathcal{Z}}$. If M is indecomposable preprojective or preinjective, then, by [R3],

$$\langle u_M\rangle^{(*s)} = \langle u_{sM}\rangle \in \mathcal{C}^*(\Lambda)_{\mathcal{Z}} \text{ for any } \; s\geq 1.$$

Hence the subset

$$\{E^{(s)}_{(m+1,m)}, E_{k\delta}, E^{(t)}_{(n,n+1)} | m \geq 0, n \geq 0, s \geq 1, t \geq 1, k \geq 1\}$$

is also contained in $\mathcal{C}^*_{\mathcal{Z}}$. Therefore, $\mathcal{L} = \mathcal{C}^*_{\mathcal{Z}}$.

Let $\mathbf{P}(n)$ the set of all partitions of n. Recall that there are no nontrivial extensions between homogeneous regular representation. For any $w = (w_1, w_2, \cdots, w_m) \in \mathbf{P}(n)$, we define

$$E_{w\delta} = E_{w_1\delta} * E_{w_2\delta} * \cdots * E_{w_m\delta}.$$

Proposition 3.14 *The set*

$$\{\langle P\rangle * E_{w\delta} * \langle I\rangle \| P \in \mathcal{P} preprojective, w \in \mathbf{P}(n), I \in \mathcal{P} preinjective, n \in \mathbb{N}\}$$

is an integral basis of $\mathcal{C}^*_{\mathcal{Z}}$. $\square$

Remarks. (1) It has been proved by Zhang in [Z] that these monomials are $\mathbb{Q}(v)$-bases of U^+, then improved by Chen in [C] that they are $\mathcal{Z}$-bases of $U^+_{\mathcal{Z}}$.

(2) It is not difficult to see that the root vectors provided here exactly correspond to the root vectors of $U_q(\hat{sl}_2)$ provided by Damiani in [Da] and by Beck in [Be].

(3) It can be proved that the set in Proposition 3.14 is an integral basis of $\mathcal{C}^*$ over $\mathcal{A} = \mathbb{Q}[v, v^{-1}]$ by an easier way, see the proofs of Proposition 7.2 and 7.3 below.

4. The canonical bases from the Kronecker quiver

4.1 Recall from 3.1 that Φ^+ is the positive root system of $\hat{sl}_2$. A function $\mathbf{c} : \Phi^+ \to \mathbb{N}$ is called support-finite, if $\mathbf{c}(\alpha) \neq 0$ only for finitely many $\alpha \in \Phi^+$. By abusing the notation, let $\mathbb{N}^{\Phi^+}$ be the set of all support-finite $\mathbb{N}$-valued functions. We will use the order in Φ^+ given in 3.1.

For $\mathbf{c} \in \mathbb{N}^{\Phi^+}$, if $\{\alpha \in \Phi^+ | \mathbf{c}(\alpha) \neq 0\} = \{\beta_1 \prec \beta_2 \prec \cdots \prec \beta_k\}$, we set

$$E^{\mathbf{c}} = E^{(*\mathbf{c}(\beta_1))}_{\beta_1} * E^{(*\mathbf{c}(\beta_2))}_{\beta_2} * \cdots * E^{(*\mathbf{c}(\beta_k))}_{\beta_k},$$

where $E^{(*\mathbf{c}(\beta_k))}_{\beta_k} \triangleq E^{*\mathbf{c}(\beta_k)}_{\beta_k}$ if $\beta_i = m\delta$. Then Proposition 3.14 is equivalent to the statement:

The set $\{E^{\mathbf{c}} | \mathbf{c} \in \mathbb{N}^{\Phi^+}\}$ *is a* $\mathcal{Z}$*-basis of* $\mathcal{C}^*$.

For $\mathbf{d} = (d_1, d_2) \in \mathbb{N}^2$, we denote

$$E(\mathbf{d}) = E^{(*d_2)}_2 * E^{(*d_1)}_1.$$

Similarly we defined

$$E(\mathbf{c}) = E(\mathbf{c}(\beta_1)\beta_1) * E(\mathbf{c}(\beta_2)\beta_2) * \cdots * E(\mathbf{c}(\beta_k)\beta_k).$$

Note that $E(\mathbf{c}) \in \mathcal{C}^*$ since it is a monomial on the Chevalley generators E_1 and E_2 in the form of divided powers. Moreover, by definition, we $\overline{E(\mathbf{d})} = E(\mathbf{d})$. Thus $\overline{E(\mathbf{c})} = E(\mathbf{c})$.

4.2 The rest of this section is devoted to giving a triangular relation between the PBW-basis and the monomial basis.

For any $\mathbf{c} \in \mathbb{N}^{\Phi^+}$ we assume that $E^{\mathbf{c}} = \langle P\rangle * E_{\omega\delta} * \langle I\rangle$, where P is a preprojective module and I is a preinjective module. Let $V_{\omega\delta}$ be a module such that

$$\dim \mathcal{O}_{V_{\omega\delta}} = max\{\dim \mathcal{O}_V | E_{w\delta} = E_{w_1\delta} * E_{w_2\delta} * \cdots * E_{w_m\delta} = \sum_V a_V \langle V\rangle \text{ with } a_V \neq 0\}.$$

We choose[2]

$$V_{\mathbf{c}} = P \oplus V_{\omega\delta} \oplus I, \quad \text{and} \quad \mathcal{O}_{\mathbf{c}} = \mathcal{O}_{P\oplus V_{\omega\delta}\oplus I}.$$

For any $\mathbf{c} \in \mathbb{N}^{\Phi^+}$ and any real root $\alpha \in \Phi^+$, define $u_{\mathbf{c}(\alpha)\alpha} = u_{[V_\alpha\oplus\cdots\oplus V_\alpha]}$, where V_α is the indecomposable representation with $\underline{\dim}\, V_\alpha = \alpha$.

Lemma 4.2 *For any $\mathbf{c} \in \mathbb{N}^{\Phi^+}$ and any real root $\beta \in \Phi^+$, we have in $\mathcal{C}^*$*

$$E(\mathbf{c}(\beta)\beta) = \langle u_{\mathbf{c}(\beta)\beta}\rangle + \sum_{\substack{\mathbf{c}'\in\mathbb{N}^{\Phi^+} \\ \dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{\mathbf{c}(\beta)\beta}}} v^{-\lambda(\mathbf{c}')}E^{\mathbf{c}'},$$

where $\lambda(\mathbf{c}') \in \mathbb{N}$.

Proof. Let $\mathbf{c}(\beta)\beta = (m,n)$. In $\mathcal{H}$ we have

$$u_2^m u_1^n = \psi_m(q)\psi_n(q)\sum_{\underline{\dim}\, N=(m,n)} u_{[N]}.$$

By Lemma 2.1

$$u_2^m u_1^n = \psi_m(q)\psi_n(q)u_{[V_\beta\oplus\cdots\oplus V_\beta]} + \psi_m(q)\psi_n(q)\sum u_{[P]}R_{l\delta}u_{[I]},$$

where P is preprojective, I is preinjective, $\underline{\dim}\, P + l\delta + \underline{\dim}\, I = (m,n)$, and $\dim\mathcal{O}_{P\oplus V_{l\delta}\oplus I} < \dim\mathcal{O}_{\mathbf{c}(\beta)\beta}$. Then in $\mathcal{C}^*$,

$$\begin{aligned} u_2^{(*m)} * u_1^{(*n)} &= \frac{v^{m(m-1)/2}v^{n(n-1)/2}}{[m]![n]!}v^{-2mn}u_2^m u_1^n \\ &= v^{m^2-m+n^2-n-2mn}u_{[V_\beta\oplus\cdots\oplus V_\beta]} + v^{m^2-m+n^2-n-2mn}\sum u_{[P]}R_{l\delta}u_{[I]} \\ &= \langle u_{\mathbf{c}(\beta)\beta}\rangle + \sum_{\substack{\mathbf{c}'\in\mathbb{N}^{\Phi^+} \\ \dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{\mathbf{c}(\beta)\beta}}} v^{-\lambda(\mathbf{c}')}E^{\mathbf{c}'}. \ \square \end{aligned}$$

Lemma 4.3 *Let $\alpha, \beta \in \Phi^+$ be real roots and $\alpha \prec \beta$. We have in $\mathcal{C}^*$*

$$E(\alpha) * E(\beta) = \langle u_{[V_\alpha]}\rangle * \langle u_{[V_\beta]}\rangle + \sum_{\substack{\mathbf{c}\in\mathbb{N}^{\Phi^+} \\ \dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_{V_\alpha\oplus V_\beta}}} h_{\mathbf{c}}^{\alpha\beta}E^{\mathbf{c}},$$

where $h_{\mathbf{c}}^{\alpha\beta} \in \mathcal{Z}$.

Proof. By Lemma 3.13 we have

$$\begin{aligned} E(\alpha) &= \langle u_\alpha\rangle + \sum_{\substack{\mathbf{c}'\in\mathbb{N}^{\Phi^+} \\ \dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{V_\alpha}}} v^{-\lambda(\mathbf{c}')}E^{\mathbf{c}'}, \\ E(\beta) &= \langle u_\beta\rangle + \sum_{\substack{\mathbf{c}''\in\mathbb{N}^{\Phi^+} \\ \dim\mathcal{O}_{\mathbf{c}''}<\dim\mathcal{O}_{V_\beta}}} v^{-\lambda(\mathbf{c}'')}E^{\mathbf{c}''}. \end{aligned}$$

Since $\mathrm{Hom}_\Lambda(V_\beta, V_\alpha) = \mathrm{Ext}_\Lambda(V_\alpha, V_\beta) = 0$ and $\dim Z_{V_\alpha\oplus V_\beta, V_\alpha, V_\beta} = 0$, by Lemma 2.3(7),

$$\dim\mathcal{O}_{\mathbf{c}} < \dim\mathcal{O}_{V_\alpha\oplus V_\beta}$$

[2]This selection is not unique, in fact we may require that $V_{m\delta}$ is absolutely indecomposable in a homogeneous tube.

for any extension $V_{\mathbf{c}}$ of $V_{\mathbf{c}'}$ by $V_{\mathbf{c}''}$ with the property:

$$\mathcal{O}_{\mathbf{c}'} \subseteq \overline{\mathcal{O}_{V_\alpha}} \setminus \mathcal{O}_{V_\alpha}, \text{or } \mathcal{O}_{\mathbf{c}''} \subseteq \overline{\mathcal{O}_{V_\beta}} \setminus \mathcal{O}_{V_\beta}.$$

Therefore, the conclusion follows from Proposition 3.14. □

Lemma 4.4 *Let* $\alpha = (n+1, n)$, $\beta = (l, l) = l\delta$, *and* $\gamma = (m, m+1)$ *be in* Φ^+. *The for all* $s \geq 1$

$$(1) \qquad E(s\alpha) * E(\beta) = \langle u_{[sV_\alpha]} \rangle * E_{l\delta} + \sum_{\substack{\mathbf{c} \in \mathbb{N}^{\Phi^+} \\ \dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_{sV_\alpha \oplus V_{l\delta}}}} h(\mathbf{c}) E^{\mathbf{c}},$$

$$(2) \qquad E(\beta) * E(s\gamma) = E_{l\delta} * \langle u_{[sV_\gamma]} \rangle + \sum_{\substack{\mathbf{c}' \in \mathbb{N}^{\Phi^+} \\ \dim \mathcal{O}_{\mathbf{c}'} < \dim \mathcal{O}_{V_{l\delta} \oplus sV_\gamma}}} h(\mathbf{c}') E^{\mathbf{c}'}.$$

Here $h(\mathbf{c}')$, $h(\mathbf{c}) \in \mathcal{Z}$.

Proof. By Lemma 2.3(7) and Lemma 3.13(3), then the proof is same as that of Lemma 4.3. □

Lemma 4.5 *Let* V *be an indecomposable regular module with* $\underline{\dim} V = n\delta$. $M = P \oplus M' \oplus I$ *with* $P \neq 0, M', I \neq 0$ *are respectively preprojective,regular and preinjective modules and* $\underline{\dim} M = n\delta$. *Then* $\dim \mathcal{O}_V > \dim \mathcal{O}_M$.

