CYCLOTOMIC q-SCHUR ALGEBRAS AND SCHUR-WEYL DUALITY

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1. Introduction

The representation theory of Hecke algebras has been an important part toward understanding (ordinary or modular) representation theories of finite groups of Lie type. Schur algebras, as endomorphism algebras, connects the representation theory of general linear groups and the representations of symmetric groups via Schur-Weyl duality. The quantum version of Schur-Weyl duality is established by Jimbo [J]. Since then, the representation theories of Hecke algebras and q-Schur algebras plays an important role in relating the representation theory of quantum groups and finite groups. Another important feature of Hecke algebras is their role in decomposing representations of algebraic groups in positive characteristics and quantum groups at roots of unit, by Lusztig's conjectures.

In [VV], Varagnolo and Vasserot used the geometric description of the representations of affine Hecke algebras and affine q-Schur algebras and reformulated Lusztig's conjecture about decomposing the Weyl modules for quantum groups at roots of unit in terms of certain canonical bases constructed on the Fock spaces using representations of Hecke algebras. This formulation enables Schiffmann [Sch] to give a completely different proof the Lusztig conjecture for quantum $\mathfrak{sl}_n$ at roots of unit. However this celebrating approach has been so far limited to type A as the symmetric groups appear as the Weyl groups on one hand and the permutation group acting on the tensor factors on the other hand. In this paper, we study the relation between the cyclotomic Hecke algebras and the corresponding q-Schur algebras in searching for a "right" tensor spaces in this case. One of the special case is the root system of type B.

The representation theory of cyclotomic Hecke algebras $H_{m,r}$ has been studied in [AK, DJM, DR1, DR2, GL], etc., mostly along the line of representations of symmetric groups. One of the important feature of the q-Schur algebra for type A, is that they are quasi-hereditary [CPS1], which has many important applications as studied by Cline-Parshall-Scott and many others. Dipper, James, and Mathas [DJM] has defined a version of cyclotomic q-Schur algebra, which is quasi-hereditary. But that version does not seem to fit into the picture of [VV] and to make a connection to the representations of affine quantum groups. In this paper, we define a differently cyclotomic q-Schur algebra $S_m(n,r)$ in Section 5. Although it may not be quasi-hereditary, but it is a finite dimensional quotient of the affine quantum group $U_q(\hat{\mathfrak{gl}}_n)$. The definition depends on choosing the right tensor space on which $U_q(\hat{\mathfrak{gl}}_n)$ and $H_{m,r}$ both act. In a recent preprint[M1], Mathas extended the definition of cyclotomic q-Schur algebra to more general setting which include the one we consider here as a special case. However the special case we treat in this paper has a direct connection to $U_q(\hat{\mathfrak{gl}}_n)$.

The paper is organized as follows. In Section 2 we discuss about the multi-compositions and the standard setting. In Section 3 we follow the setup of [DJM], and many others to discuss the cellular basis of $H_{m,r}$. In Section 4 we discuss the cyclotomic q-Schur algebras in the setting of [DJM] corresponding to each saturated set Γ. They are all quasi-hereditary and then prove an double centralizer property in general case. This generalizes [DPS1, 6.2]. When ω_r is in Γ (then

$H_{m,r}$ acts faithfully on the "tensor space"), the double centralize property is proved by Mathas in [M1]. In this proof, we had to appeal to fact that the cyclotomic q-Schur algebra satisfied the base change property and then we can follow the argument of [DPS1]. Using some of results due to Gizburg-Vasserot, Lusztig, and Varagnolo-Vasserot in [GV, Lu, VV], we establish a quantum Schur-Weyl reciprocity between $U_q(\hat{\mathfrak{gl}}_n)$ and the semi-simple cyclotomic Hecke algebras in section 5. We prove that $S_m(n,r)$ is stratified along the line of [DPS2] in section 6. The stratification is closely related to the stratification of q-Schur algebras in [DPS2].

2. Young tableaux and symmetry groups

2.1. **m-Compositions and m-Partitions.** Suppose that r is a positive integer. A composition λ of r is a sequence of nonnegative integers $(\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots)$ with $|\lambda| = \sum_i \lambda_i = r$.

If the corresponding sequence is weakly decreasing, λ is called a partition. Let $\Lambda(r)$ (resp. $\Lambda^+(r)$) be the set of all compositions (resp. partitions) of r. Let m be a positive integer. An m-composition λ of r is an m-tuple of compositions $(\lambda^{(1)}, \cdots, \lambda^{(m)})$ such that $\sum_{i=1}^m |\lambda^{(i)}| = r$. If each $\lambda^{(i)}$ is a partition, then λ is called an m-partition of r. In the subsequent, we denote by $\Lambda_m(r)$ (resp. $\Lambda_m^+(r)$) the set of all m-compositions (resp. m-partitions) of r.

It is known that $\Lambda(r)$ is a poset with dominance order $\trianglelefteq$ such that $\lambda \trianglelefteq \mu$ for $\lambda, \mu \in \Lambda(r)$ if $\sum_{j=1}^i \lambda_j \leq \sum_{j=1}^i \mu_j$. The dominance order on $\Lambda_1(r) := \Lambda(r)$ extends to a partial order $\trianglelefteq$ on $\Lambda_m(r)$. More precisely, write $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ with $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots,)$ and $\mu = (\mu^{(1)}, \cdots, \mu^{(m)})$ with $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}, \cdots,)$. Then $\lambda \trianglelefteq \mu$ if, for any $1 \leq k \leq m$ and l, the following inequality holds

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{j=1}^{l} \lambda_j^{(k)} \leq \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{j=1}^{l} \mu_j^{(k)}.$$

Let $\Lambda(m,r) = \{(a_1, \ldots, a_m) \in \mathbb{N}^m \mid \sum_i a_i = r\}$, where $\mathbb{N}$ is the set of all nonnegative integers. Each element in $\Lambda(m,r)$ can be thought as a composition of r with all ith components being zero for all $i > m$. Similarly, $\Lambda^+(m,r)$ is the set of all partitions with at most m-parts. Each m-composition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ defines a composition $[\lambda] \in \Lambda(m,r)$ by $[\lambda] := (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|)$. It follows from the definition that $\lambda \trianglelefteq \mu$ implies $[\lambda] \trianglelefteq [\mu]$. For a given m-composition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$, rearranging the entries within each component $\lambda^{(i)}$, one obtains a set of m-compositions $\Lambda(\lambda)$. In $\Lambda(\lambda)$ there is a unique maximal composition λ^+ under the dominance order. λ^+ is the unique m-partition in the set $\Lambda(\lambda)$.

2.2. **Young Diagrams and Tableaux.** A Young diagram $Y(\lambda)$ of a composition λ of r consists of r boxes placed at the matrix entries $\{(i,j) | \ 1 \leq j \leq \lambda_i\}$. If $\lambda_i = 0$, then there is no box in the ith row. Thus each box is determined by its coordinates (i,j). A λ-tableau $\mathbf{s}$ is a bijective map from the set of boxes in $Y(\lambda)$ to $\{1, 2, \ldots, r\}$. Thus a λ-tableau can be thought as colored Young diagram $Y(\lambda)$ by placing the integers $1, 2, \cdots, r$ into the boxes of the Young diagram without repetition. A λ-tableau $\mathbf{s}$ is called row standard if the entries in $\mathbf{s}$ are increasing from left to right. When λ is a partition, a λ-tableau $\mathbf{s}$ is said to be standard if the entries $\mathbf{s}$ are increasing from left to right down columns. The Young diagram $Y(\lambda)$ of an m-composition λ is an m-tuple of Young diagrams $(Y(\lambda^{(1)}), Y(\lambda^{(2)}), \cdots, Y(\lambda^{(m)}))$. A λ-tableau $\mathbf{t}$ is a bijective map from the set of all boxes in $Y(\lambda)$ to $\{1, 2, \ldots, r\}$. $\mathbf{t}(k,i,j)$ is the number at the (i,j)-entry of the k-th component with $1 \leq j \leq \lambda_i^{(k)}$. When the λ-tableau $\mathbf{t}$ is understood, for $1 \leq i \leq r$, we write $(\mathrm{comp}(i), \mathrm{row}(i), \mathrm{col}(i))$ for the box that contains i in $\mathbf{t}$. A λ-tableau $\mathbf{t}$ has m components $\mathbf{t}_1, \ldots, \mathbf{t}_m$. Note that each $\mathbf{t}_i$ is a $\lambda^{(i)}$-tableau though the numbers in boxes are not necessarily the set $\{1,\dots,|\lambda^{(i)}|\}$. If each $\mathbf{t}_i$ is row-standard, we call $\mathbf{t}$ row-standard. If λ is an m-partition, $\mathbf{t}$ is said to be standard if each component is standard. Let $T^s(\lambda)$ be the set of all standard λ-tableaux. We emphasize that standard tableaux are defined for m-partitions λ only.

2.3. **Dual Partitions.** Given a composition λ, the dual composition λ' is defined by $\lambda'_i = |\{j \mid \lambda_j \geq i\}|$. Note that λ' is necessarily a partition. If $\lambda = (\lambda^{(1)},\dots,\lambda^{(m)})$ is an m-composition, we define the dual m-composition $\lambda' = ((\lambda')^{(1)},\dots,(\lambda')^{(m)}) := ((\lambda^{(m)})',\dots,(\lambda^{(1)})')$. Note also that λ' is necessarily an m-partition of r. It follows from the definition that $(\lambda')' = \lambda$ if and only if λ is an m-partition. It is well known that for two partitions λ,μ of r, $\lambda \trianglelefteq \mu$ if and only if $\lambda' \trianglerighteq \mu'$. The following is a generalization of this fact. We state and provide a proof for later use.

Lemma 2.4. *For two m-partitions λ and μ of r, $\lambda \trianglelefteq \mu$ if and only if $\lambda' \trianglerighteq \mu'$.*

Proof. We need only to prove $\lambda \trianglerighteq \mu$ implies $\mu' \trianglerighteq \lambda'$ since $\lambda'' = \lambda$ for any m-partition λ. By the definition of dual m-partition, $(\lambda')^{(k)} = (\lambda^{(m-k+1)})'$. Thus for a given pair (k,l) we have

$$\sum_{i=1}^{k-1}|(\lambda')^{(i)}| + \sum_{j=1}^{l}(\lambda')^{(k)}_j = r - [\sum_{i=1}^{m-k}|\lambda^{(i)}| + \sum_{j>l}(\lambda^{(m-k+1)})'_j]. \tag{2.4.1}$$

If $\lambda' \ntrianglelefteq \mu'$, there exist positive integers k and l such that

$$\sum_{i=1}^{k-1}|(\lambda')^{(i)}| + \sum_{j=1}^{l}(\lambda')^{(k)}_j > \sum_{i=1}^{k-1}|(\mu')^{(i)}| + \sum_{j=1}^{l}(\mu')^{(k)}_j, \tag{2.4.2}$$

and for any pairs (k',l') such that either $k' < k$ or $k' = k$ but $l' < l$, we have

$$\sum_{i=1}^{k'-1}|(\lambda')^{(i)}| + \sum_{j=1}^{l'}(\lambda')^{(k')}_j \leq \sum_{i=1}^{k'-1}|(\mu')^{(i)}| + \sum_{j=1}^{l'}(\mu')^{(k')}_j. \tag{2.4.3}$$

In particular $(\lambda^{(m-k+1)})'_l = (\lambda')^{(k)}_l > (\mu')^{(k)}_l = (\mu^{(m-k+1)})'_l$. By the definition of dual partiton, we have

$$\begin{aligned}
\sum_{j>l}(\lambda^{(m-k+1)})'_j &= \sum_{j=1}^{(\lambda^{(m-k+1)})'_l}(\lambda^{(m-k+1)}_j - l),\\
\sum_{j>l}(\mu^{(m-k+1)})'_j &= \sum_{j=1}^{(\mu^{(m-k+1)})'_l}(\mu^{(m-k+1)}_j - l).
\end{aligned}$$

By (2.4.1) and (2.4.2), we have

$$\begin{aligned}
r - [\sum_{i=1}^{k-m}|\lambda^{(i)}| + \sum_{j=1}^{(\mu^{(m-k+1)})'_l}(\lambda^{(m-k+1)}_j - l)] &\geq r - [\sum_{i=1}^{k-m}|\lambda^{(i)}| + \sum_{j=1}^{(\lambda^{(m-k+1)})'_l}(\lambda^{(m-k+1)}_j - l)]\\
&> r - [\sum_{i=1}^{k-m}|\mu^{(i)}| + \sum_{j=1}^{(\mu^{(m-k+1)})'_l}(\mu^{(m-k+1)}_j - l)]
\end{aligned}$$

which implies

$$\sum_{i=1}^{k-m}|\lambda^{(i)}|+\sum_{j=1}^{(\mu^{(m-k+1)})'_l}\lambda_j^{(m-k+1)}<\sum_{i=1}^{k-m}|\mu^{(i)}|+\sum_{j=1}^{(\mu^{(m-k+1)})'_l}\mu_j^{(m-k+1)}$$

contradicting $\lambda \trianglerighteq \mu$. □

2.5. **Young Subgroups.** Let $\mathfrak{S}_r$ be the symmetry group in r letters. For a given m-composition, $\mathfrak{S}_r$ acts on the set of all λ-tableaux (from right) by permuting its entries. Given a λ-tableau $\mathbf{t}$, the row subgroup of $\mathbf{t}$ is defined to be $S_{\mathbf{t}} = \{w \in S_r \mid (\mathbf{t}(k,i,j))w = \mathbf{t}(k,i,j')$ for some $1 \le j' \le \lambda_i^{(k)}\}$. Let $\mathbf{t}^\lambda$ be the λ-tableau which defines the (total) lexicographic order on the set of boxes $\{(k,i,j) \mid 1 \le j \le \lambda_i^{(k)}, i \ge 1, 1 \le k \le m\}$ in $Y(\lambda)$. The row subgroup $S_{\mathbf{t}^\lambda}$ is called the Young subgroup of λ and denoted by $\mathfrak{S}_\lambda$.

Let D_λ be the set of distinguished right coset representatives of $\mathfrak{S}_\lambda\backslash\mathfrak{S}_r$ (of minimal length). It is known [DJ1] that the map $w \mapsto \mathbf{t}^\lambda w$ is a bijection from D_λ to the set of row-standard λ-tableaux. Let $D_{\lambda,\lambda'} = D_\lambda \cap D_{\lambda'}^{-1}$, where $D_{\lambda'}^{-1}$ is the distinguished left coset representatives in $\mathfrak{S}_r/\mathfrak{S}_{\lambda'}$.

We remark that the Young subgroup $\mathfrak{S}_\lambda$ defined for an m-composition λ is the same as $\mathfrak{S}_{\bar\lambda}$ defined in [DR2], where $\bar\lambda$ is the composition of r obtained from λ by concatenation of nonzero entries. In the sub-sequel, we also use the notion $\mathfrak{S}_{\bar\lambda}$ to replace $\mathfrak{S}_\lambda$.

