# A Mackey Decomposition Theorem and Cohomology for Quantum Groups at Roots of 1

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**Introduction** This is a continuation of the investigation in [L3] into the module structure of those modules for quantum enveloping algebras $U$ analogous to the cohomology groups of line bundles on the flag varieties for semisimple algebraic groups. In [L3], we used the admissible categories for Hopf algebras and their induction theory and proved a modified Borel-Bott-Weil theorem for $R^{l(w)}\mathrm{Ind}_{U^{\leq 0}}^{U} w \cdot \lambda$ when $\lambda$ is in the dominant chamber and not too close to chamber walls. In this paper we first prove some special Mackey decomposition theorems for certain Hopf subalgebras of $U$. Then we use these theorems to obtain some results on the vanishing properties of certain cohomology groups, and on all possible composition factors of these modules.

A more general form of Mackey decomposition theorem for algebraic groups is proved by Cline, Parshall, and Scott in [CPS] using the geometric method, which is available since the modules involved in the theorem are cohomology groups of line bundles over the quotient varieties. For quantum groups, the analogous result is not obvious especially when the higher derived functors are involved. This result has fundamental importance in the theory and is used in [APW1, Sect. 5] to get the Demazure character formula and the Kempf vanishing theorem for quantum groups. We prove a special Mackey decomposition theorem in Section 2 (over a field). As a consequence, we prove that the cohomology groups of $U$ (the quantum enveloping algebra) and the cohomology groups of the "parabolic" subalgebras are isomorphic if the coefficients are in a $U$-module. This resembles a result in [CPSK] by Cline, Parshall, Scott, and van der Kallen.

In Section 3, we use the Mackey decomposition theorem proved in Section 2 to study $R^1\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda$. Similar to Andersen's result on the first cohomology of line bundles [A1], we describe all $\lambda$'s such that $R^1\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda \neq 0$. The socles of such non-zero modules are simple and their highest weights can be calculated. As a consequence, the first $U^{\leq 0}$-cohomology can be described.

Another consequence of the Mackey decomposition theorem is that $Q \otimes E$ is $\mathrm{Ind}_{U^{\leq 0}}^{U}$-acyclic for any injective $B$-module $Q$ and any *rational* $U^{\leq 0}$-module $E$. Here $B$ is a Borel

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subgroup of the semisimple algebraic group $G$ of the same root system and there is a Frobenius map from $U$ to the distribution algebra of $G$ [Lu4, 8.14] (in this case the parameter is a root of 1). The same result for semisimple algebraic groups is proved by Andersen [A2] using the ampleness of certain line bundles. The proof given here also gives a non-geometric proof to Andersen's result. Many results in Andersen's paper [A2] can be stated for quantum groups (but this does not yield a short proof of Kempf's vanishing theorem). Thus we have an analogous result on the non-vanishing of certain higher derived functors at weights close to a chamber wall. This is done in Section 4.

In the last section, we use the results on cohomology groups proved in Section 2 to get a criterion for a module to have a *good filtration* which was proved by Donkin [D2] for algebraic groups. Using the results developed in this paper and those in [APW1], one can follow an argument similar to that in [DS] to get Ye's result [Y] for quantum groups at roots of 1, which describes all composition factors (not mentioning multiplicities) of $\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda$ when $\lambda$ is in the dominant chamber and not close to a chamber wall. When $\lambda$ is close to a chamber wall, one can use the cancellation principle. Ye's result for quantum groups has also been observed by J. E. Humphreys.

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# 1 Rational representations of Hopf algebras

**1.1.** We assume, throughout this section, that $A$ is a Dedekind domain and $H$ an $A$-free Hopf algebra over $A$. We denote by $\gamma : H \to H$ the antipode, by $\Delta : H \to H \otimes H$ the comultiplication, and by $\varepsilon : H \to A$ the counit. All tensor products in this paper are assumed to be taken over $A$ unless specified.