Proof. By Lemma 2.1(1), we only prove that $\dim \mathrm{End}(V) < \dim \mathrm{End}(M)$. By the structure of representations of Kronecker quiver, we have $\dim \mathrm{End}(V) = n$. Suppose

$$P = P_1 \oplus P_2 \oplus \cdots \oplus P_r, \quad \text{and} \quad I = I_1 \oplus I_2 \oplus \cdots \oplus I_t,$$

where P_j, and $I_j (j \geqslant 1)$ are respectively indecomposable preprojective and preinjective modules with $\underline{\dim} P_j = (n_j + 1, n_j)$, $\underline{\dim} I_j = (m_j, m_j + 1)$, and $\underline{\dim} M' = s\delta$. Thus $r = t$ and $n = \sum_{j=1}^{t} (n_j + 1) + s + \sum_{j=1}^{t} (m_j)$. Note that

$$\begin{aligned} \dim \mathrm{End}(P) \geqslant& t, \quad \dim \mathrm{End}(M') = s, \quad \dim \mathrm{End}(I) \geqslant t, \\ \dim \mathrm{Hom}(P, M') =& \langle \underline{\dim} P, \underline{\dim} M' \rangle = st, \quad \dim \mathrm{Hom}(M', I) = st, \\ \dim \mathrm{Hom}(P, I) =& \langle \underline{\dim} P, \underline{\dim} I \rangle = t(\sum_{j=1}^{t} n_j + \sum_{j=1}^{t} m_j). \end{aligned}$$

Using the direct sum decomposition of M, one computes

$$\dim \mathrm{End}(M) \geqslant t + s + t + st + t(\sum_{j=1}^{t} n_j + \sum_{j=1}^{t} m_j) + st \geqslant 2t + \sum_{j=1}^{t} n_j + \sum_{j=1}^{t} m_j > n.$$

This implies that $\dim \mathcal{O}_V > \dim \mathcal{O}_M$. □

Lemma 4.6 *Let* $n \geqslant 1, m \geqslant 1$. *Then*

$$E(n\delta) * E(m\delta) = E_{n\delta} * E_{m\delta} + \sum_{\substack{\mathbf{c} \in \mathbb{N}^{\Phi^+} \\ \dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_{V_{(n,m)\delta}} = \dim \mathcal{O}_{V_{(n+m)\delta}}}} h(\mathbf{c}) E^{\mathbf{c}},$$

where $V_{(n,m)\delta}$ *is defined in* 4.2 *and* $h(\mathbf{c}) \in \mathcal{Z}$.

Proof. By Lemma 3.13(3), we have

$$\begin{aligned} E(n\delta) &= E_{n\delta} + \sum_{P\neq 0, I\neq 0} v^{l(\langle P\rangle * E_{l\delta} * \langle I\rangle)} \langle P\rangle * E_{l\delta} * \langle I\rangle, \\ E(m\delta) &= E_{m\delta} + \sum_{P\neq 0, I\neq 0} v^{l(\langle P\rangle * E_{l\delta} * \langle I\rangle)} \langle P\rangle * E_{l\delta} * \langle I\rangle, \end{aligned}$$

where $l(\langle P\rangle * E_{l\delta} * \langle I\rangle) \in \mathbb{Z}$.

We then have

$$E(n\delta) * E(m\delta) = E_{n\delta} * E_{m\delta} + \sum h(\mathbf{c}) E^{\mathbf{c}}.$$

To prove the lemma, it is sufficient to prove that $V_{\mathbf{c}}$, which is defined in 4.2, is decomposable. This is easy to see because the structure of the AR-quiver of Kronecker quiver. □

Remark. By Lemma 4.6 we can get

$$E(\omega\delta) = E(\omega_1\delta) * \cdots * E(\omega_m\delta) = E_{\omega\delta} + \sum_{\dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_{\omega\delta}} h(\mathbf{c}) E^{\mathbf{c}},$$

where $h(\mathbf{c}) \in \mathcal{Z}$.

Let $\varphi : \mathbb{N}^{\Phi^+} \to \mathbb{N}^2$ be defined by $\varphi(\mathbf{c}) = \sum_{\alpha\in\Phi^+} \mathbf{c}(\alpha)\alpha$. Then for any $\mathbf{d} \in \mathbb{N}^2$, $\varphi^{-1}(\mathbf{d})$ is a finite set. We define a (geometric) order in $\varphi^{-1}(\mathbf{d})$ as follows: $\mathbf{c}' \preceq \mathbf{c}$ *if and only if* $\mathbf{c}' = \mathbf{c}$ *or* $\mathbf{c}' \neq \mathbf{c}$ *but* $\dim \mathcal{O}_{\mathbf{c}'} < \dim \mathcal{O}_{\mathbf{c}}$.[3]

From Lemma 2.3(7) and above lemmas, we may summarize our results of this subsection as follows.

Proposition 4.7 *For any* $\mathbf{c} \in \mathbb{N}^{\Phi^+}$, *we have*

$$E(\mathbf{c}) = \sum_{\mathbf{c}' \in \varphi^{-1}(\varphi(\mathbf{c}))} h^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'}$$

such that (1) $h^{\mathbf{c}}_{\mathbf{c}'} \in \mathcal{Z}$, (2) $h^{\mathbf{c}}_{\mathbf{c}} = 1$, (3) *if* $h^{\mathbf{c}}_{\mathbf{c}'} \neq 0$ *then* $\mathbf{c}' \preceq \mathbf{c}$, *(4)* $\overline{E(\mathbf{c})} = E(\mathbf{c})$. □

For any $\mathbf{c}, \mathbf{c}' \in \mathbb{N}^{\Phi^+}$ we define $\omega^{\mathbf{c}}_{\mathbf{c}'} \in \mathcal{Z}$ such that

$$\overline{E^{\mathbf{c}}} = \sum_{\mathbf{c}' \in \mathbb{N}^{\Phi^+}} \omega^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'}.$$

Proposition 4.8 $\omega^{\mathbf{c}}_{\mathbf{c}} = 1$ *and, if* $\omega^{\mathbf{c}}_{\mathbf{c}'} \neq 0$ *and* $\mathbf{c}' \neq \mathbf{c}$ *then* $\mathbf{c}' \prec \mathbf{c}$.

Proof. Using $\overline{E(\mathbf{c})} = E(\mathbf{c})$ and the fact that $\{E^{\mathbf{c}} \mid \mathbf{c} \in \mathbb{N}^{\Phi^+}\}$ is a $\mathcal{Z}$-bases of $\mathcal{C}^*$, we have

$$h^{\mathbf{c}}_{\mathbf{c}''} = \sum_{\mathbf{c}'} \overline{h^{\mathbf{c}}_{\mathbf{c}'}} \omega^{\mathbf{c}'}_{\mathbf{c}''}, \quad \text{for } \mathbf{c}, \mathbf{c}'' \in \varphi^{-1}(\mathbf{d}).$$

By Lemma 4.5, the matrices $(h^{\mathbf{c}}_{\mathbf{c}''})$ as well as $(\overline{h^{\mathbf{c}}_{\mathbf{c}'}})$, where the index set is $\varphi^{-1}(\mathbf{d})$, are triangular with 1 on diagonal. Hence, by the equation above, the matrix $(\omega^{\mathbf{c}}_{\mathbf{c}''})$ has the same property. □

Consider the bar involution $\overline{(\ )} : \mathcal{C}^* \to \mathcal{C}^*$. For any $\mathbf{c} \in \mathbb{N}^{\Phi^+}$,

$$E^{\mathbf{c}} = \overline{\overline{E^{\mathbf{c}}}} = \overline{\sum_{\mathbf{c}'} \omega^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'}} = \sum_{\mathbf{c}', \mathbf{c}''} \overline{\omega^{\mathbf{c}}_{c'}} \omega^{\mathbf{c}'}_{\mathbf{c}''} E^{\mathbf{c}''}.$$

[3]There are selection $V_{C'}$ and V_C as in 4.2 such that $\mathcal{O}_{V_{C'}} \subseteq \overline{\mathcal{O}}_{V_C} \setminus \mathcal{O}_{V_C}$.

implies the orthogonal relation

$$\sum_{\mathbf{c}'} \overline{\omega_{\mathbf{c}'}^{\mathbf{c}}} \omega_{\mathbf{c}''}^{\mathbf{c}'} = \delta_{\mathbf{c}\mathbf{c}''}.$$

Therefore one can solve uniquely the system of equations

$$\zeta_{\mathbf{c}'}^{\mathbf{c}} = \sum_{\mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}} \omega_{\mathbf{c}'}^{\mathbf{c}''} \overline{\zeta_{\mathbf{c}''}^{\mathbf{c}}}$$

with unknowns $\zeta_{\mathbf{c}'}^{\mathbf{c}} \in \mathbb{Z}[v^{-1}]$, $\mathbf{c}' \preceq \mathbf{c}$ and $\mathbf{c}'$, $\mathbf{c} \in \varphi^{-1}(\mathbf{d})$, such that

$$\zeta_{\mathbf{c}}^{\mathbf{c}} = 1 \quad \textit{and} \quad \zeta_{\mathbf{c}'}^{\mathbf{c}} \in v^{-1}\mathbb{Z}[v^{-1}] \quad \textit{for all } \mathbf{c}' \prec \mathbf{c}.$$

For any $\mathbf{d} \in \mathbb{N}^2$ and $\mathbf{c} \in \varphi^{-1}(\mathbf{d})$, we set

$$\mathcal{E}^{\mathbf{c}} = \sum_{\mathbf{c}' \in \varphi^{-1}(\mathbf{d})} \zeta_{\mathbf{c}'}^{\mathbf{c}} E^{\mathbf{c}'} \text{ and } \mathbf{J} = \{\mathcal{E}^{\mathbf{c}} | \mathbf{c} \in \varphi^{-1}(\mathbf{d}), \mathbf{d} \in \mathbb{N}^2\}.$$

Our final result of this section is as follows.

Proposition 4.9 *The set $\mathbf{J}$ is a canonical basis of $\mathcal{C}_{\mathcal{Z}}^*$.*
Proof. We verify the following two properties of $\mathbf{J}$. The first is

$$\begin{aligned}
\overline{\mathcal{E}^{\mathbf{c}}} &= \sum_{\mathbf{c}'} \overline{\zeta_{\mathbf{c}'}^{\mathbf{c}}} \overline{E^{\mathbf{c}'}} = \sum_{\mathbf{c}'} \overline{\zeta_{\mathbf{c}'}^{\mathbf{c}}} \sum_{\mathbf{c}''} \omega_{c''}^{\mathbf{c}'} E^{\mathbf{c}''} \\
&= \sum_{\mathbf{c}''} (\sum_{\mathbf{c}'} \overline{\zeta_{\mathbf{c}'}^{\mathbf{c}}} \omega_{c''}^{\mathbf{c}'}) E^{\mathbf{c}''} = \sum_{\mathbf{c}''} \zeta_{\mathbf{c}''}^{\mathbf{c}} E^{\mathbf{c}''} = \mathcal{E}^{\mathbf{c}}.
\end{aligned}$$

So the elements $\mathcal{E}^{\mathbf{c}}$ are bar-invariant. The second, obviously the set $\mathbf{J}$ is a $\mathbb{Z}[v^{-1}]$-basis of the lattice $\mathcal{L} = \text{span}_{\mathbb{Z}[v^{-1}]}\{E^{\mathbf{c}} | \mathbf{c} \in \mathbb{N}^{\phi^+}\}$ and $\pi(\mathcal{E}^{\mathbf{c}}) = \pi(E^{\mathbf{c}})$ for the canonical projection $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$. Therefore $\mathbf{J}$ is the canonical basis of $\mathcal{C}_{\mathcal{Z}}^*$. □

5. The integral and canonical bases arising from a tube

The main results we present in this section are taken from [DDX], in which the canonical bases of $U_q(\hat{sl}_n)$ and $U_q(\hat{gl}_n)$ are constructed by a linear algebra method from the category of finite dimensional nilpotent representations of a cyclic quiver, i.e, from a tube. However in an preliminary version of the present paper we treat it by assuming the existence and the structure of Lusztig's canonical basis for the composition algebra of a tube (see[L3] and [VV]).

5.1 Let $\Delta = \Delta(n)$ be the cyclic quiver with vertex set $\Delta_0 = \mathbb{Z}/n\mathbb{Z} = \{1, 2., \cdots, n\}$ and arrow set $\Delta_1 = \{i \to i+1 | i \in \mathbb{Z}/n\mathbb{Z}\}$. We consider the category $\mathcal{T} = \mathcal{T}(n)$ of finite dimensional nilpotent representations of $\Delta(n)$ over $\mathbb{F}_q$. For the reason of the shape of its Auslander-Reiten quiver, $\mathcal{T}(n)$ is called a tube of rank n. Let $S_i,\ i \in \Delta_0$ be the irreducible objects in $\mathcal{T}(n)$ and $S_i[l]$ the (unique) indecomposable object in $\mathcal{T}(n)$ with top S_i and length l. Again in this section, we let $\mathcal{P}$ be the set of isomorphism classes of objects in $\mathcal{T}(n)$ and $\mathcal{H}$ the Ringel-Hall algebra of $\mathcal{T}(n)$, $\mathcal{H}^*$ the twisted Ringel-Hall algebra and $\mathbf{L}$ the Lusztig's form of Hall algebra of $\mathcal{T}(n)$. Because that the Hall polynomials always exist in this case (see [R2]), we may regard the algebras $\mathcal{H}$, $\mathcal{H}^*$ and $\mathbf{L}$ in their generic form. So they all are defined generically over $\mathbb{Q}(v)$, where v is an indeterminate. By Proposition 1.1, we may identify $\mathbf{L}$ with $\mathcal{H}^*$ via the morphism φ.