If λ is an m-partition, let $\mathbf{t}_\lambda$ be the standard λ-tableau obtained from $Y(\lambda)$ by putting the integers $1, 2, \cdots, r$ from top to bottom down successive columns of $Y(\lambda^{(m)})$, then the component $Y(\lambda^{(m-1)})$, and so on.

For each $\lambda \in \Lambda_m(r)$, each λ-tableau $\mathbf{t}$ defines an $d_\lambda(\mathbf{t}) \in \mathfrak{S}_r$ such that $\mathbf{t} = \mathbf{t}^\lambda d(\mathbf{t})$. d_λ defines a bijection between the set of all λ-tableaux and $\mathfrak{S}_r$. Set $w_\lambda = d_\lambda(\mathbf{t}^\lambda)$. We omit the subscript λ in d_λ unless there is a confusion. If λ is an m-partition and $\mathbf{t}$ is a λ-tableau, let $\mathbf{t}'$ be the λ'-tableau defined by $\mathbf{t}'(k,i,j) = \mathbf{t}(m-k+1, j, i)$. It follows from the definition that $\mathbf{t}_{\lambda'} = (\mathbf{t}^\lambda)'$. Since $(\mathbf{t}w)' = \mathbf{t}'w$ for all $w \in \mathfrak{S}_r$, we have $w_{\lambda'} = w_\lambda^{-1}$.

2.6. For any m-partition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(m)})$, let $\boldsymbol{a} = [a_1, a_2, \cdots, a_m]$ with $a_i = \sum_{j=1}^i |\lambda^{(j)}|$. Following [DR1, 1.6], we define $w_{\boldsymbol{a}}$ by

(2.6.1) $$(a_{i-1}+l)w_{\boldsymbol{a}} = r - a_i + l \text{ for all } i \text{ with } a_{i-1} < a_i, 1 \le l \le a_i - a_{i-1}.$$

For example, if $\boldsymbol{a} = [4, 8, 9]$, then

$$w_{\boldsymbol{a}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

In the subsequent, we also denote $w_{\boldsymbol{a}}$ by $w_{[\lambda]}$ since $\boldsymbol{a}$ corresponds to the composition $(a_1, a_2 - a_1, \cdots, a_m - a_{m-1})$, which is equal to $[\lambda]$. In fact, viewing $[\lambda]$ as an m-partition $\widetilde{[\lambda]}$ with only one part $|\lambda^i|$ in the ith component, then $w_{[\lambda]} = w_{\widetilde{[\lambda]}}$ (cf. Cor. 2.8).

Let $\mathbf{t}^i$ be the i-th sub-tableau of $\mathbf{t}^\lambda$ and $w(i) \in \mathfrak{S}_r$ permuting the entries of $\mathbf{t}^i$ (and leaving all entries of $\mathbf{t}^j$ invariant for $i \ne j$) such that $\mathbf{t}^i w(i)$ is the tableau by rearranging the entries of $\mathbf{t}^i$ in increasing order in the first column down and then second column etc. Then $w(1), \cdots, w(m)$ commute each other and,

(2.6.2) $$w_\lambda = w(1)w(2)\cdots w(m)w_{[\lambda]}.$$

Let $s_i = (i, i+1)$ be the basic transposition. Then $S = \{s_1, \cdots, s_{r-1}\}$ is the set of Coxeter generators of the symmetry group $\mathfrak{S}_r$ as a Coxeter group. We will use the standard notion of reduced expressions of elements in $\mathfrak{S}_r$ and the length function $l(w)$ for elements $w \in \mathfrak{S}_r$. Denote by $\geq_w$ the weak Bruhat order on $\mathfrak{S}_r$, i.e., $x \geq_w y$ if there is a reduced expression $y = s_{i_1}s_{i_2}\cdots s_{i_l}$ of y such that $x = s_{i_1}s_{i_2}\cdots s_{i_k}$ for some $k \leq l$. Note that the weak Bruhat order should not be confused with the usual Bruhat order. In fact, if $x \geq_w y$ implies that x"$\leq$"y under the usual Bruhat order. (We keep the terminology from [DJ1].)

The following result was proved by Dipper, James, and Murphy in [DJ1] and [DJMu] for $m = 1, 2$.

Proposition 2.7. *Let λ be an m-partition of r.*

(a) *d_λ defines a bijection from $T^s(\lambda)$ to from $\{w \in \mathfrak{S}_r \mid w \geq_w w_\lambda\}$.*
(b) *For any $\mathbf{t} \in T^s(\lambda)$, $d_\lambda(\mathbf{t})d_{\lambda'}(\mathbf{t}')^{-1} = w_\lambda$ and $l(d_\lambda(\mathbf{t})) + l(d_{\lambda'}(\mathbf{t}')) = l(w_\lambda)$.*

Proof. (a) Note that the map $w \mapsto \mathbf{t}^\lambda w$ defines a one-to-one correspondence from $\mathfrak{S}_r$ to the set of all λ-tableaux. We only show that $\mathbf{t}^\lambda w$ is standard if and only if $w \geq_w w_\lambda$.

Suppose that $w >_w w_\lambda$. Then there is an $s_i \in S$ such that $w >_w ws_i \geq_w w_\lambda$. By induction, we assume that $\mathbf{t}^\lambda ws_i = \mathbf{t}$ is a standard λ-tableau. Since $l(ws_i) = l(w) + 1$, we have $j = (i+1)(ws_i)^{-1} < (i)(ws_i)^{-1} = l$. If i and $i+1$ are in a same row (or same column) of a component of $\mathbf{t}$, then $col(j) > col(l)$ (or $row(j) > row(l)$) in $\mathbf{t}^\lambda$ which is impossible. Thus i and $i+1$ have to be in different rows and columns or in different components and $\mathbf{t}s_i$ remains a standard λ-tableau exchanging i and $i+1$ in $\mathbf{t}$.

Suppose $\mathbf{t} = \mathbf{t}^\lambda w \neq \mathbf{t}_\lambda$ is a standard λ-tableau. We claim $w \geq_w w_\lambda$ if there is a component $\mathbf{t}_i$ of $\mathbf{t}$ which does not contain all entries of $(\mathbf{t}_\lambda)_i$. We take i maximal. Thus numbers in the set $A = \{|\lambda^{(i+1)}| + \cdots + |\lambda^{(m)}| + 1, \ldots, r\}$ will be the entries of the components of $\mathbf{t}_1, \ldots, \mathbf{t}_i$. There exists $k > |\lambda^{(m)}| + \cdots + |\lambda^{(i)}|$ which appears in $\mathbf{t}_i$ while there is a number $l \in A$ such that $l < k$ which is an entry of $\mathbf{t}_j$ for some $j < i$. By choosing maximal such l we have $l \in \mathbf{t}_j$ and $l+1$ is in $\mathbf{t}_i$. Since $\mathbf{t}^\lambda = \mathbf{t}w^{-1}$, it follows from $(l)w^{-1} \in \mathbf{t}^\lambda_j$ and $(l+1)w^{-1} \in \mathbf{t}^\lambda_i$ that $(l)w^{-1} < (l+1)w^{-1}$, $l(ws_l) = l(w) + 1$ and $w \geq_w ws_l$. Since l and $l+1$ are not in the same row or column, $\mathbf{t}s_l$ is standard. By induction assumption, $ws_l \geq_w w_\lambda$, and hence the claim follows.

Now suppose that $\mathbf{t}_i$ and $(\mathbf{t}_\lambda)_i$ have the same entry set for all $i = 1, \ldots, m$. If $\mathbf{t} \neq \mathbf{t}_\lambda$, say $\mathbf{t}_j \neq (\mathbf{t}_\lambda)_j$ for a j, then exist k, l with $k+1 < l$ such that k, l appear in two consecutive rows of $\mathbf{t}_j$. In this case, each number p with $k \leq p \leq l$ must be in this component $\mathbf{t}_j$. So, one can find a p with $row(p) < row(p+1)$ and $col(p) > col(p+1)$. Thus $\mathbf{t}s_p$ is standard and $(p)w^{-1} < (p+1)w^{-1}$. The latter is equivalent to say $l(ws_p) = l(w) + 1$. By induction assumption, $ws_p \geq_w w_\lambda$, and hence $w \geq_w w_\lambda$.

(b). It follows from definition that for any λ-tableau $\mathbf{t}$

$$\mathbf{t}^\lambda d_\lambda(\mathbf{t}) = (\mathbf{t}')' = (\mathbf{t}^{\lambda'} d_{\lambda'}(\mathbf{t}'))' = \mathbf{t}_\lambda d_{\lambda'}(\mathbf{t}') = \mathbf{t}^\lambda w_\lambda d_{\lambda'}(\mathbf{t}')$$

and $w_\lambda = d_\lambda(\mathbf{t})d_{\lambda'}(\mathbf{t}')^{-1}$. If $\mathbf{t} \in T^s(\lambda)$, then $d_\lambda(\mathbf{t}) \geq_w w_\lambda$ by (a) and (b) follows. □

Corollary 2.8. (a) *For any composition $\lambda = (\lambda_1, \cdots, \lambda_m)$ of r (with $\lambda_i = 0$ for all $i > m$), $\tilde{\lambda} = ((\lambda_1, 0, \ldots), (\lambda_2, 0, \ldots), \cdots, (\lambda_m, 0, \ldots))$ is an m-partition. Then $w_{\tilde{\lambda}}$ is the longest element in $D_\lambda = D_{\tilde{\lambda}}$ such that for any $x \in D_\lambda$, $x \geq_w w_{\tilde{\lambda}}$, i.e., there is a $y \in \mathfrak{S}_r$ such that $w_{\tilde{\lambda}} = xy$ and $l(xy) = l(x) + l(y)$.*

(b) *For any m-partition λ, the element $w_{[\lambda]}$ is the longest element in $D_{[\lambda],[\lambda']}$.*

Proof. There is a bijection between the set of row standard λ-tableaux and the set of standard $\tilde{\lambda}$-tableaux. So, (a) follows from Proposition 2.7. In order to prove (b), we have to verify $w_{[\lambda]} \in D^{-1}_{[\lambda']}$ for any m-partition λ, which has been proved in [DR1, 1.8]. □

3. Cyclotomic Hecke algebras of type $G(m,1,r)$

3.1. Let R be a commutative ring with identity element 1 and fixed $q, u_1, \cdots, u_m \in R$ such that $q^{-1} \in R$. In [AK], Ariki and Koike defined the cyclotomic Hecke algebra $H_{m,r}$ of type $G(m,1,r)$ as an associative algebra over R generated by $T_i, 0 \le i \le r-1$ satisfying the following conditions:

$$\begin{cases} (T_0-u_1)(T_0-u_2)\cdots(T_0-u_m)=0, \\ (T_i-q^2)(T_i+1)=0, \text{ for any } 1\le i\le r-1, \\ T_0T_1T_0T_1=T_1T_0T_1T_0, \\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \text{ for } 1\le i\le r-2, \\ T_iT_j=T_jT_i, \text{ if } |j-i|>1. \end{cases}$$

Following [AK], write $t_1 = T_0$ and $t_i = q^{-2}T_{i-1}t_{i-1}T_{i-1}$ if $i > 1$. Then t_i and t_j are commuting. In order to use Kazhdan-Lusztig bases for type A affine Hecke algebras and Hecke algebras, we use q^2 to replace q in the definition of cyclotomic Hecke algebras.

3.2. Let $s_i = (i, i+1)$ be the basic transposition. Denote by $T_{s_i} := T_i$. If $w = s_{i_1}\cdots s_{i_k}$ is a reduced expression of $w \in \mathfrak{S}_r$ write $T_w := T_{i_1}T_{i_2}\cdots T_{i_k}$. It is known that T_w is independent of the reduced expression of w. For any m-composition λ of r, we set

$$x_{\bar\lambda} = \sum_{w\in\mathfrak{S}_{\bar\lambda}} T_w \text{ and } y_{\bar\lambda} = \sum_{w\in\mathfrak{S}_{\bar\lambda}} (-q^2)^{-l(w)}T_w.$$

Write $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(m)})$. Following [DJM] and [DR1], we define $a_i = \sum_{j=1}^{i} |\lambda^{(j)}|$ and $\pi_{[\lambda]} = \prod_{i=1}^{m-1} \pi_{a_i}(u_{i+1})$, and $\tilde\pi_{[\lambda]} = \prod_{i=1}^{m-1} \pi_{a_i}(u_{m-i})$, where $\pi_a(x) = \prod_{j=1}^{a}(t_j - x)$ for any $x \in R$ and any positive integer a. Let $x_\lambda = \pi_{[\lambda]}x_{\bar\lambda}$ and $y_\lambda = \tilde\pi_{[\lambda]}y_{\bar\lambda}$. It is known that there is an R-linear anti-involution $*$ on $H_{m,r}$ such that $T_i^* = T_i, 0 \le i \le r-1$. For any standard λ-tableaux $\mathbf{s}, \mathbf{t} \in T^s(\lambda)$, set

$$x^\lambda_{\mathbf{s},\mathbf{t}} = T^*_{d(\mathbf{s})}x_\lambda T_{d(\mathbf{t})} \quad \text{and} \quad y^\lambda_{\mathbf{s},\mathbf{t}} = T^*_{d(\mathbf{s})}y_\lambda T_{d(\mathbf{t})}. \tag{3.2.1}$$

Note that $T_w^* = T_{w^{-1}}$. Then $x_\lambda^* = x_\lambda$ and $y_\lambda^* = y_\lambda$. Moreover, $(x^\lambda_{\mathbf{s},\mathbf{t}})^* = x^\lambda_{\mathbf{t},\mathbf{s}}$ and $(y^\lambda_{\mathbf{s},\mathbf{t}})^* = y^\lambda_{\mathbf{s},\mathbf{t}}$. The following result is about the Specht modules for the cyclotomic Hecke algebras.