Let $M, V$ be two (left) $H$-modules. Then $M \otimes V$ is also an $H$-module with the action defined by $h(m \otimes v) = \sum h_{(1)}m \otimes h_{(2)}v$ for all $h \in H$ (using the $\sum$-notation). The (left) $H$-module structure on $M^*$ is defined by $(hf)(m) = f(\gamma(h)m)$ for all $h \in H$ and $f \in M^*$. An $H$-module $M$ is called trivial if $hm = \varepsilon(h)m$ for all $h \in H$ and $m \in M$. A full subcategory $\mathcal{C}$ of $H$-modules is called *admissible* if it satisfies the following conditions:

(i) Each module in $\mathcal{C}$ is locally finite in $\mathcal{C}$, i.e., for $M$ in $\mathcal{C}$ and any $m \in M$, $Hm$ is finitely generated over $A$;

(ii) $\mathcal{C}$ is closed under direct limit;

(iii) $\mathcal{C}$ is closed under tensor product;

(iv) If $M \in \mathrm{Ob}(\mathcal{C})$ is finitely generated and projective over $A$, then $M^*$ is also in $\mathcal{C}$;

(v) The trivial module $A$ is in $\mathcal{C}$;

(vi) For each exact sequence $0 \to K \to F \to M \to 0$ of $H$-modules, $F$ is in $\mathcal{C}$ if and only if both $K$ and $M$ are in $\mathcal{C}$;

(vii) Each $A$-finite module $M$ in $\mathcal{C}$ has a projective $A$-finite cover $F$ in $\mathcal{C}$, i.e., there is a surjective $H$-map $F \to M$ in $\mathcal{C}$ such that $F$ is finitely generated and projective over $A$.

A set $\mathcal{I}$ of cofinite 2-sided ideals of $H$ is called admissible if

(i) for each $I \in \mathcal{I}$, there is $J \in \mathcal{I}$ such that $J \subseteq \gamma^{-1}(I)$,

(ii) there is an $I \in \mathcal{I}$ such that $I \subseteq \ker(\varepsilon)$,

(iii) for any $I_1, I_2 \in \mathcal{I}$, there is $J \in \mathcal{I}$ such that $J \subseteq I_1 \cap I_2$,

(iv) for any $I_1, I_2 \in \mathcal{I}$, there is a $J \in \mathcal{I}$ such that $J \subseteq I_1 \wedge I_2$. Here $I_1 \wedge I_2$ is the kernel of the composition map $H \xrightarrow{\Delta} H \otimes H \to H/I_1 \otimes H/I_2$.

$\mathcal{I}$ is called flat if each $I \in \mathcal{I}$ contains an ideal $J \in \mathcal{I}$ such that $H/J$ is finitely generated and projective as $A$-module. Each $\mathcal{I}$ defines a linear topology on $H$ with $\mathcal{I}$ as a basis at 0. Two admissible sets of ideals are called equivalent if they define the same topology in $H$.

By (i)–(iv), one can see that all operations of Hopf algebras are continuous and we have a topological Hopf algebra. (For two topological Hopf algebras $H$ and $G$, $H \otimes G$ is also a topological Hopf algebra with the tensor product topology on $H \otimes G$ being defined by the set of ideals $I \otimes G + H \otimes J$ with $I$'s and $J$'s defining the topology on $H$ and $G$ respectively.)

It is proved in [L3, 1.7] that the admissible categories of $H$-modules are in one-to-one correspondence with the equivalence classes of flat admissible sets of ideals of $H$. If $H^0$ is the continuous dual of $H$ with respect to the given topology, then $H^0$ is also a Hopf algebra and the admissible category of $H$-modules is equivalent to the category of all (right) $H^0$-comodules [L3, 1.8]. Thus each admissible category has enough injective objects.

**1.2.** From now on we fix an admissible category $\mathcal{C}$ for $H$ and a flat admissible set $\mathcal{I}$ of ideals corresponding to $\mathcal{C}$. We will call the modules in $\mathcal{C}$ *rational* $H$-modules. If any confusion may arise, we will add the subscript $H$ to appropriate notations to indicate that they are related to $H$. For a rational $H$-module $M$, $M^H = \{m \in M \mid hm = \varepsilon(h)m, \forall h \in H\}$ is called the fixed point set. Thus we have $M^H \cong \mathrm{Hom}_H(A, M)$ and the cohomology groups $H^i(H, M)$ of $H$ in the category $\mathcal{C}$ with coefficients in $M$ are naturally isomorphic to the groups $\mathrm{Ext}^i_H(A, M)$. Here the extensions are taken in the category of rational $H$-modules and $A$ is the trivial $H$-module.