In this section, all properties we obtain are generic and independent of the base field $\mathbb{F}_q$, so the subscripts $\mathbb{F}_q$ will be omitted for simplicity. Since the number n is fixed, sometimes it is omitted, too.

5.2 Let Π be the set of n-tuples of partitions $\pi = (\pi^{(1)}, \pi^{(2)}, \cdots, \pi^{(n)})$ with each $\pi^{(i)} = (\pi_1^{(i)} \geq \pi_2^{(i)} \geq \dots)$ being a partition of an integer. For each $\pi \in \Pi$, we define an object in $\mathcal{T}$

$$M(\pi) = \bigoplus_{\substack{i \in \Delta_0 \\ j \geq 1}} S_i[\tilde{\pi}_j^{(i)}].$$

In this way we obtain a bijection between Π and the set $\mathcal{P}$. So we simply denote $u_\pi, \pi \in \Pi$ for $u_{[M(\pi)]}$ in $\mathcal{H}^*$.

An n-tuple $\pi = (\pi^{(1)}, \pi^{(2)}, \cdots \pi^{(n)})$ of partition in Π is called aperiodic (in the sense of Lusztig [L3]), or separated (in the sense of Ringel [R2]), if for each $l \geq 1$ there is some $i = i(l) \in \Delta_0$ such that $\tilde{\pi}_j^{(i)} \neq l$ for all $j \geq 1$. By Π^a we denote the set of aperiodic n-tuples of partitions. An object M in $\mathcal{T}$ is called aperiodic if $M \simeq M(\pi)$ for some $\pi \in \Pi^a$. For any dimension vector $\alpha \in \mathbb{N}^n (= \mathbb{N}I)$, we let

$$\Pi_\alpha = \{\lambda \in \Pi | \, \underline{\dim}\, M(\lambda) = \alpha\} \quad \text{and} \quad \Pi_\alpha^a = \Pi^a \cap \Pi_\alpha.$$

Given any two modules M, N in $\mathcal{T}$, there exists a unique (up to isomorphism) extension L of M by N with minimal $\dim \mathrm{End}(L)$. This extension L is called the generic extension of M by N and we denote by $L = M \diamond N$. If we define the operation in $\mathcal{P}$ by $[M] \diamond [N] = [M \diamond N]$, then $(\mathcal{P}, \diamond)$ is a monoid with identity $[0]$.

Let Ω denote the set of all words on the alphabet Δ_0. For each $w = i_1 i_2 \cdots i_m \in \Omega$, we set

$$M(w) = S_{i_1} \diamond S_{i_2} \diamond \cdots \diamond S_{i_m}.$$

Then there is a unique $\pi \in \Pi$ such that $M(\pi) \simeq M(w)$, we define $\wp(w) = \pi$. It has been proved in [R2] that $\pi = \wp(w) \in \Pi^a$ and $\wp$ induces a surjection $\wp : \Omega \twoheadrightarrow \Pi^a$.

We have a geometric order in $\mathcal{P}$, or equivalently in Π, as follows: for $\mu, \lambda \in \Pi$, $\mu \preceq \lambda$ if and only if $\mathcal{O}_{M(\mu)} \subseteq \overline{\mathcal{O}}_{M(\lambda)}$, or equivalently, $\dim \mathrm{Hom}(M, M(\lambda)) \leq \dim \mathrm{Hom}(M, M(\mu))$ for all modules M in $\mathcal{T}$.

For each module M in $\mathcal{T}$ and integer $s \geq 1$,, by sM we denote direct sum of s copies of M. For $w \in \Omega$, write w the tight form $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ with $j_{r-1} \neq j_r$ for all r and define $\mu_r \in \Pi$ such that $M(\mu_r) = e_r S_{j_r}$. For any $\lambda \in \Pi_{\sum_{r=1}^t e_r j_r}$, write g_w^λ for the Hall polynomial $g_{\mu_1 \cdots \mu_t}^\lambda$. A word w is called distinguished if the Hall polynomial $g_w^{\wp(w)} = 1$. This means that $M(\wp(w))$ has a unique reduced filtration of type w, i.e., a filtration

$$M(\wp(w)) = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0$$

with $M_{r-1}/M_r \simeq e_r S_{j_r}$ for all r.

Proposition 5.1 *For any $\pi \in \Pi^a$, there exists a distinguished word $w_\pi = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \wp^{-1}(\pi)$ in tight form.*

In $\mathcal{H}^*$, let $u_i^{(*m)} = E_i^{(*m)} = \frac{u_i^{*m}}{[m]!}, i \in \Delta_0, m \geq 1$. The $\mathcal{Z}$-subalgebra $\mathcal{C}_\mathcal{Z}^*$ of $\mathcal{H}^*$ generated by $u_i^{(*m)}, i \in \Delta_0, m \geq 1$ is called the twisted composition algebra of $\mathcal{T}$.

5.3 For each $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ in tight form, define in $\mathcal{C}^*$

$$\mathfrak{m}^{(w)} = E_{j_1}^{(*e_1)} * \cdots * E_{j_t}^{(*e_t)}.$$

For each $\pi \in \Pi_\alpha^a$, we from now on fix a distinguished word $w_\pi \in \wp^{-1}(\pi)$. Thus we have a section $\mathcal{D} = \{w_\pi | \pi \in \Pi^a\}$ of $\wp$ over Π^a. $\mathcal{D}$ is called a section of distinguished words in [DDX].

For each $\pi \in \Pi^a$ with the fixed distinguished word $w_\pi = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}$ in tight form, define $L_0 = e_{j_1}S_{j_1}, L_1 = e_{j_1}S_{j_1} \diamond e_{j_2}S_{j_2}, L_2 = L_1 \diamond e_{j_3}S_{j_3}, \cdots, L_{t-1} = L_{t-2} \diamond e_{j_t}S_{j_t}$. As L_i is the generic extension of L_{i-1} by $e_{j_{i+1}}S_{j_{i+1}}$ and thus $\dim \mathrm{End}(L_i)$ is minimal, we have $M(\pi) \simeq L_{t-1}$. Since

$$1 = g^\pi_{w_\pi} = g^{L_1}_{e_{j_1}S_{j_1}, e_{j_2}S_{j_2}} g^{L_2}_{L_1, e_{j_3}S_{j_3}} \cdots g^\pi_{L_{t-2}, e_{j_t}S_{j_t}},$$

we get $g^{L_i}_{L_{i-1}, e_{j_{i+1}}S_{j_{i+1}}} = 1, 1 \le i \le t-2$. Furthermore, by Lemma 2.3(6) and Proposition 1.1, we have

$$\langle L_{i-1} \rangle * \langle e_{j_{i+1}}S_{j_{i+1}} \rangle = \langle L_i \rangle + \sum_{X, dim\mathcal{O}_X < dim\mathcal{O}_{L_i}} a_X \langle X \rangle,$$

with $a_X \in \mathcal{Z}$. Recall that $\langle M \rangle = v^{-\dim M + \dim \mathrm{End}(M)} u_{[M]}$. Thus

$$\mathfrak{m}^{(w_\pi)} = \langle M(\pi) \rangle + \sum_{\lambda \prec \pi} \xi^\lambda_{w_\pi} \langle M(\lambda) \rangle,$$

where $\xi^\lambda_{w_\pi} \in \mathcal{Z}$ and of course if $\xi^\lambda_{w_\pi} \neq 0$ then $\underline{\dim}\, M(\lambda) = \underline{\dim}\, M(\pi) = \alpha$. Define E_π inductively by the relation

$$E_\pi = \mathfrak{m}^{(w_\pi)} - \sum_{\lambda \prec \pi, \lambda \in \Pi^a_\alpha} v^{-\dim M(\pi) + \dim \mathrm{End}\, M(\pi) + \dim M(\lambda) - \dim \mathrm{End}\, M(\lambda)} g^\lambda_{w_\pi}(v^2) E_\lambda.$$

Therefore we have the relations

$$E_\pi = \langle M(\pi) \rangle + \sum_{\lambda \in \Pi_\alpha \setminus \Pi^a_\alpha, \lambda \prec \pi} \eta^\pi_\lambda \langle M(\lambda) \rangle$$

with $\eta^\pi_\lambda \in \mathcal{Z}$.

Proposition 5.2 *Let $\mathcal{D} = \{w_\pi | \pi \in \Pi^a\}$ be a section of distinguished words of Ω over Π^a. Then both $\{\mathfrak{m}^{(w_\pi)} | \pi \in \Pi^a\}$ and $\{E_\pi | \pi \in \Pi^a\}$ are $\mathcal{Z}$-bases of $\mathcal{C}^*_\mathcal{Z}$. Furthermore, for any $\pi \in \Pi^a_\alpha$,*

$$\mathfrak{m}^{(w_\pi)} = E_\pi + \sum_{\lambda \in \Pi^a_\alpha, \lambda \prec \pi} v^{-\dim M(\pi) + \dim \mathrm{End}\, M(\pi) + \dim M(\lambda) - \dim \mathrm{End}\, M(\lambda)} g^\lambda_{w_\pi}(v^2) E_\lambda.$$

Remark. The definition of the basis $\{E_\pi | \pi \in \Pi^a\}$ is given according to the choice of the section $\mathcal{D}$ of distinguished words, but eventually it has been proved in [DDX] that it is independent of the selection of the sections of distinguished words.

We may regard $\{\mathfrak{m}^{(w_\pi)} | \pi \in \Pi^a\}$ as a monomial $\mathcal{Z}$-basis of $\mathcal{C}^*_\mathcal{Z}$ and $\{E_\pi | \pi \in \Pi^a\}$ as a "PBW"-basis of $\mathcal{C}^*_\mathcal{Z}$. With the triangular relation between the two bases, we can follow an idea of Lusztig by the standard linear algebra method, as we did as in Section 4, to obtain the canonical bases $\{\mathcal{E}_\pi | \pi \in \Pi^a\}$ of $\mathcal{C}^*_\mathcal{Z}$ by

$$\mathcal{E}_\pi = \sum_{\lambda \preceq \pi, \lambda \in \Pi^a_\alpha} p_{\lambda\pi} E_\lambda, \quad \text{for} \quad \pi \in \Pi^a_\alpha, \alpha \in \mathbb{N}^n,$$

with $p_{\lambda\lambda} = 1$ and $p_{\lambda\pi} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $\lambda \prec \pi$.

6. Integral bases arising from preprojective and preinjective components

In this this section consider connected tame quiver Q without oriented cycles. For the projective and injective components, the argument in this section is essentially as same as in the case of finite type.

6.1 Let U be the quantized affine enveloping algebra associated the quiver Q, with the Chevalley generators: E_i, F_i and $K_i^{\pm}$. Lusztig in [L5] has introduced the symmetries $T''_{i,1}: U \to U$ for $i \in I$, which are automorphisms of U and satisfy the braid group relation.

$$\begin{aligned} T''_{i,1}(K_\beta) &= K_{s_i(\beta)}, \quad T''_{i,1}(E_i) = -F_iK_i, \qquad T''_{i,1}(F_i) = -K_iE_i, \\ T''_{i,1}(E_j) &= \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)} \quad \text{for } j \neq i \text{ in } I, \\ T''_{i,1}(F_j) &= \sum_{r+s=-a_{ij}} (-1)^r v^{r} F_i^{(r)} F_j F_i^{(s)} \quad \text{for } j \neq i \text{ in } I. \end{aligned}$$

Here $a_{ij} = (i,j)$ for $i, j \in I$, and $\beta \in \mathbb{Z}I$ and $s_i(\beta) = \beta - (\beta, i)i$. For each $i \in I$, define

$$U^+[i] = \{x \in U^+ | T''_{i,1}(x) \in U^+\}.$$

Then $T''_{i,1} : U^+[i] \to U^+[i]$ is an automorphism. Moreover, if we consider the Lusztig form $U^+_{\mathcal{Z}}$ and let $U^+_{\mathcal{Z}}[i] = U^+_{\mathcal{Z}} \cap U^+[i]$, then $T''_{i,1} : U^+_{\mathcal{Z}}[i] \to U^+_{\mathcal{Z}}[i]$.