Proposition 3.3. *Suppose $\lambda \in \Lambda^+_m(r)$.*

(a) [DR2, 2.10] *The right $H_{m,r}$-module $S^\lambda = x_\lambda T_{w_\lambda} y_{\lambda'} H_{m,r}$, called the Specht module with respect to the m-partition λ, is a non-zero free R-module with a basis $\{x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})} \mid \mathbf{t} \in T^s(\lambda')\}$.*

(b) *The left $H_{m,r}$-module $H_{m,r}x_\lambda T_{w_\lambda} y_{\lambda'}$ is a non-zero free R-module with a basis $\{T^*_{d(\mathbf{t})} x_\lambda T_{w_\lambda} y_{\lambda'} \mid \mathbf{t} \in T^s(\lambda)\}$.*

Proof. The result (a) is proved in [DR2, 2.10]. Let $\Phi : H_{m,r} \to H_{m,r}$ be the ring automorphism defined by setting $\Phi(q) = q^{-1}$, $\Phi(u_i) = u_{m-i+1}$, for $1 \le i \le m$, $\Phi(T_j) = (-q^2)^{-1}T_j$ for $1 \le j \le r-1$ and $\Phi(T_0) = T_0$. It is known that $\Phi(x_\lambda) = y_\lambda$. So,

$$\Phi(S^{\lambda'}) = \Phi(x_{\lambda'}T_{w_{\lambda'}}y_\lambda H_{m,r}) = y_{\lambda'}T_{w_{\lambda'}}x_\lambda H_{m,r}.$$

Applying the anti-involution $* : H_{m,r} \to H_{m,r}$ to both sides of the above equality and using $w_{\lambda'}^{-1} = w_\lambda$, we have

$$* \circ \Phi(S^{\lambda'}) = H_{m,r}x_\lambda T_{w_\lambda} y_{\lambda'}.$$

Now, the result (b) follows from (a) immediately. □

3.4. For an m-partition λ, an m-tableau $\mathbf{S}$ of shape λ is a map $\mathbf{S} : Y(\lambda) \to \{1, \dots, m\} \times \mathbb{N}$. This is equivalent to coloring the boxes of $Y(\lambda)$ by ordered pairs of numbers (i, j) such that $1 \leq i \leq m$ and $j \geq 1$. The type of $\mathbf{S}$ is the m-composition $\mu = (\mu^{(1)}, \dots, \mu^{(m)})$ such that $\mu^{(k)}_l = |\mathbf{S}^{-1}(k, l)|$. In [DJM] $\mathbf{S}$ is called a λ-tableau of shape μ. We say $\mathbf{S}$ an m-tableau of shape λ and type μ to distinguish with the tableaux defined earlier. When $m = 1$, they are all called λ-tableaux.

If $\mathbf{S}$ is an m-tableau of shape λ and type μ, then $\mathbf{S}'(k, i, j) = \mathbf{S}(m - k + 1, j, i)$ defines an m-tableau of shape λ' and type μ. The lexicographic order on the set $\{1, \dots, m\} \times \mathbb{N}$ is a total order. For an m-partition λ, we say an m-tableau $\mathbf{S}$ of shape λ is semi-standard if

(a) $\mathbf{S}(k, i, j)$ is weakly increasing in j;
(b) $\mathbf{S}(k, i, j)$ is strictly increasing in i;
(c) If $\mathbf{S}(k, i, j) = (p, l)$, then $p \geq k$.

For $\lambda \in \Lambda^+_m(r)$ and $\mu \in \Lambda_m(r)$, let $T^{ss}(\lambda, \mu)$ be the set of all semi-standard m-tableaux of shape λ and of type μ. Note that $T^{ss}(\lambda, \mu) \neq \emptyset$ implies $\lambda \trianglerighteq \mu$.

An m-tableau $\mathbf{T}$ is called column semi-standard [M1] if $\mathbf{T}'$ is semi-standard. Let $T^{cs}(\lambda, \mu)$ be the set of column semi-standard λ-tableaux of type μ. Then $T^{cs}(\lambda, \mu) \neq \emptyset$ implies $\lambda' \trianglerighteq \mu$.

Let $\omega = ((0), \dots, (0), (1^r)) \in \Lambda^+_m(r)$. For any $\lambda \in \Lambda^+_m(r)$, and any λ-tableau $\mathbf{t}$ we can define an m-tableau $\mathbf{T}(\mathbf{t})$ by $\mathbf{T}(\mathbf{t})(k, i, j) = (m, \mathbf{t}(k, i, j))$ of shape λ and type ω. When $\mathbf{t} \in T^s(\lambda)$, then $\mathbf{T}(\mathbf{t}) \in T^{ss}(\lambda, \omega)$.

3.5. Suppose $\lambda \in \Lambda^+_m(r)$ and $\mu \in \Lambda_m(r)$. For any $\mathbf{t} \in T^s(\lambda)$, let $\mu(\mathbf{t})$ be the m-tableau of shape λ and type μ defined by $\mu(\mathbf{t})(k, i, j) = (p, l)$ if $\mathbf{t}(k, i, j) = \mathbf{t}^\mu(p, l, j')$. For any m-tableau $\mathbf{S}$ of shape λ and type μ, let $\mu^{-1}(\mathbf{S})$ be the set of all $\mathbf{t} \in T^s(\lambda)$ such that $\mu(\mathbf{t}) = \mathbf{S}$. It is shown in [M1] that there is a unique standard λ-tableau $\dot{\mathbf{S}} \in \mu^{-1}(\mathbf{S})$ such that $d(\dot{\mathbf{S}}) \geq_w d(\mathbf{t})$ for any $\mathbf{t} \in \mu^{-1}(\mathbf{S})$ [M1]. The following lemma follows immediately.

Lemma 3.6. *Suppose* $\mathbf{S} \in T^{ss}(\lambda, \mu)$.

(a) $\mathbf{s} \in \mu^{-1}(\mathbf{S})$ *if and only if its conjugate* $\mathbf{s}' \in \mu^{-1}(\mathbf{S}') \subset T^s(\lambda')$.
(b) $l(d((\dot{\mathbf{S}})')) > l(d(\mathbf{t}))$ *for any* $\mathbf{t} \in \mu^{-1}(\mathbf{S}')$ *with* $\mathbf{t} \neq (\dot{\mathbf{S}})'$.

Proof. (a) follows from the definition of the map μ in 3.5. By (2.7), $d(\mathbf{t})d(\mathbf{t}')^{-1} = w_\lambda$ with $l(d(\mathbf{t})d(\mathbf{t}')) = l(d(\mathbf{t})) + l(d(\mathbf{t}')^{-1})$ for any standard λ-tableau $\mathbf{t}$. Now, (b) follows from the fact $\dot{\mathbf{S}} \geq_w \mathbf{t}$ for any $\mathbf{t} \in \mu^{-1}(\mathbf{S})$. □

The following result will be used repeatedly, we quote it here for convenience.

Lemma 3.7. (a) [M1] *For* $\lambda, \mu \in \Lambda_m(r)$, $x_\mu H_{m,r} y_\lambda \neq 0$ *if and only if there is a* $\nu \in \Lambda^+_m(r)$ *such that* $T^{cs}(\nu, \lambda) \neq \emptyset$ *and* $T^{ss}(\nu, \mu) \neq \emptyset$. *In particular,* $\lambda' \trianglerighteq \mu$ *if* $\lambda \in \Lambda^+_m(r)$.
(b) *If* $\lambda, \mu \in \Lambda^+_m(r)$ *and* $x^\mu_{\mathbf{u},\mathbf{v}} H_{m,r} y^{\lambda'}_{\mathbf{s},\mathbf{t}} \neq 0$ *for some* $\mathbf{s}, \mathbf{t} \in T^s(\lambda'), \mathbf{u}, \mathbf{v} \in T^s(\mu)$, *then* $\mu \trianglelefteq \lambda$.
(c) *If* $\lambda, \mu \in \Lambda^+_m(r)$ *and* $y^{\lambda'}_{\mathbf{s},\mathbf{t}} H_{m,r} x^\mu_{\mathbf{u},\mathbf{v}} \neq 0$ *for some* $\mathbf{s}, \mathbf{t} \in T^s(\lambda'), \mathbf{u}, \mathbf{v} \in T^s(\mu)$, *then* $\mu \trianglelefteq \lambda$.

Proof. (a) is due to Mathas in [M1, 6.8]. Under the assumption of (b), $x_\mu H_{m,r} y_{\lambda'} \neq 0$. By (a), there is a $\nu \in \Lambda^+_m(r)$ such that $\nu' \trianglerighteq \lambda'$ and $\nu \trianglerighteq \mu$. Now, Lemma 2.4 implies that $\lambda \trianglerighteq \nu$. (c) follows from (b) by applying the anti-involution $*$. □

The following result will be useful in the subsequent.

Lemma 3.8. *Suppose* $\lambda \in \Lambda^+_m(r)$.

(a) *For any* $w \in \mathfrak{S}_r$ *with* $l(w) \leq l(w_\lambda)$ *then* $x_\lambda T_w y_{\lambda'} \neq 0$ *only when* $w = w_\lambda$.

(b) *For any* $x, y \in \mathfrak{S}_r$ *with* $l(x) + l(y) \le l(w_\lambda)$,

$$x_\lambda T_x T_y y_{\lambda'} = \begin{cases} x_\lambda T_{w_\lambda} y_{\lambda'}, & \text{if } xy = w_\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (a) By writing $w = w_1 w_2 w_3$ with $w_1 \in \mathfrak{S}_\lambda$ and $w_3 \in \mathfrak{S}_{\lambda'}$ and $w_2 \in D_{\lambda,\lambda'}$, we can assume that $w \in D_{\lambda,\lambda'}$ with $l(w) \le l(w_\lambda)$. Suppose $x_\lambda T_w y_{\lambda'} \ne 0$. Write $w = w_2 w_1$ with $w_1 \in D_{[\lambda]}$ and $w_2 \in \mathfrak{S}_{\bar\lambda}/\mathfrak{S}_{[\lambda]}$. Then $x_\lambda T_w y_{\lambda'} = x_{\bar\lambda} \pi_\lambda T_{w_2} T_{w_1} \tilde\pi_{\lambda'} y_{\bar\lambda'}$. Since $w_2 \in \mathfrak{S}_{[\lambda]}$, T_{w_2} commute with π_λ. Therefore, $x_\lambda T_w y_{\lambda'} \ne 0$ implies $\pi_\lambda T_{w_1} \tilde\pi_{\lambda'} \ne 0$. By the proof of [DR1, 3.1], $\pi_\lambda T_{w_1} \tilde\pi_{\lambda'} = 0$ for $w_1 \in D_{[\lambda][\lambda']}$ unless $w_1 = w_{[\lambda]}$. Note that $w_{[\lambda]}$ is the longest element in $D_{[\lambda][\lambda']}$ (Corollary 2.7(b)). Now, write $w_2 = x_1 x_2 \cdots x_m$ such that $x_i \in \mathfrak{S}_{|\lambda^i|}$ and $\mathfrak{S}_{|\lambda^i|}$ is the symmetry group on the set $\{\sum_{j=1}^{i-1} |\lambda^{(j)}| + 1, \cdots, \sum_{j=1}^{i} |\lambda^{(j)}|\}$. By the definition of $y_{\bar\lambda'}$, we have

$$y_{\lambda^{(1)'} \vee \lambda^{(2)'} \vee \cdots \vee \lambda^{(m)'}} T_{w_{[\lambda]}} = T_{w_{[\lambda]}} y_{\bar\lambda'},$$

where the composition $\lambda^{(1)'} \vee \lambda^{(2)'} \vee \cdots \vee \lambda^{(m)'}$ is obtained by concatenating the components $\lambda^{(1)'}, \lambda^{(2)'}, \cdots, \lambda^{(m)'}$. We have

$$x_{\bar\lambda} T_{w_2} \pi_\lambda T_{w_{[\lambda]}} \tilde\pi_{[\lambda']} y_{\bar\lambda'} = \left(\prod_{i=1}^{m} x_{\lambda^i} T_{x_i} y_{\lambda^{i'}} \right) \pi_\lambda T_{w_{[\lambda]}} \tilde\pi_{[\lambda']} \ne 0.$$

We can similary assume that x_i is a (minimal) double coset representative of $\mathfrak{S}_{\lambda^{(i)}} \backslash \mathfrak{S}_{|\lambda^{(i)}|} / \mathfrak{S}_{\lambda^{(i)'}}$. Since $l(w) \le l(w_\lambda)$, there is an i such that $l(x_i) < l(w(i))$, the latter is defined in Section 2.5. Thus, $x_{\lambda^{(i)}} T_{x_i} y_{\lambda^{(i)'}} \ne 0$. This is a contradiction since [DJ1, 4.1] implies that $w(i)$ is the unique element in $D_{\lambda^{(i)},\lambda^{(i)'}}$ such that $x_{\lambda^{(i)}} T_{x_i} y_{\lambda^{(i)'}} \ne 0$. This completes the proof of (a).

For any $x, y \in \mathfrak{S}_r$, $T_x T_y$ is a linear combination of the element T_z with $l(z) \le l(x) + l(y)$. By (a), we can assume that $l(z) = l(w_\lambda)$. This happens only when $z = xy$. Now, the result follows from (a) immediately. □

3.9. For any $\mathbf{S} \in T^{ss}(\lambda, \mu)$ and $\mathbf{T} \in T^{ss}(\lambda, \nu)$, following [DJM] set

$$m_{\mathbf{ST}} = \sum_{\mathbf{s} \in \mu^{-1}(\mathbf{S}), \mathbf{t} \in \nu^{-1}(\mathbf{T})} x_{\mathbf{st}}.$$

Set $\mathbf{T}^\lambda = \lambda(\mathbf{t}^\lambda)$. Then $\lambda^{-1}(\mathbf{T}^\lambda) = \{\mathbf{t}^\lambda\}$. Therefore, for any $\mathbf{T} \in T^{ss}(\lambda, \mu)$

$$m_{\mathbf{T}^\lambda \mathbf{T}} = \sum_{\mathbf{t} \in \mu^{-1}(\mathbf{T})} x_\lambda T_{d(\mathbf{t})}. \tag{3.9.1}$$

Lemma 3.10. *For any* $\lambda \in \Lambda_m^+(r)$ *and* $\mu \in \Lambda_m(r)$, *suppose* $\mathbf{T}, \mathbf{T}_1 \in T^{cs}(\lambda', \mu)$ *with* $l(d(\mathbf{T}_1)) \le l(d(\mathbf{T}))$ *and* $t \in T^s(\lambda)$. *Then*

$$m_{\mathbf{T}^\lambda \mathbf{T}'} y^{\lambda'}_{\mathbf{T}_1 \mathbf{t}} = \delta_{\mathbf{T},\mathbf{T}_1} x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})}. \tag{3.10.1}$$

Proof. By Lemma 3.6, we have

- (a) $l(d(\mathbf{s})) + l(d(\dot{\mathbf{T}})^{-1}) \le l(w_\lambda)$ for any $\mathbf{s} \in \mu^{-1}(\mathbf{T}')$ with equality holds only if $d(\mathbf{s}) d(\dot{\mathbf{T}})^{-1} = w_\lambda$, i.e., $\mathbf{s} = (\dot{\mathbf{T}})'$.
- (b) $l(d(\mathbf{s})) + l(d(\dot{\mathbf{T}}_1)^{-1}) \le l(w_\lambda)$ and $d(\mathbf{s}) d(\dot{\mathbf{T}})^{-1} \ne w_\lambda$ for any $\mathbf{s} \in \mu^{-1}(\mathbf{T}_1')$.

Using (3.9.1), (3.10.1) follows from Lemma 3.8 immediately. □

3.11. (Graham-Lehrer) An algebra A over a ring R with respect to the poset $(\Lambda, \leq)$ is called a cellular algebra if A is free as an R-module of finite rank and for each $\lambda \in \Lambda$, there is an index set $I(\lambda)$ such that

- (a) $\dot{\cup}_{\lambda\in\Lambda} B^\lambda$ is an R-basis of A, where $B^\lambda = \{c^\lambda_{ij} \mid i, j \in I(\lambda)\}$;
- (b) There is an anti-involution σ on A such that $\sigma(c^\lambda_{ij}) = c^\lambda_{ji}$;
- (c) For each $a \in A$, $a \cdot c^\lambda_{ij} \equiv \sum_{k\in I(\lambda)} f_{i,\lambda}(a, k)c^\lambda_{kj} \pmod{A^{>\lambda}}$, where (1) $f_{i,\lambda}(a, k) \in R$ is independent of j, and (2) $A^{>\lambda}$ is the free R-submodule of A spanned by $\dot{\cup}_{\mu>\lambda} B^\mu$.