6.2 We define $\sigma_i Q$ to be the quiver obtained from Q by reversing the direction of every arrow connected to the vertex i. If i is a sink of Q, one may define the BGP reflection functor (see[BGP] or[DR]):

$$\sigma_i^+ : \operatorname{mod}\Lambda \longrightarrow \operatorname{mod}\sigma_i\Lambda$$

where $\Lambda = \mathbb{F}_q(Q)$ and $\sigma_i\Lambda = \mathbb{F}_q(\sigma_i Q)$. Therefore we have the homomorphism:

$$\sigma_i : \mathcal{H}^*(\Lambda)[i] \longrightarrow \mathcal{H}^*(\sigma_i\Lambda)[i]$$

defined by

$$\sigma_i(u_{[M]}) = u_{[\sigma_i^+(M)]} \quad \text{for any} \quad M \in \operatorname{mod}\Lambda[i].$$

Here $\operatorname{mod}\Lambda[i]$ is the subcategory of all representations which do not have S_i as a direct summand and $\mathcal{H}^*(\Lambda)[i]$ is the subalgebra of $\mathcal{H}^*(\Lambda)$ generated by $u_{[M]}$ with $M \in \operatorname{mod}\Lambda[i]$. Note that $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ is canonically isomorphic to $\mathcal{C}^*(\sigma_i\Lambda)_{\mathcal{Z}}$ by fixing the Chevalley generators which correspond to the simple modules of Λ and $\sigma_i\Lambda$ respectively. Furthermore, we may regard that the functor σ_i^+ induces the homomorphism:

$$\sigma_i : \mathcal{C}^*(\Lambda)_{\mathcal{Z}}[i] \longrightarrow \mathcal{C}^*(\Lambda)_{\mathcal{Z}}[i],$$

where $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}[i] = \{x \in \mathcal{C}^*(\Lambda)_{\mathcal{Z}} | \sigma_i(x) \in \mathcal{C}^*(\Lambda)_{\mathcal{Z}}\}$. It is known that $\sigma_i = T''_{i,1}$ under the identification $\mathcal{C}^*(\Lambda) = U^+$ (for example, see[XY]) .

Dually, if i is a source of Q, we have the similar results.

We call an indecomposable representation M to be exceptional if $\operatorname{Ext}^1(M, M) = 0$. Then it has been proved (for example in [CX]) that

$$\langle sM \rangle \in \mathcal{C}^*(\Lambda)_{\mathcal{Z}} \text{ for any } \ s \geq 1$$

if M is exceptional. In fact,

$$\langle M \rangle^{(*s)} = \frac{1}{[s]!} v^{-s\dim M + s} u_{[M]}^{*s} = \frac{1}{[s]!} v^{-s\dim M + s} (v^{\binom{s}{2}} \psi_s(q)) u_{[sM]} = v^{-s\dim M + s^2} u_{[sM]} = \langle sM \rangle.$$

We denote by *Prep* and *Prei* the isomorphism classes of indecomposable representations in the preprojective and in the preinjective components of $\operatorname{mod}\Lambda$, respectively. In particular, $\mathcal{C}^*_{\mathcal{Z}}$ contains the set

$$\{\langle u_{[sM]} \rangle | M \text{ is indecomposable in } \ Prep \text{ or } Prei \text{ and } s \geq 1\}.$$

6.3 Let $i_m, \cdots, i_1$ be an admissible sink sequence of Q, i.e., i_m is a sink of Q and for any $1 \le t < m$, the vertex i_t is a sink for the orientation $\sigma_{i_{t+1}} \cdots \sigma_{i_m} Q$. Let M be an indecomposable in $Prei$. Then there exists an admissible sink sequence of Q such that

$$M = \sigma^+_{i_1} \cdots \sigma^+_{i_m}(S_{i_{m+1}}),$$

where $S_{i_{m+1}}$ is a simple representation in $\mathrm{mod}\, \sigma_{i_m} \cdots \sigma_{i_1} \Lambda$.

Lemma 6.1 *Let M be an indecomposable preinjective representation. Then*

$$\langle M \rangle = T''_{i_1,1} \cdots T''_{i_m,1}(E_{i_{m+1}}),$$

where $M = \sigma^+_{i_1} \cdots \sigma^+_{i_m}(S_{i_{m+1}})$, for an admissible sink sequence $i_m, \cdots, i_1$ of Q.
Proof. See [R3].□

Since $Prei$ is representation-directed, we may give an enumeration to the set

$$\Phi^+_{Prei} = \{\cdots, \beta_3, \beta_2, \beta_1\}$$

of all positive real roots appearing in $Prei$ with $\{\cdots, M(\beta_3), M(\beta_2), M(\beta_1)\}$ being the corresponding indecomposables and define total ordering $\preceq$ on Φ^+_{Prei} such that

$$\mathrm{Hom}(M(\beta_i), M(\beta_j)) \neq 0 \text{ implies } \beta_i \preceq \beta_j \text{ and } i \ge j.$$

Then such an ordering has the property

$$\langle \beta_i, \beta_j \rangle > 0 \text{ implies } \beta_i \preceq \beta_j \text{ and } i \ge j$$

and

$$\langle \beta_i, \beta_j \rangle < 0 \text{ implies } \beta_j \prec \beta_i \text{ and } i < j$$

and

$$\mathrm{Ext}(M(\beta_i), M(\beta_j)) = 0 \text{ for } i \ge j.$$

Therefore $\beta_i \preceq \beta_j$ if and only if $i \ge j$. Similarly, $Prep$ is representation-directed, we may give a total ordering on the

$$\Phi^+_{Prep} = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}$$

of all positive real roots appearing in $Prep$, with

$$\{M(\alpha_1), M(\alpha_2), M(\alpha_3), \cdots\}$$

be the corresponding indecomposables in $Prep$ and define a total ordering $\preceq$ on Φ^+_{Prep} such that

$$\mathrm{Hom}(M(\alpha_i), M(\alpha_j)) \neq 0 \text{ implies } \alpha_i \preceq \alpha_j \text{ and } i \le j.$$

Then such an ordering has the property

$$\langle \alpha_i, \alpha_j \rangle > 0 \text{ implies } \alpha_i \preceq \alpha_j \text{ and } i \le j$$

and

$$\langle \alpha_i, \alpha_j \rangle < 0 \text{ implies } \alpha_j \prec \alpha_i \text{ and } j < i$$

and

$$\mathrm{Ext}(M(\alpha_i), M(\alpha_j)) = 0 \text{ for } i \le j.$$

We denote by $\mathbb{N}^{Prei}$ the set of all support-finite functions $\mathbf{b} : \Phi^+_{Prei} \to \mathbb{N}$. Each $\mathbf{b} \in \mathbb{N}^{Prei}$ defines a preinjective representation

$$M(\mathbf{b}) = \bigoplus_{\beta_i \in \Phi^+_{Prei}} \mathbf{b}(\beta_i) M(\beta_i)$$

and any preinjective representation is isomorphic to one of the form. By Ringel (Proposition 1' of [R3]) we have

Lemma 6.2 *For any* $\mathbf{b} \in \mathbb{N}^{Prei}$,

$$\langle M(\mathbf{b})\rangle = \langle \mathbf{b}(\beta_{i_m})M(\beta_{i_m})\rangle * \cdots * \langle \mathbf{b}(\beta_{i_1})M(\beta_{i_1})\rangle,$$

where $\{\beta_{i_m} \prec \beta_{i_{m-1}} \prec \cdots \prec \beta_{i_1}\}$ *are those* $\beta \in \Phi^+_{Prei}$ *such that* $\mathbf{b}(\beta) \neq 0$. $\Box$

Thus, by 6.2, $\langle M(\mathbf{b})\rangle \in \mathcal{C}^*_{\mathcal{Z}}$ for all $\mathbf{b} \in \mathbb{N}^{Prei}$. We now define $\mathcal{C}^*(Prei)$ to be the $\mathcal{Z}$-submodule of $\mathcal{C}^*_{\mathcal{Z}}$ generated by

$$\{\langle M(\mathbf{b})\rangle | \mathbf{b} \in \mathbb{N}^{Prei}\}.$$

We have

Lemma 6.3 *The* $\mathcal{Z}$*-submodule* $\mathcal{C}^*(Prei)$ *is an subalgebra of* $\mathcal{C}^*_{\mathcal{Z}}$ *and* $\{\langle M(\mathbf{b})\rangle | \mathbf{b} \in \mathbb{N}^{Prei}\}$ *is a* $\mathcal{Z}$*-basis of* $\mathcal{C}^*(Prei)$.

Proof. If $\mathbf{b}, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{N}^{Prei}$, then the Hall polynomial $g^{M(\mathbf{b})}_{M(\mathbf{b}_1)M(\mathbf{b}_2)}$ always exists (see Ringel [R5]). Then it is easy to see that $\mathcal{C}^*(Prei)$ is closed under the multiplication $*$. $\Box$

With similar definitions for $Prep$, we have

Lemma 6.4 *For any* $\mathbf{a} \in \mathbb{N}^{Prep}, M(\mathbf{a}) = \oplus_{\alpha_i \in \Phi^+_{Prep}} \mathbf{a}(\alpha_i)M(\alpha_i)$, *then*

$$\langle M(\mathbf{a})\rangle = \langle \mathbf{a}(\alpha_{i_1})M(\alpha_{i_1})\rangle * \cdots * \langle \mathbf{a}(\alpha_{i_m})M(\alpha_{i_m})\rangle,$$

where $\{\alpha_{i_1} \prec \alpha_{i_2} \prec \cdots \prec \alpha_{i_m}\}$ *is the support of* $\mathbf{a}$. $\Box$

Lemma 6.5 *Let* $\mathcal{C}^*(Prep)$ *be the* $\mathcal{Z}$*-submodule of* $\mathcal{C}^*_{\mathcal{Z}}$ *generated by*

$$\{\langle M(\mathbf{a})\rangle | \mathbf{a} \in \mathbb{N}^{Prep}\}.$$

Then $\mathcal{C}^*(Prep)$ *is an subalgebra of* $\mathcal{C}^*_{\mathcal{Z}}$ *and* $\{\langle M(\mathbf{a})\rangle | \mathbf{a} \in \mathbb{N}^{Prep}\}$ *is a* $\mathcal{Z}$*-basis of* $\mathcal{C}^*(Prep)$. $\Box$

6.4 Since Q is a tame quiver without oriented cycles, we can order a complete set $\{S_1, S_2, \cdots, S_n\}$ of non-isomorphic simple modules of $\mathrm{mod}\,\Lambda$ such that

$$\mathrm{Ext}^1(S_i, S_j) = 0 \text{ for } i \geq j.$$

We can now identify $I = \{1, 2, \dots, n\}$ and $\mathbb{N}I = \mathbb{N}^n$. Any module M with dimension vector $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ has a unique filtration

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$$

with factors M_{i-1}/M_i isomorphic to d_iS_i, since $\mathrm{Ext}(S_i, S_j) = 0$ for $i \geq j$. This shows that the Hall polynomial $g^M_{d_1S_1\cdots d_nS_n} = 1$. Then we have, in $\mathcal{H}$ and $\mathcal{H}^*$ respectively,

$$\begin{aligned} u^{d_1}_{[S_1]}u^{d_2}_{[S_2]}\cdots u^{d_n}_{[S_n]} &= \psi_{d_1}(q)\psi_{d_2}(q)\cdots\psi_{d_n}(q)\sum u_{[M(\mathbf{a})\oplus M(\mathbf{t})\oplus M(\mathbf{b})]}, \\ u^{(*d_1)}_{[S_1]} * u^{(*d_2)}_{[S_2]} * \cdots * u^{(*d_n)}_{[S_n]} &= v^{-(d_1+d_2+\cdots+d_n)+\langle \mathbf{d},\mathbf{d}\rangle}\sum u_{[M(\mathbf{a})\oplus M(\mathbf{t})\oplus M(\mathbf{b})]}, \end{aligned}$$

where the summation is over the triples $(M(\mathbf{a}), M(\mathbf{t}), M(\mathbf{b}))$ with $M(\mathbf{a})$ preprojective, $M(\mathbf{t})$ regular, $M(\mathbf{b})$ preinjective, and $\underline{\dim}\, M(\mathbf{a}) + \underline{\dim}\, M(\mathbf{t}) + \underline{\dim}\, M(\mathbf{b}) = (d_1, \cdots, d_n) = \mathbf{d}$.

For any $\mathbf{a} \in \mathbb{N}^{Prep}$, let $\{\alpha_{i_1} \prec \alpha_{i_2} \prec \cdots \prec \alpha_{i_m}\}$ be the support of $\mathbf{a}$ and, for $1 \leq t \leq m$, define

$$\begin{aligned} \mathbf{a}_t &= \mathbf{a}(\alpha_{i_t})\alpha_{i_t} = (a_{1t}, a_{2t}, \cdots, a_{nt}) \in \mathbb{N}^n; \\ \mathfrak{m}_{\mathbf{a}_t} &= u^{(*a_{1t})}_{[S_1]} * u^{(*a_{2t})}_{[S_2]} * \cdots * u^{(*a_{nt})}_{[S_n]}; \\ \mathfrak{m}_{\mathbf{a}} &= \mathfrak{m}_{\mathbf{a}_1} * \mathfrak{m}_{\mathbf{a}_2} * \cdots * \mathfrak{m}_{\mathbf{a}_m}. \end{aligned}$$

Similarly for $\mathbf{b} \in \mathbb{N}^{Prei}$ define

$$\begin{aligned} \mathfrak{m}_{\mathbf{b}_t} &= u_{[S_1]}^{(*b_{1t})} * u_{[S_2]}^{(*b_{2t})} * \cdots * u_{[S_n]}^{(*b_{nt})}; \\ \mathfrak{m}_{\mathbf{b}} &= \mathfrak{m}_{\mathbf{b}_m} * \mathfrak{m}_{\mathbf{b}_{m-1}} * \cdots * \mathfrak{m}_{\mathbf{b}_1}. \end{aligned}$$

Lemma 6.6 *For any $\mathbf{a} \in \mathbb{N}^{Prep}$ and $\mathbf{b} \in \mathbb{N}^{Prei}$, in $\mathcal{H}^*$, we have*

(1) $$\mathfrak{m}_{\mathbf{a}} = \langle M(\mathbf{a})\rangle + \sum_{\dim \mathcal{O}_{M(\mathbf{a}')\oplus M(\mathbf{t}')\oplus M(\mathbf{b}')} < \dim \mathcal{O}_{\mathbf{a}}} c^{\mathbf{a}}_{\mathbf{a}'\mathbf{t}'\mathbf{b}'q} u_{[M(\mathbf{a}')\oplus M(\mathbf{t}')\oplus M(\mathbf{b}')]}.$$

Here the sum ranges over all triples $M(\mathbf{a}'), M(\mathbf{t}'), M(\mathbf{b}')$ with $M(\mathbf{a}')$ preprojective, $M(\mathbf{t}')$ regular, $M(\mathbf{b}')$ preinjective, and $\underline{\dim}\, M(\mathbf{a}') + \underline{\dim}\, M(\mathbf{t}') + \underline{\dim}\, M(\mathbf{b}') = \sum_{\alpha \in Prep} \mathbf{a}(\alpha)\alpha$, and $c^{\mathbf{a}}_{\mathbf{a}'\mathbf{t}'\mathbf{b}'q} \in \mathbb{Z}[v, v^{-1}]$;

(2) $$\mathfrak{m}_{\mathbf{b}} = \langle M(\mathbf{b})\rangle + \sum_{\dim \mathcal{O}_{M(\mathbf{a}'')\oplus M(\mathbf{t}'')\oplus M(\mathbf{b}'')} < \dim \mathcal{O}_{\mathbf{b}}} d^{\mathbf{b}}_{\mathbf{a}''\mathbf{t}''\mathbf{b}''q} u_{[M(\mathbf{a}'')\oplus M(\mathbf{t}'')\oplus M(\mathbf{b}'')]},$$

where the sum is over all triples $M(\mathbf{a}''), M(\mathbf{t}''), M(\mathbf{b}'')$ with $M(\mathbf{a}'')$ preprojective, $M(\mathbf{t}'')$ regular, $M(\mathbf{b}'')$ preinjective, and $\underline{\dim}\, M(\mathbf{a}'') + \underline{\dim}\, M(\mathbf{t}'') + \underline{\dim}\, M(\mathbf{b}'') = \sum_{\beta \in Prep} \mathbf{b}(\beta)\beta$, and $d^{\mathbf{b}}_{\mathbf{a}''\mathbf{t}''\mathbf{b}''q} \in \mathbb{Z}[v, v^{-1}]$.

Proof. (1) Since $M(\alpha_{i_t})$ is exceptional, then by Lemma 2.1, $\mathcal{O}_{\mathbf{a}(\alpha_{i_t})M(\alpha_{i_t})}$ is a unique orbit of maximal dimension in $\mathbb{E}_{\mathbf{a}(\alpha_{i_t})\alpha_{i_t}}$. Note that all simple modules are exceptional. We have

$$\begin{aligned} \mathfrak{m}_{\mathbf{a}_t} &= u_{[S_1]}^{(*a_{1t})} * u_{[S_2]}^{(*a_{2t})} * \cdots * u_{[S_n]}^{(*a_{nt})} \\ &= \langle a_{1t}S_1\rangle * \langle a_{2t}S_2\rangle * \cdots * \langle a_{nt}S_n\rangle \\ &= v^{-\dim(\mathbf{a}(\alpha_{i_t})M(\alpha_{i_t})) + \dim \mathrm{End}(\mathbf{a}(\alpha_{i_t})M(\alpha_{i_t}))} \sum_{\underline{\dim}\, M = \mathbf{a}(\alpha_{i_t})\alpha_{i_t}} u_{[M]} \\ &= \langle \mathbf{a}(\alpha_{i_t})M(\alpha_{i_t})\rangle + \sum_{\dim \mathcal{O}_M < \dim \mathcal{O}_{\mathbf{a}(\alpha_{i_t})M(\alpha_{i_t})}} v^{-\dim \mathrm{Ext}(M,M)} \langle M\rangle. \end{aligned}$$

Because $\mathrm{Ext}(M(\alpha_{i_t}), M(\alpha_{i_s})) = 0$ and $\mathrm{Hom}(M(\alpha_{i_s}, M(\alpha_{i_t})) = 0$ for $i_t \le i_s$, by Lemma 2.3(7) we have

$$\begin{aligned} \mathfrak{m}_{\mathbf{a}} &= \mathfrak{m}_{\mathbf{a}_1} * \mathfrak{m}_{\mathbf{a}_2} * \cdots * \mathfrak{m}_{\mathbf{a}_m} \\ &= \langle M(\mathbf{a})\rangle + \sum_{\dim \mathcal{O}_{M(\mathbf{a}')\oplus M(\mathbf{t}')\oplus M(\mathbf{b}')} < \dim \mathcal{O}_{\mathbf{a}}} c^{\mathbf{a}}_{\mathbf{a}'\mathbf{t}'\mathbf{b}'q} u_{[M(\mathbf{a}')\oplus M(\mathbf{t}')\oplus M(\mathbf{b}')]}, \end{aligned}$$

which satisfies the condition. The proof for (2) is dual, so the proof is completed. $\square$

Remark. In Lemma 6.6, the element v is equal to $\sqrt{q}$, but the degree of v^{-1} in $c^{\mathbf{a}}_{\mathbf{a}'\mathbf{t}'\mathbf{b}'q}$ or in $d^{\mathbf{b}}_{\mathbf{a}''\mathbf{t}''\mathbf{b}''q}$ is bounded and independent of $\mathbb{F}_q$.(see Lemma 1.2)

7. Integral bases for the generic composition algebras

7.1 In this section, we still assume that Q is connected tame quiver without oriented cycles. We first consider the embedding of the representation category of the Kronecker quiver into the representation category of Q.

Let e be an extending vertex of Q and $\Lambda = \mathbb{F}_q Q$: the path algebra of Q over $\mathbb{F}_q$. Let $P = P(e)$ be projective module cover of the simple module S_e. Set $\mathfrak{p} = \underline{\dim}\, P(e)$. Clearly $\langle \mathfrak{p}, \mathfrak{p}\rangle = 1 = \langle \mathfrak{p}, \delta\rangle$ and there exists a unique indecomposable preprojective module L with $\underline{\dim}\, L = \mathfrak{p} + \delta$. Moreover we have $\mathrm{Hom}_\Lambda(L,P)=0$ and $\mathrm{Ext}_\Lambda(L,P)=0$. This means that (P,L) is an exceptional pair. Let $\mathfrak{C}(P,L)$ be the smallest full subcategory of $\mathrm{mod}\,\Lambda$ which contains P and L and is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. Also we have $\dim_{\mathbb{F}_q}\mathrm{Hom}_\Lambda(P,L)=2$, therefore $\mathfrak{C}(P,L)$ is equivalent to the module category of the Kronecker quiver over $\mathbb{F}_q$. Thus it induces an exact embedding $F:\mathrm{mod}\,K\hookrightarrow\mathrm{mod}\,\Lambda$, where K is the path algebra of the Kronecker quiver over $\mathbb{F}_q$. We note here that the embedding functor F is essentially independent of the field $\mathbb{F}_q$. This gives rise to an injection of algebras, still denoted by $F:\mathcal{H}^*(K)\hookrightarrow\mathcal{H}^*(\Lambda)$. In $\mathcal{H}^*(K)$ we have defined the element $E_{m\delta}$ for $m\geq 1$. We may still denote by $E_{m\delta}$ for its image $F(E_{m\delta})$. Since $E_{m\delta}$ is in $\mathcal{C}^*(K)$, so $E_{m\delta}$ is in $\mathcal{C}^*(\Lambda)$, in fact in $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$. Let $\mathcal{K}$ be the subalgebra of $\mathcal{C}^*(\Lambda)$ generalized by $E_{m\delta}$ for $m\in\mathbb{N}$, it is a polynomial ring on infinitely many variables $\{E_{m\delta}|m\geq 1\}$, and its integral form is the polynomial ring on variables $\{E_{m\delta}|m\geq 1\}$ over $\mathcal{Z}$.

7.2 We may list all non-homogeneous tubes $\mathcal{T}_1,\mathcal{T}_2,\cdots,\mathcal{T}_s$ in $\mathrm{mod}\,\Lambda$ (in fact, $s\leq 3$). For each $\mathcal{T}_i$, let $r_i=r(\mathcal{T}_i)$ be the period of $\mathcal{T}_i$, i.e., the number of quasi-simple modules in $\mathcal{T}_i$.. Then $r_i>1$. It is well-known that (for example see [CB])

Lemma 7.1 *We have the equation* $\sum_{i=1}^s(r_i-1)=|I|-2$ *and the multiplicity of each imaginary root* $m\delta$ *is equal to* $|I|-1$, *where* $|I|$ *is the number of vertices of* Q. $\square$

7.3 For each non-homogeneous tube $\mathcal{T}_i$, as we did in Section 5, we have the generic composition algebra $\mathcal{C}^*(\mathcal{T}_i)$ of $\mathcal{T}_i$ and its integral form $\mathcal{C}^*(\mathcal{T}_i)_{\mathcal{Z}}$. For each $\mathcal{T}_i$ we have the set Π_i^a of aperiodic r_i-tuples of partitions such that for any $\pi\in\Pi_i^a$, $M(\pi)$ is a aperiodic module in $\mathcal{T}_i$. We have constructed in 5.3 the element

$$E_\pi=\langle M(\pi)\rangle+\sum_{\lambda\in\Pi_i\setminus\Pi_i^a,\lambda\prec\pi}\eta_\lambda^\pi\langle M(\lambda)\rangle$$

such that $\{E_\pi|\pi\in\Pi_i^a\}$ is a $\mathcal{Z}$-basis of $\mathcal{C}^*(\mathcal{T}_i)_{\mathcal{Z}}$.

Let $\mathcal{M}$ be the set of quadruples $\mathbf{c}=(\mathbf{a_c},\mathbf{b_c},\pi_{\mathbf{c}},w_{\mathbf{c}})$ such that $\mathbf{a_c}\in\mathbb{N}^{Prep}$, $\mathbf{b_c}\in\mathbb{N}^{Prei}$, $\pi_{\mathbf{c}}=(\pi_{1\mathbf{c}},\ldots,\pi_{s\mathbf{c}})\in\Pi_1^a\times\cdots\times\Pi_s^a$, and $w_{\mathbf{c}}=(w_1\geq w_2\geq\cdots\geq w_t)$ is a partition.