Fix an index $j_0 \in I(\lambda)$. Let $\Delta(\lambda, j_0)$ be the free R-module generated by $\{\bar{c^\lambda_{ij_0}} = c^\lambda_{ij_0} + A^{>\lambda} \mid i \in I(\lambda)\}$. Then $\Delta(\lambda, j_0)$ is a left A-module such that $\Delta(\lambda, j_0) \cong \Delta(\lambda, j)$ for any $j \in I(\lambda)$. This module, denoted by $\Delta(\lambda)$, is called a cell module with respect to λ.

Theorem 3.12. (a) [DJM] *The set $\{x^\lambda_{\mathbf{st}} \mid \lambda \in \Lambda^+_m(r), \mathbf{s}, \mathbf{t} \in T^s(\lambda)\}$ is a cellular basis of $H_{m,r}$ with respect to the dominance order on $\Lambda^+_m(r)$.*

(b) [DR2] *The set $\{y^\lambda_{\mathbf{st}} \mid \lambda \in \Lambda^+_m(r), \mathbf{s}, \mathbf{t} \in T^s(\lambda)\}$ is another cellular basis of $H_{m,r}$ with respect to the dominance order on $\Lambda^+_m(r)$.*

For any subset $\Gamma \subset \Lambda_m(r)$, let $\Gamma^+ = \{\lambda \in \Lambda^+_m(r) \mid \lambda \trianglerighteq \mu$ for some $\mu \in \Gamma\}$. We say that Γ is *saturated* [M1] if $\Gamma^+ \subset \Gamma$.

Proposition 3.13. *If $\Gamma \subseteq \Lambda_m(r)$ is saturated, then $I(\Gamma^+) = \sum_{\lambda\in\Lambda^+_m(r)\setminus\Gamma^+} R\{y^{\lambda'}_{\mathbf{st}} \mid \mathbf{s}, \mathbf{t} \in T^s(\lambda)\}$ is a two-sided ideal of $H_{m,r}$.*

Proof. Suppose $y^{\lambda'}_{\mathbf{st}} \in I(\Gamma^+)$ $(\lambda \notin \Gamma^+)$. Since $\{x^\mu_{\mathbf{uv}} \mid \mathbf{u}, \mathbf{v} \in T^s(\mu), \mu \in \Lambda^+_m(r)\}$ is a basis of $H_{m,r}$, it is enough to prove that $x^\mu_{\mathbf{uv}} y^{\lambda'}_{\mathbf{st}} \in I(\Gamma^+)$ for any $\mu \in \Lambda^+_m(r)$ and $\mathbf{u}, \mathbf{v} \in T^s(\mu)$.

By Theorem 3.12(b), we can write

$$x^\mu_{\mathbf{uv}} y^{\lambda'}_{\mathbf{st}} = \sum_{\lambda'\trianglelefteq\nu'\in\Lambda^+_m(r),\mathbf{s}_1,\mathbf{t}_1\in T^s(\nu')} r^{\nu'}_{\mathbf{s}_1\mathbf{t}_1} y^{\nu'}_{\mathbf{s}_1\mathbf{t}_1}. \tag{3.13.1}$$

For any ν with $r^{\nu'}_{\mathbf{s}_1\mathbf{t}_1} \neq 0$, we have $\lambda' \trianglelefteq \nu'$. By Lemma 2.4, we have $\lambda \trianglerighteq \nu$. Since Γ is saturated, ν must not be in Γ^+ since $\lambda \notin \Gamma^+$. Thus $I(\Gamma^+)$ is a left ideal. Applying the anti-involution $*$, we get that $I(\Gamma^+)$ is a right ideal as well since $*(I(\Gamma^+)) = I(\Gamma^+)$. ☐

Definition 3.14. For any saturated subset $\Gamma \subset \Lambda^+_m(r)$, set $H(\Gamma) = H_{m,r}/I(\Gamma^+)$.

4. Quasi-hereditary cyclotomic q-Schur algebras

Definition 4.1. [M2] For a finite subset Γ of $\Lambda_m(r)$, set $T_\Gamma = \oplus_{\lambda\in\Gamma} x_\lambda H_{m,r}$, which is a right $H_{m,r}$-module and $S(\Gamma) = \mathrm{End}_{H_{m,r}}(T_\Gamma)$. T_Γ is an $S(\Gamma)$-$H_{m,r}$-bimodule.

The following results are due to Dipper, James, Mathas [DJM].

Proposition 4.2. [DJM, 6.3] *For any m-compositions μ, ν, $x_\mu H_{m,r} \cap H_{m,r} x_\nu$ is a free R-module with a basis $\{m_{\mathbf{ST}} \mid \mathbf{S} \in T^{ss}(\lambda, \mu), \mathbf{T} \in T^{ss}(\lambda, \nu),$ for some $\lambda \in \Lambda^+_m(r)\}$.*

For any $\Gamma \subseteq \Lambda_m(r)$, define $\bar{\Gamma} = \{\lambda \in \Lambda^+_m(r) \mid T^{ss}(\lambda, \mu) \neq \emptyset$ for some $\mu \in \Gamma\}$. Note that $\bar{\Gamma} = \Gamma^+$ if Γ is saturated. But they are different when Γ is not saturated (see Remark 5.9).

For any $\mu, \nu \in \Lambda_m(r)$, $\lambda \in \Lambda^+_m(r)$, $\mathbf{S} \in T^{ss}(\lambda, \mu)$ and $\mathbf{T} \in T^{ss}(\lambda, \nu)$, we define $\Phi^\lambda_{\mathbf{ST}} \in \mathrm{Hom}_{H_{m,r}}(x_\nu H_{m,r}, x_\mu H_{m,r})$ by

$$\Phi^\lambda_{\mathbf{ST}}(x_\nu h) = m_{\mathbf{ST}} h \tag{4.2.1}$$

$\Phi^\lambda_{\mathbf{ST}}$ can be regarded as an element in $S(\Gamma)$ by letting $\Phi^\lambda_{\mathbf{ST}}(x_\mu h) = \delta_{\lambda\mu} m_{\mathbf{ST}} h$. The following result was proved in [DJM].

Theorem 4.3. [DJM, 6.6, 6.18] *Assume $\Gamma \subseteq \Lambda_m(r)$ is finite.*

(1) *$S(\Gamma)$ is a cellular algebra over R with a cellular basis $\{\Phi^\lambda_{\mathbf{ST}} \mid \mathbf{S} \in T^{ss}(\lambda,\mu), \mathbf{T} \in T^{ss}(\lambda,\nu), \mu,\nu \in \Gamma, \lambda \in \bar\Gamma\}$.*
(2) *When Γ is saturated, the cyclotomic q-Schur algebra $S(\Gamma)$ is a quasi-hereditary algebra in the sense of* [CPS1]*, with standard modules $\Delta(\lambda)$ ($\lambda \in \Gamma^+$) being the cell modules with R-bases $\{\bar\Phi^\lambda_{\mathbf{ST}} \| \mathbf{S} \in T^{ss}(\lambda,\mu), \mathbf{T} \in T^{ss}(\lambda,\nu), \mu, \nu\Gamma\}$.*

4.4. Suppose Γ is saturated. For each $\lambda \in \Gamma^+ \subset \Gamma$, by 3.9, $m_{\mathbf{t}^\lambda \mathbf{T}^\lambda} = x_\lambda$ and $\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda} \in S(\Gamma)$ is the identity map in $\mathrm{Hom}_{H_{m,r}}(x_\lambda H_{m,r}, x_\lambda H_{m,r})$. Thus $\Phi^\nu_{\mathbf{ST}}\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda} = \Phi^\nu_{\mathbf{ST}}$ for any $\mathbf{S} \in T^{ss}(\nu,\mu)$ and $\mathbf{T} \in T^{ss}(\nu,\lambda)$. This implies $\Delta(\lambda) = S(\Gamma)\bar\Phi^\lambda_{T^\lambda T^\lambda}$ for any $\lambda \in \Gamma^+$.

Lemma 4.5. *For any $\lambda \in \Gamma^+ \subseteq \Gamma$ and $\mathbf{t} \in T^s(\lambda)$, there is a $S(\Gamma)$-module homomorphism $f_\mathbf{t} : \Delta(\lambda) \to T_\Gamma$ such that*

$$f_\mathbf{t}(\bar\Phi^\lambda_{T^\lambda T^\lambda}) = x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})}.$$

Proof. We need verify that, for any $\phi \in S(\Gamma)$, $\phi\bar\Phi^\lambda_{T^\lambda T^\lambda} = 0$ implies $\phi(x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})}) = 0$. Note that $\Phi^\nu_{\mathbf{ST}}\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda} = 0$ and $\Phi_{\mathbf{ST}}(x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})}) = 0$ if $\mathbf{T} \in T^{ss}(\nu,\mu_1)$ with $\mu_1 \neq \lambda$. By Theorem 4.3(a), we can assume

$$\phi = \sum_{\substack{\mathbf{S}\in T^{ss}(\nu,\mu), \mathbf{T}\in T^{ss}(\nu,\lambda) \\ \lambda \unlhd \nu \in \Gamma^+, \mu\in\Gamma}} r^\nu_{\mathbf{ST}} \Phi^\nu_{\mathbf{ST}}.$$

Since $\phi\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda} \in S(\Gamma)^{>\lambda}$, we have $\nu \rhd \lambda$ for all ν with $r^\nu_{\mathbf{ST}} \neq 0$ (see 3.11). For $\nu \rhd \lambda$, Lemma 3.7 and 3.9 imply $\Phi^\nu_{\mathbf{ST}}(x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})}) = m_{\mathbf{ST}} T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})} = 0$ for all $\mathbf{S}, \mathbf{T}$. This prove our claim. □

Theorem 4.6. *Suppose Γ is saturated. For any $\lambda \in \Gamma^+$, $\mathrm{Hom}_{S(\Gamma)}(\Delta(\lambda), T_\Gamma)$ is a free R-module and the map $x_\lambda T_{w_\lambda} y_{\lambda'} T_{d(\mathbf{t})} \mapsto f_\mathbf{t}$ ($\mathbf{t} \in T^s(\lambda)$) defines an isomorphism of right $H_{m,r}$-modules*

$$S^\lambda \cong \mathrm{Hom}_{S(\Gamma)}(\Delta(\lambda), T_\Gamma).$$

Proof. For any $f \in \mathrm{Hom}_{S(\Gamma)}(\Delta(\lambda), T_\Gamma)$, we compute $f(\bar\Phi^\lambda_{T^\lambda T^\lambda}) \in T_\Gamma$. By [M1, 5.9], $x_\mu H_{m,r}$ is free over R with a basis $\{x_\mu y^{\nu'}_{\dot{\mathbf{S}}\mathbf{t}} \mid \mathbf{S} \in T^{cs}(\nu',\mu), \mathbf{t} \in T^s(\nu'), \nu \in \Lambda^+_m(r), \nu \unrhd \mu\}$. Therefore,

$$f(\bar\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda}) = \sum_{\substack{\mu\in\Gamma, \nu\in\Gamma^+ \\ \mathbf{S}\in T^{ss}(\nu',\mu), \mathbf{t}\in T^s(\nu')}} r^{\nu'}_{\mathbf{St}} x_\mu y^{\nu'}_{\dot{\mathbf{S}}\mathbf{t}},$$

where $r^{\nu'}_{\mathbf{St}} \in R$. Noting that $\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda}(x_\mu) = \delta_{\lambda\mu} x_\lambda$, we have $\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda} f(\bar\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda}) = f(\bar\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda})$ and $r^{\nu'}_{\mathbf{St}} = 0$ for any $\mathbf{S} \in T^{cs}(\nu',\mu)$ with $\mu \neq \lambda$. Thus we have

$$f(\bar\Phi^\lambda_{\mathbf{T}^\lambda\mathbf{T}^\lambda}) = \sum_{\substack{\lambda \unlhd \nu \in \Gamma^+ \\ \mathbf{S}\in T^{ss}(\nu',\lambda), \mathbf{t}\in T^s(\nu')}} r^{\nu'}_{\mathbf{St}} x_\lambda y^{\nu'}_{\dot{\mathbf{S}}\mathbf{t}} \tag{4.6.1}$$

We claim $r^{\nu'}_{\mathbf{St}} = 0$ for any $\nu \rhd \lambda$ and $\mathbf{S} \in T^{cs}(\nu',\lambda),$. Otherwise, take ν_0 be the maxiaml among all $\nu \in \Gamma^+$ with $\nu \unrhd \lambda$ and $r^{\nu'}_{\mathbf{St}} \neq 0$ for some $\mathbf{S}$ and $\mathbf{t}$. Also, let $\mathbf{S}_0$ be such that $l(d(\dot{\mathbf{S}}_0))$ is maxiaml among all $\mathbf{S}\in T^{ss}(\nu_0',\lambda)$ with $r^{\nu_0'}_{\mathbf{St}}\neq 0$ for some $\mathbf{t}$. Since $\nu_0\in\Gamma^+\subseteq\Gamma$, $\Phi^{\nu_0}_{\mathbf{T}^{\nu_0}\mathbf{S}_0'}\in S(\Gamma)$. By Theorem 4.3(a), $\Phi^{\nu_0}_{\mathbf{T}^{\nu_0}\mathbf{S}_0'}(\bar\Phi^{\lambda}_{\mathbf{T}^\lambda\mathbf{T}^\lambda})=0$. On the other hand, Lemma 3.10 implies

$$\Phi^{\nu_0}_{\mathbf{T}^{\nu_0}\mathbf{S}_0'}(f(\bar\Phi^{\lambda}_{T^\lambda T^\lambda}))=\sum_{\mathbf{t}\in T^s(\nu_0')} r^{\nu_0}_{\mathbf{S}_0\mathbf{t}}x_{\nu_0}T_{w_{\nu_0}}y_{\nu_0'}T_{d(\mathbf{t})}\in x_{\nu_0}H_{m,r}.$$

Now, Proposition 3.3(a) shows that $r^{\nu_0}_{\mathbf{S}_0\mathbf{t}}=0$ for any $\mathbf{t}\in T^s(\nu_0')$, a contradiction. Note that $x_\lambda=m_{\mathbf{T}^\lambda\mathbf{T}^\lambda}$. Applying Lemma 3.10 again we have

$$f(\bar\Phi^{\lambda}_{\mathbf{T}^\lambda\mathbf{T}^\lambda})=\sum_{\mathbf{t}\in T^s(\lambda')} r_{\mathbf{t}}x_\lambda T_{w_\lambda}y_{\lambda'}T_{d(\mathbf{t})},$$

where $r_{\mathbf{t}}\in R$. Therefore, by Lemma 4.5,

$$f=\sum_{\mathbf{t}\in T^s(\lambda')} r^{\lambda}_{w_\lambda,\mathbf{t}}f_{\mathbf{t}}.$$

Using Proposition 3.3 again, one will see immediately that $\{f_{\mathbf{t}}\mid \mathbf{t}\in T^s(\lambda')\}$ is an R-basis of $\mathrm{Hom}_{S(\Gamma)}(\Delta(\lambda),T_\Gamma)$ which is therefore a free R-module . Obviously, the R-linear isomorphism sending $f_{\mathbf{t}}$ to $x_\lambda T_{w_\lambda}y_{\lambda'}T_{d(\mathbf{t})}$ is a right $H_{m,r}$-module isomorphism. $\square$

Proposition 4.7. *For any $\Gamma\subseteq\Lambda_m(r)$ (not necessarily saturated), the left $S(\Gamma)$-module T_Γ has a Δ-filtration such that the multiplicity of $\Delta(\lambda)$ is $\#T^s(\lambda)$ for any $\lambda\in\bar\Gamma$.*

Proof. By [DJM, 4.12], T_Γ is a free R-module with a basis

$$\{m_{\mathbf{ST(t)}}\mid \mathbf{S}\in T^{ss}(\lambda,\mu),\mathbf{t}\in T^s(\lambda),\mu\in\Gamma,\lambda\in\bar\Gamma\}.$$

Note that $\mathbf{T}(\mathbf{t})\in T^{ss}(\lambda,\omega)$ defined in 3.4. For notational simplicity, we will simply write $\mathbf{t}$ for $\mathbf{T}(\mathbf{t})$. Its meaning will be clear from the context. Write $\bar\Gamma=\{\lambda_1,\lambda_2\cdots,\}$ such that $\lambda_j\lhd\lambda_i$ implies $j>i$. Let M_i be the R-submodule of T_Γ generated by

$$\{m_{\mathbf{St}}\mid \mathbf{S}\in T^{ss}(\lambda_j,\mu),\mathbf{t}\in T^s(\lambda_j),\mu\in\Gamma \text{ and } j\le i\}.$$

For any $\lambda\in\bar\Gamma,\mu,\nu\in\Gamma$, and $\mathbf{S}\in T^{ss}(\lambda,\mu),\mathbf{T}\in T^{ss}(\lambda,\nu)$, $\mu,\nu\in\Gamma$ we show that $\Phi^{\lambda}_{\mathbf{ST}}(m_{\mathbf{S}_1,\mathbf{t}})\in M_i$ for any $\mathbf{S}_1\in T^{ss}(\lambda_j,\mu_1)$ and $\mathbf{t}\in T^s(\lambda_j)$ $(j\le i)$ for some $\mu_1\in\Gamma$. By (4.2.1),

$$m_{\mathbf{S}_1\mathbf{t}}=\Phi^{\lambda_j}_{\mathbf{S}_1\mathbf{t}}(x_\omega).$$

If $\mu_1\neq\nu$, then $\Phi^{\lambda}_{\mathbf{ST}}\circ\Phi^{\lambda_j}_{\mathbf{S}_1,\mathbf{t}}=0$. So we can assume $\mu_1=\nu$. Since $\Phi^{\lambda}_{\mathbf{ST}}\circ\Phi^{\lambda_j}_{\mathbf{S}_1,\mathbf{t}}\in\mathrm{Hom}_{H_{m,r}}(x_\omega H_{m,r},x_\mu H_{m,r})$, we consider the abvious embedding $S(\Gamma)\subseteq S(\Gamma\cup\omega)$, and use cellular basis in Theorem 4.3(1) (for $S(\Gamma\cup\{\omega\})$ to

$$\Phi^{\lambda}_{\mathbf{ST}}\circ\Phi^{\lambda_j}_{\mathbf{S}_1,\mathbf{t}}=\sum_{\substack{\lambda_j\unlhd\eta\in\overline{\Gamma\cup\{\omega\}}\\ \mathbf{S}_2\in T^{ss}(\eta,\mu),\mathbf{s}\in T^s(\eta)}} f^{\eta}_{\mathbf{S}_2,\mathbf{s}}\Phi^{\eta}_{\mathbf{S}_2,\mathbf{s}}. \tag{4.7.1}$$

Here $f^{\eta}_{\mathbf{S}_2\mathbf{s}}\in R$. In the summation, $T^{ss}(\eta,\mu)\neq\emptyset$ implies $\eta\in\bar\Gamma$. Thus $\eta=\lambda_l\unlhd\lambda_j$ for some $l\le j\le i$. Acting on x_ω we get $\Phi^{\eta}_{\mathbf{S}_2,\mathbf{s}}(\omega)\in M_i$. Therefore $\Phi^{\lambda}_{\mathbf{ST}}(m_{\mathbf{S}_1\mathbf{t}})\in M_i$ and M_i is an $S(\Gamma)$-submodule of T_Γ. In the summation of (4.7.1), If $\eta=\lambda_j$, then $\mathbf{s}=\mathbf{t}$ (again by the cellular basis property for $S(\Gamma\cup\{\omega\})$ as in Theorem 4.3(1)). Thus for each $\mathbf{t}\in T^s(\lambda_i)$, the R-submodule $M_i^{(\mathbf{t})}$ of M_i/M_{i-1} generated by $\{m_{\mathbf{St}}+M_{i-1}\mid \mathbf{s}\in T^{ss}(\lambda_i,\mu),\mu\in\Gamma\}$ is an $S(\Gamma)$-submodule and free over R. Fix $T_0\in T^{ss}(\lambda_i,\mu)$ for some $\mu\in\Gamma$. Using Theorem 4.3(1) again, one can verify similarly as above, that the R-linear map $f : M_i^{(\mathbf{t})} \to \Delta(\lambda_i)$ with $f(m_{\mathbf{S},\mathbf{t}}) = \Phi_{\mathbf{ST}_0}$ is a left $S(\Gamma)$-module isomorphism. Thus $M_i/M_{i-1} \cong \Delta(\lambda_i)^{\# T^s(\lambda_i)}$ as $S(\Gamma)$-modules. □

Lemma 4.8. *Suppose that $u_1, u_2, \cdots, u_m$ are invertible in $R = \mathbb{Z}[q, q^{-1}]$. If Γ is saturated, then the endomorphism algebra $\mathrm{End}_{S(\Gamma)}(T_\Gamma)$ is a free R-module.*

Proof. Since $u_1, u_2, \cdots, u_m$ are invertible, by [M1, 5.13], there is a non-degenerated bilinear from $\langle\ ,\ \rangle_\lambda : x_\lambda H_{m,r} \times x_\lambda H_{m,r} \to R$. It induces a non-degenerated bilinear from $\langle , \rangle$ on T_Γ with

$$\langle h_1, h_2\rangle = \begin{cases} \langle h_1, h_2\rangle_\lambda, & \text{if } h_1, h_2 \in x_\lambda H_{m,r}, \lambda \in \Gamma,\\ 0, & \text{otherwise.}\end{cases}$$

The bilinear form $\langle , \rangle$ associative in the sense $\langle xh, y\rangle = \langle x, y(h^*)\rangle$ for all $h \in H_{m,r}$. Thus T_Γ is self-dual as a right $H_{m,r}$-module. Since $* : H_{m,r} \to H_{m,r}$ is an anti-automorphism, [DJ2, (1.5), (1.6)] implies that T_Γ is also self-dual as a left $S(\Gamma)$-module. By Proposition 4.7, T_Γ has a Δ-filtration. This implies that T_Γ is a tilting $S(\Gamma)$-module. Hence $\mathrm{Ext}_{S(\Gamma)}(\Delta(\lambda), T_\Gamma) = 0$ for all $i > 0$ (since $S(\Gamma)$ is quasi-hereditary algebra). Theorem 4.6 now implies that $S(\Gamma)$ has a S^λ-filtration of right $H_{m,r}$-module. This implies that $\mathrm{End}_{S(\Gamma)}(T_\Gamma)$ is a free R-module. □

To pass the result over to other base rings, we need the following

Lemma 4.9. *Suppose $u_1, u_2, \cdots, u_m$ are invertible elements in $R = \mathbb{Z}[q^{-1}, q]$. For any commutative R-algebra S, and any saturated subset $\Gamma \subseteq \Lambda_m(r)$, let $S(\Gamma)_S = S(\Gamma) \otimes_R S$ and $(T_\Gamma)_S = \oplus_{\lambda\in\Gamma} x_\lambda H_{m,r\,S}$. Then $\mathrm{End}_{S(\Gamma)}(T_\Gamma) \otimes_R S \cong \mathrm{End}_{S(\Gamma)\otimes_R S}((T_\Gamma)_S)$. In particular, if S is a field, then*

$$\dim \mathrm{End}_{S(\Gamma)\otimes_R S}((T_\Gamma)_S) = \sum_{\lambda\in\Gamma^+} \# T^(\lambda)^2.$$

Proof. In the proof of Lemma 4.8, we have proved that T_Γ is a tilting $S(\Gamma)$-module. By [DPS1, 4.2c], the base change property holds for $\mathrm{End}_{S(\Gamma)}(\oplus_{\lambda\in\Gamma} x_\lambda H_{m,r})$. Now, the formula about the dimension follows from Theorem 4.6 and Proposition 4.7. □

Theorem 4.10. *If Γ is a finite saturated subset of $\Lambda_m^+(r)$, and R is any commutive ring such that $u_1, \ldots, u_m$ are invertible in R, then $H(\Gamma) \cong \mathrm{End}_{S(\Gamma)}(\oplus_{\lambda\in\Gamma} x_\lambda H_{m,r})$ and $S(\Gamma) \cong \mathrm{End}_{H(\Gamma)}(\oplus_{\lambda\in\Gamma} x_\lambda H_{m,r})$.*

Proof. The natural map $H \to \mathrm{End}_{S(\Gamma)}(T_\Gamma)$ by right action defines a homomorphism of R-algebras. For any $\mu \in \Lambda_m^+(r)\backslash\Gamma^+$, by Lemma 3.7(b), $y_{\mathbf{s},\mathbf{t}}^{\mu'}$ annihilates T_Γ. Thus the map $H \to \mathrm{End}_{S(\Gamma)}(T_\Gamma)$ factors through $\phi : H(\Gamma) \to \mathrm{End}_{S(\Gamma)}(T_\Gamma)$.

We claim that the map ϕ is injective. Suppose we have

$$\phi(\sum_{\lambda\in\Gamma^+, \mathbf{s},\mathbf{t}\in T^s(\lambda')} r_{\mathbf{s},\mathbf{t}}^{\lambda'} y_{\mathbf{s},\mathbf{t}}^{\lambda'}) = 0, \text{ and not all } r_{\mathbf{s},\mathbf{t}}^{\lambda'} \text{ are zero.}$$

We can choose $\nu \in \Gamma^+$ such that ν is maximal among all $\lambda \in \Gamma^+$ with $r_{\mathbf{s},\mathbf{t}}^{\lambda'} \neq 0$ for some $\mathbf{s}, \mathbf{t} \in T^s(\lambda')$. For such a ν, choose $\mathbf{s}_0 \in T^s(\nu')$ such that $l(d(\mathbf{s}_0))$ is maximal among all $\mathbf{s} \in T^s(\nu')$ with $r_{\mathbf{s},\mathbf{t}}^{\nu'} \neq 0$ for some $\mathbf{t}$. By Proposition 2.7, there is a $y \in \mathfrak{S}_r$ with $w_{\nu'} = d(\mathbf{s}_0)y$ and $l(w_{\nu'}) = l(d(\mathbf{s}_0)) + l(y)$. Acting on $x_\nu T_y^* \in x_\nu H_{m,r}$, we have

$$0 = \sum_{\lambda\in\Gamma^+, \mathbf{s},\mathbf{t}\in T^s(\lambda')} r_{\mathbf{s},\mathbf{t}}^{\lambda'} x_\nu T_y^* y_{\mathbf{s},\mathbf{t}}^{\lambda'} = \sum_{\lambda\in\Gamma^+, \mathbf{s},\mathbf{t}\in T^s(\lambda')} r_{\mathbf{s},\mathbf{t}}^{\lambda'} x_\nu T_y^* T_{d(\mathbf{s})}^* y_{\lambda'} T_{d(\mathbf{t})}. \tag{4.10.1}$$

Lemma 3.7(b) implies that $x_\nu T_y^* T_{d(\mathbf{s})}^* y_{\lambda'} \neq 0$ only if $\nu \trianglelefteq \lambda$. By the choice of ν, the only possible nonzero terms on the right hand side of (4.10.1) are when $\lambda = \nu$.

Now, we deal with terms $x_\nu T_y^* T_{d(\mathbf{s})}^* y_{\nu'} T_{d(\mathbf{t})}$ with $\mathbf{s} \neq \mathbf{s}_0$. Since $l(d(\mathbf{s}_0))$ is maximal, Lemma 3.8(b) implies that $x_\nu T_y^* T_{d(\mathbf{s})}^* y_{\nu'} T_{d(\mathbf{t})} = 0$ for all $\mathbf{s} \neq \mathbf{s}_0$.

If $\mathbf{s} = \mathbf{s}_0$, then $x_\nu T_y^* T_{d(\mathbf{s})}^* y_{\nu'} T_{d(\mathbf{t})} = x_\nu T_{w_\nu} y_{\nu'} T_{d(\mathbf{t})}$ (using $w_{\nu'} = w_\nu^{-1}$), which is a basis element of the Specht module S^ν (see Proposition 3.3b). This forces $r_{\mathbf{s}_0,\mathbf{t}}^{\nu'} = 0$ which contradicts the choice. Hence, all $r_{\mathbf{s},\mathbf{t}}^{\lambda'}$ are zero and ϕ is injective.

Note that $H(\Gamma)$ is a free R-module of finite rank. The proof of the injectivity of ϕ is for arbitrary commutive ring R.

To show the surjectivity, we use the assumption on R, which makes R an $\mathbb{Z}[q, q^{-1}]$-algebra. For any field S and any ring homomorphism $R \to S$, S is automatically a $\mathbb{Z}[q, q^{-1}]$-algebra. Now we can apply Theorem 3.13 and Lemmas 4.8– 4.9, to get

$$\dim_S(H(\Gamma) \otimes_R S) = \sum_{\lambda \in \Gamma^+} \# T^s(\lambda')^2 = \sum_{\lambda \in \Gamma^+} \# T^s(\lambda)^2 = \dim_S(\mathrm{End}_{S(\Gamma)}(\oplus_{\lambda\in\Gamma} x_\lambda H_{m,r}) \otimes_R S).$$

Thus $\phi \otimes_R 1 : H(\Gamma) \otimes_R S \to \mathrm{End}_{S(\Gamma)}(\oplus_{\lambda\in\Gamma} x_\lambda H_{m,r}) \otimes_R S$ is an isomorphism for any field S. Now, Nakayama lemma implies that ϕ is an isomorphism over R. □

4.11. In [M2, 5.3], Mathas proved the double centralizer property for finite saturated set Γ such that $\omega = ((0), \ldots, (0), (1^r)) \in \Gamma$, by using the cyclotomic Schur functor. In this case, $x_\omega = 1$ and T_Γ is thus a faithful $H_{m,r}$-module. In our case, T_Γ is not faithful.