Then for each $\mathbf{c}\in\mathcal{M}$ we define

$$E^{\mathbf{c}}=\langle M(\mathbf{a_c})\rangle*E_{\pi_{1\mathbf{c}}}*E_{\pi_{2\mathbf{c}}}*\cdots*E_{\pi_{s\mathbf{c}}}*E_{w_{\mathbf{c}}\delta}*\langle M(\mathbf{b_c})\rangle,$$

where $\langle M(\mathbf{a_c})\rangle$ and $\langle M(\mathbf{b_c})\rangle$ are defined in 6.3, $E_{\pi_{i\mathbf{c}}}$ is defined above and $E_{w_{\mathbf{c}}\delta}$ is defined in 3.5. Obviously, $\{E^{\mathbf{c}}|\mathbf{c}\in\mathcal{M}\}$ lies in $\mathcal{C}^*(\Lambda)$, in fact in $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$, and are linearly independent over $\mathbb{Q}(v)$.

Proposition 7.2 *The set* $\{E^{\mathbf{c}}|\mathbf{c}\in\mathcal{M}\}$ *is a* $\mathbb{Q}(v)$*-basis of* $\mathcal{C}^*(\Lambda)$.

The proof of Proposition 7.2 will be given in 7.4. We first need some preparation.

Lemma 7.3 *Let* $\{S_j\mid 1\leq j\leq r_i\}$ *be a complete set of non-isomorphic quasi-simple modules of a non-homogeneous tube* $\mathcal{T}_i$ *such that* $S_j=\tau^{(j-1)}S_1$ *and let* $\mathcal{H}^*(\mathcal{T}_i)$ *be the integral form of the twisted Ringel-Hall algebra of* $\mathcal{T}_i$ *over* $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$. *For any* $l\in\mathbb{N}$ *and* $1\leq jleqr_i$, *let* $\pi,\pi'\in\Pi_i^a$ *such that* $S_j[l]=M(\pi)$ *and* $S_{j+1}[l]=M(\pi')$. *Then*

$$u_{[S_j[l]]}\equiv\sum_{\lambda\preceq\pi,\lambda\in\Pi_i^a}a_\lambda E_\lambda\qquad(\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i))\quad \text{if } r_i\nmid l,\tag{1}$$

$$u_{[S_j[l]]}-u_{[S_{j+1}[l]]}\equiv\sum_{\lambda\preceq\pi(or\ \pi'),\lambda\in\Pi_i^a}a_\lambda E_\lambda\qquad(\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i))\quad \text{if } r_i\mid l.\tag{2}$$

Here $a_\lambda\in\mathbb{Q}$.

Proof. Without loss of generality, we may take $j=1$. When $l=1$, we have $u_{[S_1]}=E_{S_1}$. The conclusion follows. We suppose that the conclusion is true when $1\leqslant l\leqslant r_i-2$. Then the assumption

$$u_{[S_1[l]]}\equiv\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i))$$

and $u_{[S_1[l]]}u_{[S_{l+1}]}-u_{[S_{l+1}]}u_{[S_1[l]]}\equiv u_{[S_1[l+1]]} \quad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i))$ imply

$$\begin{aligned}u_{[S_1[l+1]]}\equiv&(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda)E_{S_{l+1}}-E_{S_{l+1}}(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda) \quad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i))\\ \equiv&(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda)E_{S_{l+1}}-E_{S_{l+1}}(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda)=\sum_{\lambda\preceq\pi,\lambda\in\Pi_i^a}a'_\lambda E_\lambda\end{aligned}$$

since $\{E_\lambda\mid\lambda\in\Pi_i^a\}$ is a basis of $\mathcal{T}_i$. Thus the conclusion is true for $l+1$. For $l=r_i$, by assumption, we have

$$u_{[S_2[l-1]]}\equiv\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i)).$$

Thus we have

$$\begin{aligned}u_{[S_1[l]]}-u_{[S_2[l]]}\equiv&u_{[S_1]}u_{[S_2[l-1]]}-u_{[S_2[l-1]]}u_{[S_1]} \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i)).\\ \equiv&E_{S_1}(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda)-(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda)E_{S_1} \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i))\\ \equiv&\sum_{\lambda\preceq\pi(\text{or }\pi')\lambda\in\Pi_i^a}a_\lambda E_\lambda \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i)).\end{aligned}$$

Now we consider the general case. Let $l=kr_i+m, 0<m\leq r_i-1$, if $m=1$, by assumption, we have

$$u_{[S_1[l-1]]}-u_{[S_2[l-1]]}\equiv\sum_{\lambda\preceq\pi_1(\text{or }\pi_1')\lambda\in\Pi_i^a}a_\lambda E_\lambda \quad (\mathrm{mod}(v-1)\mathcal{H}^*(\mathcal{T})).$$

Hence

$$\begin{aligned}u_{[S_1[l]]}\equiv&(u_{[S_1[l-1]]}-u_{[S_2[l-1]]})u_{[S_1]}-u_{[S_1]}(u_{[S_1[l-1]]}-u_{[S_2[l-1]]})\\ \equiv&(\sum_{\lambda\preceq\pi_1(\text{or }\pi_1')\lambda\in\Pi_i^a}a_\lambda E_\lambda)E_{S_1}-E_{S_1}(\sum_{\lambda\preceq\pi_1(\text{or }\pi_1')\lambda\in\Pi_i^a}a_\lambda E_\lambda)\\ \equiv&\sum_{\lambda\preceq\pi(\text{or }\pi')\lambda\in\Pi_i^a}a'_\lambda E_\lambda \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i)).\end{aligned}$$

If $2\leq m\leq r_i-1$, by assumption,

$$u_{[S_1[l-1]]}\equiv\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a}a_\lambda E_\lambda \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i)).$$

Hence

$$
\begin{aligned}
u_{[S_1[l]]} \equiv & u_{[S_1[l-1]]}u_{[S_l]} - u_{[S_l]}u_{[S_1[l-1]]} \\
\equiv & (\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a} a_\lambda E_\lambda)E_{S_l} - E_{S_l}(\sum_{\lambda\preceq\pi_1,\lambda\in\Pi_i^a} a_\lambda E_\lambda) \\
\equiv & \sum_{\lambda\preceq\pi,\lambda\in\Pi_i^a} a_\lambda E_\lambda \qquad (\mathrm{mod}\ (v-1)\mathcal{H}^*(\mathcal{T}_i)).
\end{aligned}
$$

Then the conclusion is true. When $r_i \mid l$, it can be proved by a similar method for $l = r_i$. $\square$

Remark. Of course we may replace $\mathcal{H}^*(\mathcal{T})$ in Lemma 7.3 by $\mathcal{C}^*(\Lambda)$ since the natural embedding $\mathcal{C}^*(\mathcal{T})/(v-1)\mathcal{C}^*(\mathcal{T})$ into $\mathcal{H}^*(\mathcal{T})/(v-1)\mathcal{H}^*(\mathcal{T})$, here we consider the integral forms over $\mathcal{Z}$.

Lemma 7.4 *In* $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$,

$$
\tilde{E}_{n\delta} = \sum_{\substack{m_1\leq\cdots\leq m_s \\ m_1+\cdots+m_s=n}} b_{m_1,\cdots,m_s} E_{m_1\delta} * \cdots * E_{m_s\delta},
$$

where $b_{m_1,\cdots,m_s} \in \mathcal{Z}$.

Proof. By the relation

$$
E_{0\delta} = 1, \quad E_{k\delta} = \frac{1}{[k]}\sum_{s=1}^{k} v^{s-k}\tilde{E}_{s\delta} * E_{(k-s)\delta},
$$

we can solve the equation inductively to get the relation in Lemma 7.4. $\square$

It is well known from Ringel that the Lie subalgebra $\mathfrak{n}^+ \subseteq \mathcal{C}^*(\Lambda)_{\mathcal{Z}}/(v-1)\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ generated by $u_{[S_i]}$ $(i \in I)$ over $\mathbb{Q}$ is the positive part of the corresponding affine Kac-Moody Lie algebra over $\mathbb{Q}$, and $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}/(v-1)\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ is the universal enveloping algebra of $\mathfrak{n}^+$.

For each non-homogeneous tube $\mathcal{T}_i$ of rank r_i, we denote $u_{\alpha,i} = u_{[S_j[l]]}$ where $S_j[l]$ is indecomposable in $\mathcal{T}_i$ and $\underline{\dim}\, S_j[l] = \alpha$ is a real root; and $u_{j,m\delta,i} - u_{j+1,m\delta,i} = u_{[S_j[l]]} - u_{[S_{j+1}[l]]}$ where $S_j[l]$ is indecomposable in $\mathcal{T}_i$ and $\underline{\dim}\, S_j[l] = m\delta$ an imaginary root. Let $\Psi : \mathcal{C}^*(\Lambda)_{\mathcal{Z}} \to \mathcal{C}^*(\Lambda)/(v-1)\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ be the canonical projection. Then one of the main results in [FMV] is the following of which the proof depends on Lemma 7.1.

Proposition 7.5 *The vectors* $\Psi(u_{[M(\alpha)]})$ *for* $\alpha \in \Phi^+_{Prep}$; $\Psi(u_{\alpha,i})$ *for* $\alpha \in \mathcal{T}_i$ *real root,* $i = 1, \cdots, s$; $\Psi(u_{j,m\delta,i} - u_{j+1,m\delta,i})$, $m \geq 1$, $1 \leq j \leq r_i$, $i = 1, \cdots, s$; $\Psi(\tilde{E}_{n\delta})$, $n \geq 1$ *and* $\Psi(u_{[M(\beta)]})$ *for* $\beta \in \Phi^+_{Prei}$ *form a* $\mathbb{Z}$*-basis of* $\mathfrak{n}^+$.

Note that it is easy to see that all vectors in Proposition 7.5 belong to the Lie algebra $\mathfrak{n}^+$, and they are linearly independent over $\mathbb{Q}$. For example, $\Psi(\tilde{E}_{n\delta}), n \geq 1$, lie in $\mathfrak{n}^+$ by Lemma 3.9. Then by Lemma 7.1, one can prove that those vectors give rise to a $\mathbb{Z}$-basis of $\mathfrak{n}^+$.

7.4 Proof of Proposition 7.2. By the definition of $\{E^{\mathbf{c}} | \mathbf{c} \in \mathcal{M}\}$, we see that they are linearly independent over $\mathbb{Q}(v)$. For any weight (or, dimension vector) $w \in \mathbb{N}I$, we define the $\mathbb{Q}(v)$-space V_w to be spanned by those $E^{\mathbf{c}}$, $\mathbf{c} \in \mathcal{M}$, such that $E^{\mathbf{c}} \in \mathcal{C}^*(\Lambda)_w$. It is well-known from Lusztig that

$$
\dim_{\mathbb{Q}(v)} \mathcal{C}^*(\Lambda)_w = \dim_{\mathbb{Q}}(\mathcal{C}^*(\Lambda)_{\mathcal{Z}}/(v-1)\mathcal{C}^*(\Lambda)_{\mathcal{Z}})_w
$$

and the monomials in a fixed order on the basis elements of $\mathfrak{n}^+$ in Proposition 7.5 form a PBW basis of $\mathcal{C}^*(\Lambda)_{\mathcal{Z}}/(v-1)\mathcal{C}^*(\Lambda)_{\mathcal{Z}}$ over $\mathbb{Q}$. However, Lemma 7.3 and 7.4 implies that those PBW basis elements can be obtained by applying Ψ on $\{E^{\mathbf{c}}|\mathbf{c}\in\mathcal{M}\}$. Therefore $\dim_{\mathbb{Q}(v)} V_w \geq \dim_{\mathbb{Q}(v)} \mathcal{C}^*(\Lambda)_w$ for any $w\in\mathbb{N}I$. Hence $\{E^{\mathbf{c}}|\mathbf{c}\in\mathcal{M}\}$ is a $\mathbb{Q}(v)$-basis of $\mathcal{C}^*(\Lambda)$. □

As a consequence, the canonical mapping

$$\varphi: \mathcal{C}^*(Prep)\otimes_{\mathbb{Q}(v)}\mathcal{C}^*(\mathcal{T}_1)\otimes_{\mathbb{Q}(v)}\cdots\otimes_{\mathbb{Q}(v)}\mathcal{C}^*(\mathcal{T}_s)\otimes_{\mathbb{Q}(v)}\mathcal{K}\otimes_{\mathbb{Q}(v)}\mathcal{C}^*(Prei)\to\mathcal{C}^*(\Lambda)$$

is an isomorphism of $\mathbb{Q}(v)$-spaces.

7.5 We may consider the ring $\mathcal{A}=\mathbb{Q}[v,v^{-1}]$, and $\mathcal{C}^*(\Lambda)_{\mathcal{A}}$ is the $\mathcal{A}$-subalgebra of the generic composition algebra $\mathcal{C}^*(\Lambda)$ generated by $u_i^{(*m)}=\frac{u_i^{*m}}{[m]!}, (i\in I)$.