We remark that the double centralizer property for the Hecke algebras of type A plays an important role in the proof of quantum Schur-Weyl reciprocity in [DPS1]. However, we do not know how to realize the permutation modules T_Γ as a representation of certain quantum groups. Next section, we will construct a special tensor space T_Γ on which the affine quantum group $U_q(\hat{\mathfrak{gl}}_n)$ acts and the cyclotomic q-Schur algebra will be a quotient of the affine quantum group $U_q(\hat{\mathfrak{gl}}_n)$. This enables us to setup the quantum Schur-Weyl duality in this case.

5. Quantum Schur-Weyl duality

5.1. **Affine Hecke Algebras.** Let $A = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials in indeterminate q. The extended affine Hecke algebra $\hat{H}_r$ of type $\tilde{A}_{r-1}$ is a unital associative algebra with generators $T_i^\pm, X_j^\pm$ subject to the following conditions:

$$(5.1.1)\qquad \begin{array}{ll} (a) & (T_i + 1)(T_i - q^2) = 0, \quad 1 \le i \le r-1, \\ (b) & T_i T_j = T_j T_i, \quad |j - i| \ge 2, \\ (c) & T_i T_j T_i = T_j T_i T_j, \quad \text{if } |i-j| = 1, \\ (d) & X_i X_j = X_j X_i, \\ (e) & T_i X_j = X_j T_i, \quad \text{if } j \neq i, i+1, \\ (f) & T_i X_i T_i = q^2 X_{i+1}. \end{array}$$

Let $\rho = X_1 T_1 \cdots T_{r-1}$. Then $\rho T_i \rho^{-1} = T_{i+1}$, where the lower indices should be read module r. Denote by $T_0 = \rho T_{r-1} \rho^{-1}$. Then $T_0, T_1, \cdots, T_{r-1}$ generate a subalgebra $\tilde{H}_r$, which is isomorphic to the affine Hecke algebra of type $\tilde{A}_{r-1}$. Moreover, there is an algebraic isomorphism

$$\hat{H}_r \cong \mathbb{Z}[\rho] \wr \tilde{H}_r.$$

The subalgebra H_r, which is generated by $T_1, T_2, \cdots, T_{r-1}$ is isomorphic to the Hecke algebra associated to the symmetry group $\mathfrak{S}_r$. Take certain invertible elements $u_1, \cdots, u_m$ in $\mathbb{Z}[q, q^{-1}]$. Consider the polynomial $f = \prod_{i=1}^m (X_1 - u_i)$. Then the quotient algebra $H_{m,r} := \hat{H}_r/I$ is known as the cyclotomic Hecke algebra of type $G(m, 1, r)$ [AK], where I is the two-sided ideal of $\hat{H}_r$ generated by the polynomial f.

5.2. **Tensor Spaces.** Let V be a vector space over the field $\mathbb{C}(q)$ with dimension n. The tensor space $V^{\otimes r}$ has a basis $\{v_{\mathbf{i}} \mid \mathbf{i} \in [1, n]^r\}$, where $v_{\mathbf{i}} = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$ with $\mathbf{i} = (i_1, i_2, \cdots, i_r) \in [1, n]^r$. The symmetric group $\mathfrak{S}_r$ acts on the set $[1, n]^r$ by $\mathbf{i}w = (i_{(1)w}, i_{(2)w}, \ldots, i_{(r)w})$ for all $\mathbf{i} \in [1, n]^r$ and $w \in \mathfrak{S}_r$. $\mathfrak{S}_r$ also acts on $V^{\otimes r}$ by permuting the basis elements $v_{\mathbf{i}}^w = v_{\mathbf{i}w}$. It is known that the tensor space is a right H_r-module [J] such that the following conditions hold:

$$v_{\mathbf{i}} T_k = \begin{cases} q^2 v_{\mathbf{i}}, & \text{if } i_k = i_{k+1}, \\ q v_{\mathbf{i}}^{s_k}, & \text{if } i_k > i_{k+1}, \\ q v_{\mathbf{i}}^{s_k} + (q^2 - 1) v_{\mathbf{i}}, & \text{if } i_k < i_{k+1}. \end{cases} \tag{5.2.1}$$

Recall $\Lambda(n, r) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_{j=1}^n \lambda_j = r\}$ and $\Lambda(n, r)^+$ is the set of partitions of r with at most n parts. Recall that $\mathfrak{S}_\lambda$ is the Young subgroup with respect to the composition λ and $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$, where $T_{s_i} := T_i$ for the basic transposition $s_i = (i, i+1)$. It is known that there is a right H_r-module isomorphism

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda H_r.$$

Let $x = (x_1, \ldots, x_r)$ be indeterminates. The tensor polynomial space $T(n, r) = V^{\otimes r} \otimes \mathbb{C}(q)[x_1^{\pm}, \cdots, x_r^{\pm}]$ has a basis $\{v_{\mathbf{i}} \otimes x_{\mathbf{a}} \mid \mathbf{i} \in [1, n]^r, \mathbf{a} \in \mathbb{Z}^r\}$, where $x_{\mathbf{a}} = x_1^{a_1} \cdots x_r^{a_r}$ with $\mathbf{a} = (a_1, \ldots, a_r)$. $T(n, r)$ is a right $\hat{H}_r$-module such that the action of X_i is to multiply x_i^{-1} and the action of T_k is given as follows:

$$v_{\mathbf{i}} x_{\mathbf{a}} T_k = v_{\mathbf{i}} T_k x_{\mathbf{a}}^{s_i} + (1 - q^{-2}) v_{\mathbf{i}} \frac{x_{i+1}(x_{\mathbf{a}}^{s_i} - x_{\mathbf{a}})}{x_{i+} - x_i}$$

The above action can be found in [GV]. In fact, it comes from the formula on the multiplication of T_k and X_l in $\hat{H}_r$.

Lemma 5.3. *There is a right $\hat{H}_r$-module isomorphism $V^{\otimes r} \otimes \mathbb{C}(q)[x_1^{\pm}, \cdots, x_r^{\pm}] \cong \oplus_{\lambda \in \Lambda(n,r)} x_\lambda \hat{H}_r$*

Proof. The weight of $\mathbf{i} \in [1, n]^r$ is $\lambda \in \Lambda(n, r)$ defined by $\lambda_k = |\{j \in [1, r] \mid i_j = k\}|$. Note that $\mathbf{i}w$ and $\mathbf{i}$ have the same weight λ. For a given $\lambda \in \Lambda(n, r)$, set

$$\mathbf{i}(\lambda) = (\underbrace{1, \ldots, 1}_{\lambda_1 \text{ times}}, \underbrace{2, \ldots, 2}_{\lambda_2 \text{ times}}, \ldots, \underbrace{r, \ldots, r}_{\lambda_r \text{ times}})$$

which is an element of $[1, n]^r$ of weight λ. For $\mathbf{i}$ of weight λ, there is a unique $d \in D_\lambda$ such that $\mathbf{i} = (\mathbf{i}(\lambda))d$. This defines a bijection between D_λ and $\mathbf{i}(\lambda)\mathfrak{S}_r$. Thus we have $v_{\mathbf{i}} = v_{\mathbf{i}(la)}^d$. Then the linear map $V^{\otimes r} \to \oplus_{\lambda \in \Lambda(n,r)} x_\lambda H_r$ defined by $v_{\mathbf{i}} \mapsto x_\lambda T_d$ is the right H_r-module isomorphism mentioned earlier. Here $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$. Now, define the linear map

$$T(n, r) \to \otimes_{\lambda \in \Lambda(n,r)} x_\lambda \hat{H}_r$$

by $v_{\mathbf{i}}x_{\mathbf{a}} \mapsto x_\lambda T_d X_{-\mathbf{a}}$ if $\mathbf{i}$ has weight λ. One will see that this is a bijection. Also, by Bernstein formula

$$X_{\mathbf{a}}T_k = T_k X_{\mathbf{a}}^{s_i} + (1-q^{-2})\frac{X_{i+1}(X_{\mathbf{a}}^{s_i} - X_{\mathbf{a}})}{X_{i+1} - X_i} \tag{5.3.1}$$

one will check easily that it is a right $\hat{H}_r$-module homomorphism. $\square$

5.4. **Cyclotomic Tensor Spaces.** We are in a position to introduce cyclotomic tensor polynomial space

$$T_m(n,r) = (V^{\otimes r} \otimes \mathbb{C}(q)[x_1^{\pm}, \cdots, x_r^{\pm}]) \otimes_{\hat{H}_r} H_{m,r}$$

where $H_{m,r}$ is the cyclotomic Hecke algebra of type $G(m,1,r)$ mentioned above. For each $\lambda \in \Lambda(n,r)$, we associate an m-composition with mth component being λ and all other components being 0. Let $\Lambda(n,r)^{(m)} \subseteq \Lambda_m(r)$ be the subset of all such m-compositions. Then for $\lambda \in \Lambda(n,r)^{(m)}$, x_λ defined in 3.2 for m-partitions is the same as we just defined above. Tensoring $H_{m,r}$ over $\hat{H}_r$ to the isomorphism in Lemma 5.3, we have the following isomorphism of right $H_{m,r}$-modules

$$T_m(n,r) \cong \bigoplus_{\lambda \in \Lambda(n,r)^{(m)}} x_\lambda H_{m,r} \tag{5.4.1}$$

In [AK], it is proved that $H_{m,r}$ is a left H_r-module with basis $\{t_{\mathbf{b}} \mid \mathbf{b} \in \mathbb{Z}_m^r\}$. The following result can be proved easily.

Lemma 5.5. *The tensor space* $T_m(n,r) = V^{\otimes r} \otimes \mathbb{C}(q)[x_1^{\pm}, \cdots, x_r^{\pm}] \otimes_{\hat{H}_r} H_{m,r}$ *has a* $\mathbb{C}(q)$*-basis* $\{v_{\mathbf{i}}x_{\mathbf{a}}t_{\mathbf{b}} \mid \mathbf{i} \in [1,n]^r, \mathbf{a} \in \mathbb{Z}^r, \mathbf{b} \in \mathbb{Z}_m^r\}$.

5.6. **Quantum Groups.** We recall the definition of quantum affine $U_q(\hat{\mathfrak{gl}}_n)$ introduced by Drinfeld in [D], called Drinfeld new realization. $U_q(\hat{\mathfrak{gl}}_{\mathfrak{n}})$ is an associative algebra over $\mathbb{C}(q)$ with generators

$$C, C^{-1}, K_\alpha, K_\alpha^{-1}, H_\alpha(i), i \in \mathbb{Z}\backslash\{0\} \text{ and } E_\alpha(j), F_\alpha(j), j \in \mathbb{Z}$$

with relations, which can be found in section 3 of [GV]. It is proved in [GV, Lu, VV] that, if q is not a root of unity, there is a surjective algebra homomorphism

$$U_q(\hat{\mathfrak{gl}}_{\mathfrak{n}}) \twoheadrightarrow \operatorname{End}_{\hat{H}_r}(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \hat{H}_r) \tag{5.6.1}$$

The latter is called affine q-Schur algebra defined in [Gr1] We have the following proposition.

Proposition 5.7. *Let* $S_m(n,r) = \operatorname{End}_{H_{m,r}}(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda H_{m,r})$. *If q is not a root of unity, then there is a surjective algebra homomorphism*

$$U_q(\hat{\mathfrak{gl}}_{\mathfrak{n}}) \twoheadrightarrow S_m(n,r)$$

Proof. Under the canonical homomorphism $\pi : \hat{H}_r \twoheadrightarrow H_{m,r}$, the elements $\hat{x}_\lambda$ defined in $\hat{H}_r$ has image x_λ for each composition λ of r with at most n-parts. By [Gr1, 2.2.4], the affine q-Schur algebra has a basis $\phi_{\lambda\mu}^d \mid \lambda, \mu \in \Lambda(n,r), d \in D_{\lambda\mu}\}$, where $D_{\lambda\mu}$ is the set of double coset representatives of $\mathfrak{S}_\lambda\backslash\hat{S}_r/\mathfrak{S}_\mu$. Moreover,

$$\phi_{\lambda\mu}^d(x_\mu) = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w.$$

Let $\tilde{\phi}^d_{\lambda\mu}(x_\mu) = \pi(\sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w)$. It is easy to see that $\pi(\sum_{w\in\mathfrak{S}_\lambda d\mathfrak{S}_\mu} T_w \in x_\lambda H_{m,r} \cap H_{m,r}x_\mu$. Thus $\tilde{\Phi}^d_{\lambda\mu} \in S_m(n,r)$. Since π is a homomorphism, the induced map sending $\phi^d_{\lambda\mu}$ to $\tilde{\phi}^d_{\lambda\mu}$ is a homomorphism from affine q-Schur algebra to $S_m(n,r)$. Now, we prove that it is a surjective. Consider the element $\phi_{\mathbf{ST}} \in \mathrm{Hom}_{H_{m,r}}(x_\lambda H_{m,r}, x_\mu H_{m,r})$, where $H_{m,r}$ is the Ariki-Koike Hecke algebra, $\mathbf{S} \in T^{ss}(\nu,\lambda)$ and $\mathbf{T} \in T^{ss}(\nu,\mu)$. Then ϕ is determined uniquely by $\phi_{\mathbf{ST}}(x_\lambda) = m_{\mathbf{ST}} \in x_\mu H_{m,r} \cap H_{m,r}x_\lambda$. Since π is a surjective algebra map $\hat{H}_{m,r} \to H_{m,r}$, one can find at least an element $\tilde{m}_{\mathbf{ST}}$, which is obtained from $m_{\mathbf{ST}}$ by replacing t_i by X_i for all i. By the proof of [DJM, 6.3], $\tilde{m}_{\mathbf{ST}} \in x_\mu \hat{H}_r \cap \hat{H}_r x_\lambda$. So one can define a $\hat{\Phi}_{\mathbf{ST}} \in \mathrm{Hom}_{\hat{H}_r}(x_\lambda \hat{H}_r, x_\mu \hat{H}_r)$ with $\hat{\phi}(x_\lambda) = \tilde{m}_{\mathbf{ST}}$. This shows that the canonical map from affine q-Schur algebra to $S_m(n,r)$ is surjective. □

We are going to prove that there is a surjective map from a cyclotomic Hecke algebra to the endomorphism ring associated to $U_q(\hat{\mathfrak{gl}}_n)$. First, we need some notions.