Proposition 7.6 *The set $\{E^{\mathbf{c}}|\mathbf{c}\in\mathcal{M}\}$ is an $\mathcal{A}$-basis of $\mathcal{C}^*(\Lambda)_{\mathcal{A}}$.*
Proof. For any monomial $\mathfrak{m}$ on the divided powers of $u_{[S_i]}$ ($i\in I$) by Proposition 7.2,

$$\mathfrak{m}=\sum_{\mathbf{c}\in\mathcal{M}} f_{\mathfrak{m},\mathbf{c}}(v)E^{\mathbf{c}} \quad \text{(finite sum)}$$

in $\mathcal{C}^*(\Lambda)$, where $f_{\mathfrak{m},\mathbf{c}}(v)\in\mathbb{Q}(v)$ and v is an indeterminate. Note that $E_{\pi_{i\mathbf{c}}}$ in the definition of $E^{\mathbf{c}}$ has the form (cf. 5.3)

$$E_\pi=\langle M(\pi)\rangle+\sum_{\lambda\in\Pi_\alpha\setminus\Pi_\alpha^a,\lambda\prec\pi}\eta_\lambda^\pi\langle M(\lambda)\rangle$$

with $\eta_\lambda^\pi\in\mathcal{Z}$. The formula $\mathfrak{m}=\sum_{\mathbf{c}\in\mathcal{M}} f_{\mathfrak{m},\mathbf{c}}(v)E^{\mathbf{c}}$ still holds in $\mathcal{H}^*$ for taking $v=\sqrt{q}$. Thus, by Lemma 1.2, for each $\mathbf{c}\in\mathcal{M}$, there exists $N(\mathbf{c})\in\mathbb{N}$ such that $(\sqrt{q})^{N(\mathbf{c})}f_{\mathfrak{m},\mathbf{c}}(\sqrt{q})\in\mathbb{Z}$ for all $q=p^l$ with p a prime number and $l\geq 1$ in $\mathbb{N}$. It is easily seen that $v^{N(\mathbf{c})}f_{\mathfrak{m},\mathbf{c}}(v)$ is a polynomial in $\mathbb{Q}[v]$. Therefore $f_{\mathfrak{m},\mathbf{c}}(v)\in\mathbb{Q}[v,v^{-1}]$. □

Corollary 7.7 *The multiplication map*

$$\varphi: \mathcal{C}^*(Prep)_{\mathcal{A}}\otimes_{\mathcal{A}}\mathcal{C}^*(\mathcal{T}_1)_{\mathcal{A}}\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\mathcal{C}^*(\mathcal{T}_s)_{\mathcal{A}}\otimes_{\mathcal{A}}\mathcal{K}_{\mathcal{A}}\otimes_{\mathcal{A}}\mathcal{C}^*(Prei)_{\mathcal{A}}\to\mathcal{C}^*(\Lambda)_{\mathcal{A}}$$

is an isomorphism of $\mathcal{A}$-modules. □

8. Affine canonical bases

8.1 The first part of this section is devoted to finding a monomial basis and a triangular relation with the basis $\{E^{\mathbf{c}}|\mathbf{c}\in\mathcal{M}\}$. We first define the variety[4]

$$\mathcal{O}_{\mathbf{c}}=\mathcal{O}_{M(\mathbf{a_c})}\star\mathcal{O}_{M_{\pi_{1\mathbf{c}}}}\star\mathcal{O}_{M_{\pi_{2\mathbf{c}}}}\star\cdots\star\mathcal{O}_{M_{\pi_{s\mathbf{c}}}}\star\mathcal{N}_{w_{\mathbf{c}}\delta}\star\mathcal{O}_{M(\mathbf{b_c})}$$

for any $\mathbf{c}\in\mathcal{M}$, where $\mathcal{N}_{w_{\mathbf{c}}\delta}=\mathcal{N}_{w_1\delta}\star\cdots\star\mathcal{N}_{w_t\delta}$ if $w_{\mathbf{c}}=(w_1,w_2,\cdots,w_t)$ and $\mathcal{N}_{w_i\delta}$ are the union of orbits of regular modules of $\mathfrak{C}(P,L)$ with dimension vector $w_i\delta$.

Then by Proposition 7.6, Lemma 6.6 can be rewritten as follows:

Lemma 8.1 *For any $\mathbf{a}\in\mathbb{N}^{Prep}$ and $\mathbf{b}\in\mathbb{N}^{Prei}$, in $\mathcal{C}^*(\Lambda)$ we have*

$$\mathfrak{m}_{\mathbf{a}}=\langle M(\mathbf{a})\rangle+\sum_{\dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_{\mathbf{a}}} f_{\mathbf{c}}^{\mathbf{a}}E^{\mathbf{c}}, \tag{1}$$

$$\mathfrak{m}_{\mathbf{b}}=\langle M(\mathbf{b})\rangle+\sum_{\dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_{\mathbf{b}}} g_{\mathbf{c}}^{\mathbf{b}}E^{\mathbf{c}}, \tag{2}$$

where $f_{\mathbf{c}}^{\mathbf{a}}, g_{\mathbf{c}}^{\mathbf{b}}\in\mathbb{Q}[v,v^{-1}]$ and $\mathbf{c}\in\mathcal{M}$.□

[4]Note that the definition of $\mathcal{O}_{\mathbf{c}}$ here is different with that in 4.2

Remark. The conclusion is also true in Lemma 8.1 if we take $M(\mathbf{a})$ to be finitely many copies of a exceptional module.

Lemma 8.2 *Let $\pi \in \Pi_i^a$ for some $\mathcal{T}_i$, then there exists a monomial $\mathbf{m}_\pi$ on the divided powers of $u_{[S_i]}$ $(i \in I)$ such that*

$$\mathbf{m}_\pi = E_\pi + \sum_{\dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_\pi} f_{\mathbf{c}}^\pi E^{\mathbf{c}},$$

where $f_{\mathbf{c}}^\pi \in \mathbb{Q}[v, v^{-1}]$.

Proof. We set $\{\theta_1, \theta_2, \cdots, \theta_{r_i}\}$ to be a complete set of non-isomorphic quasi-simple modules of $\mathcal{T}_i$ in the natural order (see Section 5). By Proposition 5.2, we then have

$$\mathfrak{m}^{(w_\pi)} = E_\pi + \sum_{\lambda \in \Pi_\alpha^a, \lambda \prec \pi} v^{-\dim M(\pi) + \dim \operatorname{End} M(\pi) + \dim M(\lambda) - \dim \operatorname{End} M(\lambda)} g_{w_\pi}^\lambda (v^2) E_\lambda,$$

where $\mathfrak{m}^{(w_\pi)} = \theta_{j_1}^{(*e_1)} * \cdots * \theta_{j_t}^{(*e_t)}$. Since each θ_{j_i} is an exceptional module, we have $\langle u_{[\theta_{j_p}]} \rangle^{(*e_p)} = \langle e_p \theta_{j_p} \rangle$ (see the proof in 6.2).

Let $\pi_{j_p} \in \Pi_i^a$ such that $M(\pi_{j_p}) = e_p \theta_{j_p}$ and $\underline{\dim}\, M(\pi_{j_p}) = (d_1, \cdots, d_n)$ with I ordered as in 6.4. By Lemma 8.1 and its remark, we define a monomial $\mathfrak{m}_{j_p}$ such that

$$\mathfrak{m}_{j_p} = \langle S_1 \rangle^{(*d_1)} * \cdots * \langle S_n \rangle^{(*d_n)} = \langle M(\pi_{j_p}) \rangle + \sum_{\dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_{M(\pi_{j_p})}} f_{\mathbf{c}}^{\pi_{j_p}} E^{\mathbf{c}}$$

where $f_{\mathbf{c}}^{\pi_{j_p}} \in \mathbb{Q}[v, v^{-1}]$.

Let $L_0 = e_1 \theta_{j_1}, L_1 = e_1 \theta_{j_1} \diamond e_2 \theta_{j_2}, L_2 = L_1 \diamond e_3 \theta_{j_3}, \cdots, L_{t-1} = L_{t-2} \diamond e_t \theta_{j_t}$. By Lemma 2.3(6), We have $M(\pi) \simeq L_{t-1}$. Similar to the argument as in 5.3, we have $g_{L_{p-1}, e_{j_{p+1}} \theta_{j_{p+1}}}^{L_p} = 1$, for $1 \le p \le t-2$. Define $\alpha_p = \underline{\dim}\, L_{p-1}$ and $\beta_p = \underline{\dim}\, M(\pi_{j_p})$. By Lemma 2.3(6), we have

$$\dim \mathcal{O}_{L_p} = \dim \mathcal{O}_{L_{p-1}} + \dim \mathcal{O}_{e_{p+1} \theta_{j_{p+1}}} + \mathbf{m}(\alpha_p, \beta_p)$$

or

$$\operatorname{codim} \mathcal{O}_{L_p} = \operatorname{codim} \mathcal{O}_{L_{p-1}} + \operatorname{codim} \mathcal{O}_{e_{p+1} \theta_{j_{p+1}}} - \langle \beta_p, \alpha_p \rangle.$$

Thus

$$\dim \mathcal{O}_{M(\pi)} = \dim \mathcal{O}_{L_{t-1}} = \sum_{p=1}^{t} \dim \mathcal{O}_{e_p \theta_{j_p}} + \sum_{p=1}^{t-1} \mathbf{m}(\alpha_p, \beta_p).$$

For any $\mathbf{c} \in \mathcal{M}$ with $\dim \mathcal{O}_{\mathbf{c}} < \dim \mathcal{O}_{e_{p+1} \theta_{j_{p+1}}}$, by Lemma 2.2, we have

$$\begin{aligned} \operatorname{codim}(\overline{\mathcal{O}}_{L_{p-1}} \star \overline{\mathcal{O}_{\mathbf{c}}}) &= \operatorname{codim}(\overline{\mathcal{O}}_{L_{p-1}}) + \operatorname{codim}(\overline{\mathcal{O}_{\mathbf{c}}}) - \langle \beta_p, \alpha_p \rangle + r \\ &> \operatorname{codim}(\overline{\mathcal{O}}_{L_{p-1}}) + \operatorname{codim}(\mathcal{O}_{e_{p+1} \theta_{j_{p+1}}}) - \langle \beta_p, \alpha_p \rangle \\ &= \operatorname{codim}(\mathcal{O}_{L_p}), \end{aligned}$$

Thereby, if we take $\mathfrak{m}_\pi = \mathfrak{m}_{\pi_{j_1}} * \cdots * \mathfrak{m}_{\pi_{j_t}}$, then

$$\begin{aligned}\mathfrak{m}_\pi &= (\langle\theta_{j_1}\rangle^{(*e_1)} + \sum_{\dim\mathcal{O}_{\mathbf{c}_1} < \dim\mathcal{O}_{e_1\theta_{j_1}}} f_{\mathbf{c}_1}^{\pi_{j_1}} E^{\mathbf{c}_1}) * \cdots * (\langle\theta_{j_t}\rangle^{(*e_t)} + \sum_{\dim\mathcal{O}_{\mathbf{c}_t} < \dim\mathcal{O}_{e_t\theta_{j_t}}} f_{\mathbf{c}_t}^{\pi_{j_t}} E^{\mathbf{c}_t})\\ &= (\langle M(\pi_{j_1})\rangle + \sum_{\dim\mathcal{O}_{\mathbf{c}_1} < \dim\mathcal{O}_{e_1\theta_{j_1}}} f_{\mathbf{c}_1}^{\pi_{j_1}} E^{\mathbf{c}_1}) * \cdots * (\langle M(\pi_{j_t})\rangle + \sum_{\dim\mathcal{O}_{\mathbf{c}_t} < \dim\mathcal{O}_{e_t\theta_{j_t}}} f_{\mathbf{c}_t}^{\pi_{j_t}} E^{\mathbf{c}_t})\\ &= E_\pi + \sum_{\dim\mathcal{O}_{\mathbf{c}} < \dim\mathcal{O}_\pi} f_{\mathbf{c}}^{\pi} E^{\mathbf{c}},\end{aligned}$$

where $f_{\mathbf{c}}^{\pi} \in \mathbb{Q}[v, v^{-1}]$. The proof is finished. $\square$

Lemma 8.3 *Let $E_{n\delta}$ be the image embedded in $\mathcal{C}^*(\Lambda)$ of the element $E_{n\delta}$ in $\mathcal{K}$,then there exists a monomial $\mathfrak{m}_{n\delta}$ on the divided powers of $u_{[S_i]}$ ($i \in I$) such that*

$$\mathfrak{m}_{n\delta} = E_{n\delta} + \sum_{\dim\mathcal{O}_{\mathbf{c}} < \dim\mathcal{O}_{n\delta}} h_{\mathbf{c}}^{n\delta} E^{\mathbf{c}},$$

where $h_{\mathbf{c}}^{n\delta} \in \mathbb{Q}[v, v^{-1}]$.