Definition 5.8. Fix positive integers m and r. Let

$$\Gamma = \{\lambda \in \Lambda^+_m(r) \mid T^{ss}(\lambda,\mu) \neq \emptyset, \text{ for some } \mu \in \Lambda(n,r)\}.$$

Theorem 5.9. *Let I be the annihilator of the right $H_{m,r}$-module $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}$. If $H_{m,r}$ is semisimple, then I is the free R-module with a basis*

$$\{y^{\lambda'}_{\mathbf{st}} \mid \lambda \in \Gamma, \mathbf{s},\mathbf{t} \in T^s(\lambda)\}. \tag{5.9.1}$$

Moreover, there is an isomorphism

$$H_{m,r}/I \cong \mathrm{End}_{S_m(n,r)}(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}) = End_{U_q(\hat{\mathfrak{gl}}_\mathfrak{n})}(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}). \tag{5.9.2}$$

Proof. Since I is the annihilator of the right $H_{m,r}$-module $\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}$, the natural map

$$H_{m,r} \to \mathrm{End}_{S_m(n,r)}(\bigoplus_{\mu\in\Lambda(n,r)} x_\lambda H_{m,r})$$

factors through the quotient $H_{m,r}/I$. Moreover, the induced map is injective. We claim that

$$\mathrm{dim} H_{m,r}/I = \ \mathrm{dim}\ \mathrm{End}_{S_m(n,r)}(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}). \tag{5.9.3}$$

Let $\Delta(\lambda)$ is the cell module of $S_m(n,r)$ with respect to the m-partition $\lambda \in \Gamma$. Using [DJM, 4.14, 6.6], the tensor space $\bigoplus_{\mu\in\Lambda(n,r)} x_\mu H_{m,r}$ has a Δ-filtration such that the multiplicity of $\Delta(\lambda)$ is equal to $\#T^s(\lambda)$. Since $H_{m,r}$ is semisimple, so is $S_m(n,r)$. By Schur's lemma, we have :

$$\mathrm{dim}\ \mathrm{End}_{S_m(n,r)}(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}) = \sum_{\lambda\in\Gamma} \#T^s(\lambda)^2. \tag{5.9.4}$$

Now, we compute the dimension of the quotient algebra $H_{m,r}/I$. Since $H_{m,r}$ is semisimple, the corresponding parameters $u_1, u_2, \cdots, u_m$ are distinct (see, e.g. [A2] or [DR1, 5.2]). Suppose that $\lambda \in \Gamma$. By definition, $y^{\lambda'}_{\mathbf{st}} = T^*_{d(\mathbf{s})} y_{\lambda'} T_{d(\mathbf{t})}$. Since $T^{ss}(\lambda,\mu) \neq \emptyset$, $T^{cs}(\lambda',\lambda) \neq \emptyset$, and $T^*_{d(\mathbf{s})}$ and $T_{d(\mathbf{t})}$ are invertible, Lemma 3.3(a) shows that $x_\mu H_{m,r} T^*_{d(\mathbf{s})} y_{\lambda'} T_{d(\mathbf{t})} \neq 0$. We claim that the set

$$\{y^{\lambda'}_{\mathbf{st}} + I \mid \lambda \in \Gamma, \mathbf{s},\mathbf{t} \in T^s(\lambda')\} \tag{5.9.5}$$

is linear independent. The claim implies $\dim H_{m,r}/I \geq \sum_{\lambda\in\Gamma} (\#T^s(\lambda))^2$ and hence

$$\mathrm{dim} H_{m,r}/I = \mathrm{dim}\mathrm{End}_{S_m(n,r)}(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r}).$$

In the rest of the proof, we prove (5.9.5). Suppose $\sum_{\substack{\lambda\in\Gamma\\ \mathbf{s},\mathbf{t}\in T^s(\lambda')}} f^{\lambda'}_{\mathbf{st}} y^{\lambda'}_{\mathbf{st}} \in I$, where $f^{\lambda'}_{\mathbf{st}}$ are certain elements in the based field. We will prove that all the coefficients must be zero, which implies (5.9.5). First, take ν such that ν is maximal among all $\lambda\in\Gamma$ with $f^{\lambda'}_{\mathbf{st}}\neq 0$ for some $\mathbf{s},\mathbf{t}\in T^s(\lambda)$. Then there is a $\mu\in\Lambda(n,r)$ such that $T^{ss}(\nu,\mu)\neq\emptyset$. By [DJM, 4.6], the element $m_{\mathbf{S},\mathbf{u}}$ is a basis element of $x_\mu H_{m,r}$ where $\mathbf{S}\in T^{ss}(\nu,\mu)$ and $\mathbf{u}\in T^s(\nu)$. Moreover, by [DJM, 4.8],

$$m_{\mathbf{S},\mathbf{u}} = \sum_{\substack{\mathbf{v}\in T^s(\nu)\\ \phi(\mathbf{v})=\mathbf{S}}} x^\nu_{\mathbf{vu}} \in T_m(n,r), \tag{5.9.6}$$

where ϕ is a map defined in [DJM, 4.2]. Since ν is maximal, Lemma 3.3(b) implies $m_{\mathbf{Su}} y^{\lambda'}_{\mathbf{st}}\neq 0$ only if $\nu\trianglelefteq\lambda$. Thus,

$$0=(m_{\mathbf{Su}})\sum_{\substack{\lambda\in\Gamma\\ \mathbf{s},\mathbf{t}\in T^s(\lambda')}} f^{\lambda'}_{\mathbf{st}} y^{\lambda'}_{\mathbf{st}} = m_{\mathbf{Su}}\sum_{\mathbf{s},\mathbf{t}\in T^s(\nu)} f^{\nu'}_{\mathbf{st}} y^{\nu'}_{\mathbf{st}}.$$

In order to compute $m_{\mathbf{Su}} y^{\nu'}_{\mathbf{st}}$, by (5.9.6), we need calculate $x^\nu_{\mathbf{vu}} y^{\nu'}_{\mathbf{st}}$. First, take $\mathbf{s}_0\in T^s(\nu)$ such that $f^{\nu'}_{\mathbf{s}_0\mathbf{t}}\neq 0$ for some $\mathbf{t}\in T^s(\nu')$ and $l(d(\mathbf{s}_0))$ is maximal. By Proposition 2.6 and [DR1, 1.8], $d(\mathbf{s}_0)\geq_w w_{\nu'} = w_\nu^{-1}$. There is an element $x\in\mathfrak{S}_r$ such that $w_\nu^{-1}=d(\mathbf{s}_0)x$ and $l(x)+l(d(\mathbf{s}_0)) = l(w_\nu^{-1}$. We have $x^{-1}d(\mathbf{s}_0)^{-1}=w_\nu$. In this case, by Proposition 2.6, $x^{-1}=d(\mathbf{u})$ for some $\mathbf{u}\in T^s(\nu)$. By Lemma 4.5(b), we have $(x^\nu_{\mathbf{vu}})y^{\nu'}_{\mathbf{st}}=0$ unless $\mathbf{s}=\mathbf{s}_0$. Thus $m_{\mathbf{S},\mathbf{u}})y^{\nu'}_{\mathbf{st}}=0$ for all $\mathbf{s}\neq\mathbf{s}_0$. We have

$$0=(m_{\mathbf{Su}})\sum_{\mathbf{s},\mathbf{t}\in T^s(\nu')} f^{\nu'}_{\mathbf{st}} y^{\nu'}_{\mathbf{st}} = (m_{\mathbf{Su}})\sum_{\mathbf{t}\in T^s(\nu)} f^{\nu'}_{\mathbf{s}_0\mathbf{t}} y^{\nu'}_{\mathbf{s}_0\mathbf{t}} = \sum_{\substack{\mathbf{v}\in T^s(\nu)\cap\phi^{-1}(\mathbf{S})\\ \mathbf{t}\in T^s(\nu')}} f^{\nu'}_{\mathbf{s}_0\mathbf{t}} T^*_{d(\mathbf{v})} x_\nu T_{w_\nu} y_{\nu'} T_{d(\mathbf{t})}. \tag{5.9.7}$$

Take $\mathbf{t}_0\in T^s(\nu')$ such that $l(d(\mathbf{t}_0))$ is maximal among all $\mathbf{t}$ with $f^{\nu'}_{\mathbf{s}_0\mathbf{t}}\neq 0$. By Proposition 2.6, there is a $y\in\mathfrak{S}_r$ such that $d(\mathbf{t})y=w_{\nu'}$. Lemma 4.5(b) implies that

$$y_{\nu'}T_{d(\mathbf{t})}T_y x_\nu = 0, \text{ unless } \mathbf{t}=\mathbf{t}_0.$$

Acting the element $T_y x_\nu T_{w_\nu} y_{\nu'}$ on the both side of (5.9.7), we have

$$0=\sum_{\mathbf{v}\in T^s(\nu)\cap\phi^{-1}(\mathbf{S})} f^{\nu'}_{\mathbf{s}_0\mathbf{t}_0} T^*_{d(\mathbf{v})} x_\nu T_{w_\nu} y_{\nu'} T_{w_{\nu'}} x_\nu T_{w_\nu} y_{\nu'}.$$

By [DR2, 2.2], $x_\nu H_{m,r} y_{\nu'}$ is of rank one with a basis $x_\nu T_{w_\nu} y_{\nu'}$. Thus $x_\nu T_{w_\nu} y_{\nu'} T_{w_{\nu'}} x_\nu T_{w_\nu} y_{\nu'} = g(q) x_\nu T_{w_\nu} y_{\nu'}$ for some element $g(q)$ in the based field. Thus

$$\sum_{\mathbf{u}\in T^s(\nu)} f^{\nu'}_{\mathbf{s}_0\mathbf{t}_0} g(q) T^*_{d(\mathbf{v})} x_\nu T_{w_\nu} y_{\nu'} = 0.$$

By Proposition 3.2(b), the element $T^*_{d(\mathbf{t})} x_\nu T_{w_\nu} y_{\nu'}$ is a basis element of the left module $H_{m,r} x_\nu H y_{\nu'}$. This implies that

$$f^{\nu'}_{\mathbf{s}_0\mathbf{t}_0} g(q) = 0. \tag{5.9.8}$$

Suppose that $H_{m,r}$ is semi-simple, the Specht module $J = x_\nu T_{w_\nu} y_{\nu'} H_{m,r}$ is projective. Therefore, $J^2=J$. Moreover, by [DR2, 2.2],

$$0\neq J^2 = x_\nu T_{w_\nu} \underline{y_{\nu'} H_{m,r} x_\nu} T_{w_\nu} y_{\nu'} H_{m,r} = x_\nu T_{w_\nu} y_{\nu'} T_{w_{\nu'}} x_\nu T_{w_\nu} y_{\nu'} H_{m,r}.$$

In particular, $x_\nu T_{w_\nu} y_{\nu'} T_{w_{\nu'}} x_\nu T_{w_\nu} y_{\nu'} \neq 0$. Thus $g(q) \neq 0$, forcing $f^{\nu'}_{\mathbf{s}_0\mathbf{t}_0} = 0$ by (5.9.8), a contradiction. This completes the proof of our claim. □

5.10. We remark that (1) if $n > r$, then $(1^r, 0, \cdots, 0) \in \Lambda(n, r)$. In this case the annihilator is zero and hence $H_{m,r} \cong \mathrm{End}_{S_{m,n}}(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda H_{m,r})$. This result was proved by Mathas in [M2, 5.3]. However, $(1^r, 0, \cdots, 0) \notin \Lambda(n, r)$ if $n < r$.

(2) The set Γ defined in definition 5.7 is not a co-ideal. For example, let $r = m = 2$ and $n = 1$. Obviously, $\lambda = ((11), 0) \trianglerighteq (0, (2)) = \mu$. But $\lambda \notin \Gamma$ since $T^{ss}(\lambda, \mu) = \emptyset$. Therefore, Theorem 5.8 are not included in [M2, 5.3] and Theorem 4.6.

(3) The Theorem 5.8 remains true if we use a subset $\Omega \in \Lambda_m(r)$ to replace $\Lambda(n, r)$ in the definition of Γ. This can be seen from the proof of Theorem 5.8 since our arguments do not depend on the set $\Lambda(n, r)$.

6. Stratifying cyclotomic q-Schur algebras

Note that the cyclotomic q-Schur algebra we defined in the previous section is not necessorily a quasi-hereditary algebra. In this section, we study its stratified structure in the sense of [CPS2] and [DPS2]. For notational simplicity, we set $R = \mathbb{Z}[q^{\pm1}, u_1^{\pm1}, \ldots, u_m^{\pm1}]$. The invertibility of $u_1, \ldots, u_m$ is only used when we applied the result of [M1, 5.13], to get $(H_{m,r}x_\lambda)^* \cong x_\lambda H_{m,r}$. We recall that

$$T_m(n, r) = V^{\otimes r} \otimes_{H_r} H_{m,r} = \oplus_{\lambda\in\Lambda(n,r)} x_\lambda H_r \otimes_{H_r} H_{m,r}.$$

where V a free R-module of rank n.

6.1. Let $s_i = (i, i + 1)$ be the basic transposition. Then $S = \{s_1, s_2, \cdots, s_{r-1}\}$ is the standard Coxeter generating set of the symmetry group $\mathfrak{S}_r$. For each composition $\lambda = (\lambda_1, \ldots, \lambda_t, \ldots)$ of r, let $I_\lambda \subseteq S$ be the generating set of S_λ (which is a parabolic subgroup of the Coxeter group $\mathfrak{S}_r$. More precisely, $s_j \in I_\lambda$ if and only if $\lambda_0 + \cdots + \lambda_{i-1} < j < \lambda_1 + \cdots + \lambda_i$ for some $i \geq 1$ assuming $\lambda_0 = 0$.

Note that removing any zero component of λ does not change S_λ and thus I_λ. Let $l(\lambda)$ be the number of non-zero components. We can assume that $\lambda = (\lambda_1, \ldots, \lambda_t)$ with $\lambda_i > 0$ for $i = 1, \ldots, t$. Then

$$S \setminus I_\lambda = \{s_{\lambda_1+\cdots+\lambda_i} | i = 1, 2, \ldots, t - 1\}. \tag{6.1.1}$$

Then $|S \setminus I_\lambda| = l(\lambda) - 1$.

Let 2^S be the power set of S which is naturally a poset under the inclusion order. The assignement $\lambda \mapsto I_\lambda$ defines a map $\Lambda(r) \to 2^S$. The map is onto and $I_\lambda = I_\mu$ if and only if $\bar\lambda = \bar\mu$. Then (6.1.1) implies that

Lemma 6.2. *The subset $\Gamma = \{I_\lambda | \lambda \in \Lambda(n, r)\}$ is coideal of the poset 2^S.*

In order to study the stratified structure on the endomorphism algebra $End_{H_{m,r}}(T_m(n, r))$, we need a quasi-poset structure on Γ, which will be defined lately.