Proof. We let θ_1, θ_2 be the two simple objects of $\mathfrak{C}(P, L)$. By Lemma 3.13(3), we then have

$$\langle\theta_2\rangle^{(*n)} * \langle\theta_1\rangle^{(*n)} = E_{n\delta} + \sum_{\dim\mathcal{O}_{\mathbf{c}} < \dim\mathcal{O}_{n\delta}} f_{\mathbf{c}}^{n\delta} E^{\mathbf{c}}, \quad \text{with } f_{\mathbf{c}}^{n\delta} \in \mathbb{Q}[v, v^{-1}].$$

Suppose that $\underline{\dim}\, n\theta_1 = \mathbf{d}' = (d_1', \cdots, d_n')$ and $\underline{\dim}\, n\theta_2 = \mathbf{d}'' = (d_1'', \cdots, d_n'')$ in $\mathbb{Z}I$. Since θ_1, θ_2 are the exceptional modules, by the remark of Lemma 8.1, we then have

$$\begin{aligned}\mathfrak{m}_1 &= \langle S_1\rangle^{(*d_1')} * \langle S_2\rangle^{(*d_2')} * \cdots * \langle S_n\rangle^{(*d_n')}\\ &= \langle\theta_1\rangle^{(*n)} + \sum_{\dim\mathcal{O}_{\mathbf{c}} < \dim\mathcal{O}_{n\theta_1}} f_{\mathbf{c}}^{n\theta_1} E^{\mathbf{c}};\end{aligned}$$

and

$$\begin{aligned}\mathfrak{m}_2 &= \langle S_1\rangle^{(*d_1'')} * \langle S_2\rangle^{(*d_2'')} * \cdots * \langle S_n\rangle^{(*d_n'')}\\ &= \langle\theta_2\rangle^{(*n)} + \sum_{\dim\mathcal{O}_{\mathbf{c}} < \dim\mathcal{O}_{n\theta_2}} g_{\mathbf{c}}^{n\theta_2} E^{\mathbf{c}}; \quad ,\end{aligned}$$

where $f_{\mathbf{c}}^{n\theta_1}, g_{\mathbf{c}}^{n\theta_2} \in \mathbb{Q}[v, v^{-1}]$. By representations of the Kronecker quiver, we know that $\mathcal{N}_{n\delta}$ is open in $\mathcal{O}_{n\theta_2} \star \mathcal{O}_{n\theta_2}$. Moreover, $\mathcal{N}_{n\delta}$ is open, then dense in $\overline{\mathcal{O}}_{n\theta_2} \star \overline{\mathcal{O}}_{n\theta_1}$, that is, $\mathcal{N}_{n\delta}$ is of maximum dimension, G-stable, irreducible and open subvariety of $\overline{\mathcal{O}}_{n\theta_2} \star \overline{\mathcal{O}}_{n\theta_1}$. Since $\mathrm{Hom}(\overline{\mathcal{O}}_{n\theta_1}, \overline{\mathcal{O}}_{n\theta_2}) = 0$, we then obtain

$$\mathrm{codim}\,\overline{\mathcal{O}}_{n\theta_2} \star \overline{\mathcal{O}}_{n\theta_1} = \mathrm{codim}\,\overline{\mathcal{O}}_{n\theta_2} + \mathrm{codim}\,\overline{\mathcal{O}}_{n\theta_1} - \langle\mathbf{d}', \mathbf{d}''\rangle$$

by Lemma 2.2. If either $\mathcal{O}_{\mathbf{c}} \subset \overline{\mathcal{O}}_{n\theta_2} \setminus \mathcal{O}_{n\theta_2}$ or $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}}_{n\theta_1} \setminus \mathcal{O}_{n\theta_1}$, then

$$\begin{aligned}\mathrm{codim}\,\overline{\mathcal{O}}_{\mathbf{c}} \star \overline{\mathcal{O}}_{\mathbf{c}'} &= \mathrm{codim}\,\overline{\mathcal{O}}_{\mathbf{c}} + \mathrm{codim}\,\overline{\mathcal{O}}_{\mathbf{c}'} - \langle\mathbf{d}', \mathbf{d}''\rangle + r\\ &> \mathrm{codim}\,\overline{\mathcal{O}}_{n\theta_2} \star \overline{\mathcal{O}}_{n\theta_1} = \mathrm{codim}\,\mathcal{N}_{n\delta}.\end{aligned}$$

We now take $\mathfrak{m}_{n\delta} = \mathfrak{m}_2 * \mathfrak{m}_1$,then

$$\begin{aligned}\mathfrak{m}_{n\delta} &= (\langle\theta_2\rangle^{(*n)} + \sum_{\dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_{n\theta_2}} g_{\mathbf{c}}^{n\delta}E^{\mathbf{c}}) * (\langle\theta_1\rangle^{(*n)} + \sum_{\dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{n\theta_1}} f_{\mathbf{c}'}^{n\delta}E^{\mathbf{c}'}) \\ &= E_{n\delta} + \sum_{\dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_{n\delta}} h_{\mathbf{c}}^{n\delta}E^{\mathbf{c}},\end{aligned}$$

where $h_{\mathbf{c}}^{n\delta} \in \mathbb{Q}[v, v^{-1}]$. □

Proposition 8.4 *For any $E^{\mathbf{c}}, \mathbf{c} \in \mathcal{M}$, there exists a monomial $\mathfrak{m}_{\mathbf{c}}$ on the divided powers of $u_{S_i}, i \in I$, such that*

$$\mathfrak{m}_{\mathbf{c}} = E^{\mathbf{c}} + \sum_{\mathbf{c}'\in\mathcal{M},\ dim\mathcal{O}_{\mathbf{c}'}<dim\mathcal{O}_{\mathbf{c}}} h_{\mathbf{c}'}^{\mathbf{c}}E^{\mathbf{c}'},$$

where $h_{\mathbf{c}'}^{\mathbf{c}} \in \mathbb{Q}[v, v^{-1}]$.

Proof. According to the structure of the Auslander-Reiten quiver of a tame quiver, if $P \in Prep, I \in Prei$ and R is a regular module, we then know that $\mathcal{O}_{P\oplus R\oplus I}$ is open in $\overline{\mathcal{O}}_P \star \overline{\mathcal{O}}_R \star \overline{\mathcal{O}}_I$ by Lemma 2.3(7). So, we need to prove the same property for $E_\pi * E_{n\delta}$ where $\pi \in \Pi_i^a$. By Lemma 8.2 and 8.3, there exist $\mathfrak{m}_\pi$ and $\mathfrak{m}_{n\delta}$ such that

$$\mathbf{m}_\pi = E_\pi + \sum_{\dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_\pi} f_{\mathbf{c}}^{\pi}E^{\mathbf{c}},$$

and

$$\mathbf{m}_{n\delta} = E_{n\delta} + \sum_{\dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{n\delta}} g_{\mathbf{c}'}^{n\delta}E^{\mathbf{c}'},$$

where $f_{\mathbf{c}}^{n\delta}, g_{\mathbf{c}'}^{n\delta} \in \mathbb{Q}[v, v^{-1}]$.

Since we can find smooth points $A \in \overline{\mathcal{O}}_\pi$ and $B \in \overline{\mathcal{O}}_{n\delta}$ such that $\mathrm{Hom}(B, A) = 0$, we have $\mathrm{Hom}(\overline{\mathcal{O}}_{n\delta}, \overline{\mathcal{O}}_\pi) = 0$. Then,

$$\mathrm{codim}\,\overline{\mathcal{O}}_\pi \star \overline{\mathcal{O}}_{n\delta} = \mathrm{codim}\,\overline{\mathcal{O}}_\pi + \mathrm{codim}\,\overline{\mathcal{O}}_{n\delta} - \langle n\delta, \alpha\rangle.$$

If ether $\mathcal{O}_{\mathbf{c}} \subset \overline{\mathcal{O}}_\pi \setminus \mathcal{O}_\pi$ or $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}}_{n\delta} \setminus \mathcal{O}_{n\delta}$, we have again that

$$\mathrm{codim}\,\overline{\mathcal{O}}_{\mathbf{c}} \star \overline{\mathcal{O}}_{\mathbf{c}'} > \mathrm{codim}\,\overline{\mathcal{O}}_\pi \star \overline{\mathcal{O}}_{n\delta} = \mathrm{codim}\,\mathcal{O}_\pi \star \mathcal{O}_{n\delta}.$$

So, we get

$$\begin{aligned}\mathfrak{m}_{\mathbf{c}} &= \mathfrak{m}_\pi * \mathfrak{m}_{n\delta} \\ &= (E_\pi + \sum_{\dim\mathcal{O}_{\mathbf{c}}<\dim\mathcal{O}_\pi} f_{\mathbf{c}}^{\pi}E^{\mathbf{c}}) * (E_{n\delta} + \sum_{\dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{n\delta}} g_{\mathbf{c}'}^{n\delta}E^{\mathbf{c}'}) \\ &= E^{\mathbf{c}} + \sum_{\mathbf{c}'\in\mathcal{M},\dim\mathcal{O}_{\mathbf{c}'}<\dim\mathcal{O}_{\mathbf{c}}} h_{\mathbf{c}'}^{\mathbf{c}}E^{\mathbf{c}'},\end{aligned}$$

where $h_{\mathbf{c}'}^{\mathbf{c}} \in \mathbb{Q}[v, v^{-1}]$. □

8.2 By argument similar that in Section 4, we can use the standard linear algebra method by Lusztig to get the relation:

$$\overline{E^{\mathbf{c}}} = \sum_{\mathbf{c}'\in\mathcal{M}} \omega_{\mathbf{c}'}^{\mathbf{c}}E^{\mathbf{c}'} \qquad \text{for any } \ \mathbf{c} \in \mathcal{M}$$

with $\omega^{\mathbf{c}}_{\mathbf{c}'} \in \mathcal{A}$ such that $\omega^{\mathbf{c}}_{\mathbf{c}} = 1$ and if $\omega^{\mathbf{c}}_{\mathbf{c}'} \neq 0$ and $\mathbf{c} \neq \mathbf{c}$ then $\dim \mathcal{O}_{\mathbf{c}'} < \dim \mathcal{O}_{\mathbf{c}}$. Thus we can solve the system of equations

$$\zeta^{\mathbf{c}}_{\mathbf{c}'} = \sum_{\dim \mathcal{O}_{\mathbf{c}'} \leq \dim \mathcal{O}_{\mathbf{c}''} \leq \dim \mathcal{O}_{\mathbf{c}}} \omega^{\mathbf{c}''}_{\mathbf{c}'} \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}$$

to get a unique solution such that

$$\zeta^{\mathbf{c}}_{\mathbf{c}} = 1 \quad \text{and} \quad \zeta^{\mathbf{c}}_{\mathbf{c}'} \in v^{-1}\mathbb{Q}[v^{-1}] \quad \text{if} \quad \dim \mathcal{O}_{\mathbf{c}'} < \dim \mathcal{O}_{\mathbf{c}}.$$

Let

$$\mathcal{E}^{\mathbf{c}} = \sum_{\mathbf{c}' \in \mathcal{M}} \zeta^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'} \quad \text{for any} \quad \mathbf{c} \in \mathcal{M}.$$

Note that this is a finite sum. Then we have the main result in this article.

Theorem 8.5 *The set $\{\mathcal{E}^{\mathbf{c}} | \mathbf{c} \in \mathcal{M}\}$ provides a canonical basis of $\mathcal{C}^*(\Lambda)_{\mathcal{A}}$, which is characterized by the two properties: (a) $\overline{\mathcal{E}^{\mathbf{c}}} = \mathcal{E}^{\mathbf{c}}$ for all $\mathbf{c} \in \mathcal{M}$. (b) $\pi(\mathcal{E}^{\mathbf{c}}) = \pi(E^{\mathbf{c}})$, where $\pi : \mathcal{C}^*(\Lambda)_{\mathcal{A}} \to \mathcal{C}^*(\Lambda)_{\mathcal{A}}/v^{-1}\mathcal{C}^*(\Lambda)_{\mathcal{A}}$ is the canonical projection.* $\square$

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