6.3. Recall that under the identification of $\Lambda(n, r)$ with the subset $\Lambda_m(n, r)_m$ of $\Lambda_m(r)$, we have $x_\lambda = x_{\bar\lambda} \in H_r \subseteq H_{m,r}$. We simply denote by x_λ by x_I if $I = I_\lambda$. Now, we can write $T = \oplus_{I\in\Gamma} x_I H_{m,r}^{\oplus m_I}$ where $m_I = |\{\lambda \in \Lambda(n, r) \mid I = I_\lambda\}|$. Recall that for any $w \in \mathfrak{S}_r$,

$$C'_w = \sum_{y\leq w} P_{y,w}(q^2) q^{-l(w)} T_y$$

where $P_{y,w}(q^2)'s$ are known as Kazhdan-Lusztig polynomials. Then $\{C'_w \mid w \in \mathfrak{S}_r\}$ is the Kazhdan-Lusztig basis of H_r. It is easy to see that $H_{m,r}$ has a basis $\{t_1^{a_1} \cdots t_r^{a_r} C'_w \mid w \in \mathfrak{S}_r, 0 \leq a_i \leq m-1\}$.

Lemma 6.4. *(a) The left $H_{m,r}$-module $H_{m,r}x_I$ is a free R-module with basis $\{t_1^{a_1} \cdots t_r^{a_r} C'_w \mid 0 \leq a_i \leq m-1, 1 \leq i \leq r, w \in \mathfrak{S}_r, R(w) \supset I\}$, where $R(w) = \{s \in S \mid ws < w\}$.*

(b) The right $H_{m,r}$-module $x_I H_{m,r}$ is a free Z-module with basis $\{C'_w t_1^{a_1} \cdots t_r^{a_r} \mid 0 \leq a_i \leq m-1, 1 \leq i \leq r, w \in \mathfrak{S}_r, L(w) \supset I\}$, where $L(w) = \{s \in S \mid sw < w\}$.

Proof. The result is known when $m = 1$, see [DPS2, 2.3.5]. In general, since $H_{m,r}x_I = H_{m,r} \otimes_{H_r} H_r x_I$ and the result follows immediately. □

Recall that $\sigma : H_{m,r} \to H_{m,r}$ is the anti-involution such that $\sigma(T_i) = T_i$ for $i = 0, \ldots, r-1$. Note that H_r is σ-invariant, and the restriction of σ on H_r is the anti-involution used in [DPS2]. σ defines an isomorphism of categories Mod- $H_{m,r} \to H_{m,r}$ -Mod such that for any M in Mod- $H_{mr,}$, $\sigma(M)$ has the left $H_{m,r}$-module structure defined by $hx := x\sigma(h)$ for all $x \in M$ and $h \in H_{m,r}$. The inverse is $\sigma : H_{m,r}$ -Mod $\to$ Mod- $H_{m,r}$. Since $\sigma(T_w) = T_{w^{-1}}$, we have $\sigma(x_I) = x_I$ and $\sigma(H_{m,r}x_I) = x_I H_{m,r}$ and $\sigma(x_I H_{m,r}) = H_{m,r}x_I$. Recall that $*$: Mod- $H_{mr} \to H_{m,r}$ -Mod is a contraviant functor. Since $H_{m,r}x_I$ is free of finte rank over R, then $\mathrm{Hom}_{H_{m,r}}(x_I H_{m,r}, x_J H_{m,r}) = \mathrm{Hom}_{H_{m,r}}((x_J H_{m,r})^*, (x_I H_{m,r})^*)$. Therefore, we have an isomorphism of algebras

$$S_m(n,r) = \mathrm{End}_{H_{m,r}}(\oplus_{I \in \Gamma} x_I H_{m,r}^{\oplus m_I}) = \mathrm{End}_{H_{m,r}}(\oplus_{I \in \Gamma} H_{m,r} x_I^{\oplus m_I}).$$

Mathas defined a functor $H_{m,r}$ -Mod $\to H_{m,r}$ -Mod by $M \mapsto \sigma(M^*) \cong (\sigma M)^*$ and proved in [M1, 5.13], under the assumption that all u_i are invertible for $i = 1, \cdots, m$, that $H_{m,r}x_I$ is self-dual, i.e., $\sigma((H_{m,r}x_I)^*) \cong H_{m,r}x_I$. Applying the functor σ, we get that $(H_{m,r}x_I)^* \cong x_I H_{m,r}$. Thus the above two description of $S_m(n,r)$ are the same.

6.5. For each $I \in \Gamma$, we list the set $\{J \in 2^S \mid J \supseteq I\} = \{J_1, \ldots, J_t\}$ such that $J_i \subset J_j$ implies $i \geq j$ (thus $J_t = I$ and $J_1 = S$). Following [DPS2, 2.3.6], we define, for each $i = 1, \ldots, t$,

$$E_i = span_R\{C_w \mid R(w) = J_j, j \leq t-i\}. \tag{6.5.1}$$

Note that E_i is an H_r-submodule of $H_r x_I$. We set $M_i = H_{m,r} \otimes_{H_r} E_i$, which is an $H_{m,r}$-submodule of $H_{m,r}x_I$. By the above description of the R-basis of $H_{m,r}x_I$, we see that E_i is a free direct summand of $H_r x_I$ and M_i is free direct summand of $H_{m,r}x_I$. We now consider the dual filtration defined in [DPS2, 2.3.6] $F^i = (H_r x_I/E_i)^*$ and $N^i = (H_{m,r}x_I/M_i)^*$ which is an ascending filtration of right $H_{m,r}$-modules of $(H_{m,r}x_I)^*$. By the above setting and the fact that E_i is a free direct summand of $H_r x_I$, we have canonical isomorphism of right $H_{m,r}$-modules

$$N^i \cong F^i \otimes_{H_r} H_{m,r}^*.$$

In [DPS2, 2.4.2], they defined $S_I = F^1$ as a right H_r-module. We define the right $H_{m,r}$-module $S^I = N^1$. Then

$$S^I \cong S_I \otimes_{H_r} H_{m,r}^*$$

as right $H_{m,r}$-modules.

Now, we are in a position to introduce a quasi-poset structure on Γ. For any $I, J \in \Gamma$, write $I \leq J$ if either $I \subset J$ or $Hom_{H_{m,r}}(S^I, S^J) \neq 0$. Note that for the semisimple $H_{m,r}$, $Hom_{H_{m,r}}(S^I, S^J) \neq 0$ if and only if $Hom_{H_{m,r}}(S^J, S^I) \neq 0$. Therefore, the quasi partial order $\leq$ results in the notion of cells on Γ. More explicitly, say $I \sim J$ if $I \leq J$ and $J \leq I$. One can easily see that $I \sim J$ for $I \neq J$ if and only only if $Hom_{H_{m,r}}(S^I, S^{I_J}) \neq 0$.

Theorem 6.6. *Let $\le$ be the quasi partial order on Γ defined above. The date $T = \otimes_{I\in\Gamma}(H_{m,r}x_I)^*$, S^I and N^i statisfy the following conditions:*

(1) Each right $H_{m,r}$-module $x_I H_{m,r}$ has a filtration such that each section is isomorphic to some S^J with $J \ge I$. In particular $J \in \Gamma$.

(2) For any $I, J \in \Gamma$, $Hom_{H_{m,r}}(S^I, x_J H) \ne 0$ implies $J \le I$.

(3) For all $I \in \Gamma$, $Ext^1_{H_{m,r}}(x_I/N_i, T) = 0$ for all i.

Proof. By (6.2), Γ is a coideal of 2^S with respect to the usual inclusion order. So each section F^i/F^{i+1} is isomorphic to some S_J as a right H_r-module ([DPS2, 1.2.9(1)]). Since $J \subset I$, $J \in \Gamma$. Using the fact that $H^*_{m,r}$ is a free H_r-module, then (1) follows.

(2) Suppose $Hom_{H_{m,r}}(S^I, x_J H_{m,r}) \ne 0$. Since $x_J H_{m,r}$ has a S^{J_i}-filtration, $Hom_{H_{m,r}}(S^I, S^{J_i}) \ne 0$ for some i. So $I \sim J_i$. Since $J_i \supset J$, $J_i \ge J$. Therefore $I \ge J$.

Now, we prove (3). By [DPS2, 1.2.13.1], we need to prove the following isomorphism

$$Hom_{H_{m,r}}(x_I H_{m,r}/N_i, x_J H_{m,r}) \otimes Z' \cong Hom_{H'_{m,r}}((x_I H_{m,r}/N_i)_{Z'}, x_J H'_{m,r}) \qquad (*)$$

where $H'_{m,r} = H_{m,r} \otimes_R Z'$ In fact, this result can be proved similarly as [DPS2, 2.3.8], which will be stated as follows. Since $H_{m,r}$ is a free R-algebra, one has

$$\mathrm{Hom}_{H_{m,r}}(x_I H_{m,r}/N_i, x_J H_{m,r}) \cong \mathrm{Hom}_{H_{m,r}}((x_J H_{m,r})^*, (x_I H_{m,r}/N_i)^*) = \mathrm{Hom}_{H_{m,r}}(H_{m,r}x_J, M_i).$$

Therefore, we need to prove

$$Hom_{H_{m,r}}(H_{m,r}x_J, M_i)_{Z'} \cong Hom_{H'_{m,r}}(H'_{m,r}x_J, (M_i)_{Z'})$$

for all i. However, each element $\phi \in \mathrm{Hom}_H(Hx_J, M_i)$ is uniquely determined by the image of $\phi(x_I) \in M_i$.

The symmetry group $\mathfrak{S}_r$ acts on the set of elements $\mathbf{a}$. Let $\mathfrak{S}_{\mathbf{a}} = stab_{\mathfrak{S}_r}(\mathbf{a})$ be the stable subgroup of $\mathfrak{S}_r$. Let $D_{\mathbf{a}}^{-1}$ be the distinguished left coset representatives with minimal length. We define

$$D_{I,\mathbf{a}}^{-1} = \{y \in \mathfrak{S}_I \cap D_{\mathbf{a}}^{-1} \mid y \cdot w \in \mathfrak{S}_I \text{ for some } w \in \mathfrak{S}_I \cap \mathfrak{S}_{\mathbf{a}}\}.$$

We claim that such a element x can be expressed as a linear combination of elements $\sum_{y\in D_{I,\mathbf{a}},\mathbf{a},w} t^y_{\mathbf{a}} C_w$ with $I \subset L(w)$, $R(w) = J_j$ with $j \le t - i$

In fact, write $x = \phi(x_I) = \sum k_{\mathbf{a},w} t_{\mathbf{a}} C_w \in M_i$, where $k_{\mathbf{a},w} \in R$. We have $T_{s_p} x = q^2 x$ for any $s_p \in I$. First of all, we prove $s_p \in L(w)$ for any $s_p \in I$. It is proved in [KL] that

$$T_s C'_w = \begin{cases} -C'_w + qC'_{sw} + q\sum_{z<w, sz<z} \mu(z,w) C'_z, & \text{if } sw > w \\ q^2 C'_w, & ws < w, \end{cases} \tag{6.6.1}$$

where $\mu(z, w)$ (an integer) is the leading coefficient of the Kazhdan-Lusztig polynomial $P_{z,w}(q)$. Here $\le$ is the Bruhat order on the Coxeter group $\mathfrak{S}_n$. Now, using **??**, we have

$$T_{s_p} t_{\mathbf{a}} C_w = t^{s_p}_{\mathbf{a}} T_{s_p} C_w + (1-q^2)\frac{t_{p+1}(t^{s_p}_{\mathbf{a}} - t_{\mathbf{a}})}{t_{p+1} - t_p}.$$

If there exist $s_p \in I \setminus L(w)$ for some w, then $t^{s_p}_{\mathbf{a}} C_{sw}$ is a term in $T_{s_p} t_{\mathbf{a}} C_w$. Now, take the element w with maximal $l(w)$. Then $T_{s_p} x$ contains the term $\sum_{\mathbf{a}} qk_{\mathbf{a},w} t^{s_p}_{\mathbf{a}} C_{s_p w}$, which must be zreo. So, $\sum_{\mathbf{a}} qk_{\mathbf{a},w} t^{s_p}_{\mathbf{a}} = 0$. Note that $t^{s_p}_{\mathbf{a}} = t^{s_p}_{\mathbf{b}}$ if and only if $\mathbf{a} = \mathbf{b}$. We have $k_{\mathbf{a},w} = 0$ for all $\mathbf{a}$, a contradiction. Using **??** again, we have

$$T_{s_p} x = \sum k_{\mathbf{a},w} T_{s_p} t_{\mathbf{a}} C'_w = \sum q^2 k_{\mathbf{a},w} t^{s_p}_{\mathbf{a}} C'_w + (1-q^2)\sum k_{\mathbf{a},w} \frac{t_{p+1}(t^{s_p}_{\mathbf{a}} - t_{\mathbf{a}})}{t_{p+1} - t_p} C'_w.$$

Since $T_{s_p}x = q^2x$, we have

$$q^2\sum k_{\mathbf{a},w}(t_{\mathbf{a}} - t_{\mathbf{a}}^{s_p})C_w = (1-q^2)\sum k_{\mathbf{a},w}\frac{t_{p+1}(t_{\mathbf{a}}^{s_p} - t_{\mathbf{a}})}{t_{p+1}-t_p}C'_w.$$

So, for any fixed w,

$$q^2\sum k_{\mathbf{a},w}(t_{\mathbf{a}} - t_{\mathbf{a}}^{s_p}) = (1-q^2)\sum k_{\mathbf{a},w}\frac{t_{p+1}(t_{\mathbf{a}}^{s_p} - t_{\mathbf{a}})}{t_{p+1}-t_p},$$

which is equivalent to the equation

$$\sum_{\mathbf{a}} k_{\mathbf{a},w}(q^2 + (q^2-1)\frac{t_{p+1}(t_{\mathbf{a}}^{s_p} - t_{\mathbf{a}})}{t_{p+1}-t_p})(t_{\mathbf{a}} - t_{\mathbf{a}}^{s_p}).$$

If $t_{\mathbf{a}} - t_{\mathbf{a}}^{s_p} = t_{\mathbf{b}} - t_{\mathbf{b}}^{s_p}$ for any $\mathbf{a} \neq \mathbf{b}$, then we must have $\mathbf{a} = \mathbf{a}^{s_p}$.

Since $q^2 \neq 1-q^2$, $\sum k_{\mathbf{a},w}(t_{\mathbf{a}} - t_{\mathbf{a}}^{s_p}) = 0$, forcing $k_{\mathbf{a},w} = k_{\mathbf{a}^{s_p},w}$ for all $s_p \in I$. This completes the proof of our claim. Therefore, there is a bijection between a basis of $\mathrm{Hom}_H(Hx_J, M_i)$ and the subset of elements $\{(\sum_{w\in\mathfrak{S}_I} t_{\mathbf{a}}^w C_w \mid R(w) \supset J_i, L(w) \supset I\}$. By similarly argument as above, $\mathrm{Hom}_{H'}((Hx_J)_{Z'}, M_{iZ'})$ is a free Z'-module with a basis, which corresponds to the subset $\{(\sum_{w\in\mathfrak{S}_I} t_{\mathbf{a}}^w C_w \otimes Z' \mid R(w) \supset J_i, L(w) \supset I\}$. Now the isomorphism $(*)$ follows from the base change property of M_i. $\square$

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