Let $M$ and $N$ be two $H$ modules. Then there is a natural $H$-module structure on $\mathrm{Hom}_A(M, N)$ given by $(hf)(m) = \sum_{(h)} h_{(1)} f(\gamma(h_{(2)})m)$ for all $h \in H$, $f \in \mathrm{Hom}_A(M, N)$ and $m \in M$. The following lemma is essentially proved in [APW1, 2.9], where $\gamma$ is assumed to bijective. The proof given here was pointed out by the referee. Using this lemma, the proof of Proposition 1.5 has been simplified and is more apparent.

**Lemma.** $\mathrm{Hom}_H(M,N) = \mathrm{Hom}_A(M,N)^H$ *for any* $H$*-modules* $M$ *and* $N$.

*Proof:* $\mathrm{Hom}_A(M,N)^H \supseteq \mathrm{Hom}_H(M,N)$ is proved in [APW1, 2.9]. To show the other inclusion, we consider the opposite Hopf algebra $H^{op}$ in which the multiplication and the comultiplication are defined to be opposite to those of $H$. Now $\mathrm{Hom}_A(M,N)$ is an $H \otimes H^{op}$-module with the action defined by $((g \otimes h)f)(m) = gf(hm)$ for $g \in H$, $h \in H^{op}$, $f \in \mathrm{Hom}_A(M,N)$, and all $m \in M$. Note that $f \in \mathrm{Hom}_H(M,N)$ if and only if $(h \otimes 1)f = (1 \otimes h)f$ for all $h \in H$. In fact, using the $\sum$-notation, we have, for each $f \in \mathrm{Hom}_A(M,N)^H$ and $h \in H$,

$$(h\otimes 1)f = \sum (h_{(1)} \otimes \varepsilon(h_{(2)}))f = \sum (h_{(1)} \otimes \gamma(h_{(2)})h_{(3)})f = \sum (\varepsilon(h_{(1)}) \otimes h_{(2)})f = (1 \otimes h)f.$$

Here the second last equality holds since $f \in \mathrm{Hom}_A(M,N)^H$. $\square$

**1.3.** Let $C$ be an $A$-flat coalgebra over $A$. Suppose that $\rho : M \to M \otimes C$ and $\delta : N \to C \otimes N$ define the right and left $C$-comodule structures on $M$ and $N$ respectively. The *cotensor product* $M\square_C N$ is defined to be the $A$-submodule $\mathrm{Ker}(\rho \otimes 1 - 1 \otimes \delta)$ of $M \otimes N$. If $B$ is another $A$-flat coalgebra over $A$, and $N$ is, in addition, a right $B$-comodule with the structure map $\beta : N \to N \otimes B$. We call $N$ a $C$-$B$-bicomodule, if $(1 \otimes \beta)\delta = (\delta \otimes 1)\beta$. If $N$ is a $C$-$B$-bicomodule, then $M \otimes N$ is a right $B$-comodule and $M\square_C N$ is a $B$-subcomodule of $M \otimes N$ (see [S]). Furthermore $-\square_C N$ is a left exact functor from the category of all $C$-comodules to the category of all $B$-comodules if $N$ is $A$-flat.

Let $G$ be another Hopf algebra satisfying the same assumptions as $H$ does and equipped with an admissible category $\mathcal{C}_G$ defined by a flat admissible set $\mathcal{I}_G$ of ideals. A homomorphism $\sigma : G \to H$ is called *compatible* if it is continuous with respect to the topologies defined by the admissible sets of ideals, i.e., for each $I \in \mathcal{I}_H$, there exists $I' \in \mathcal{I}_G$ such that $I' \subseteq \sigma^{-1}(I)$. Thus $\sigma$ defines an exact functor $\sigma^* : \mathcal{C}_H \to \mathcal{C}_G$ which is called the restriction functor. Let $C_H$ be the functor from the category of all $H$-modules to $\mathcal{C}_H$ by defining $C_H(M)$ being the maximal $H$-submodule which is in $\mathcal{C}_H$ ($\mathcal{C}_H$ is closed under direct limit). $C_G$ is defined similarly for $G$. Thus the induction functor $\sigma_* : \mathcal{C}_G \to \mathcal{C}_H$ is defined by $\sigma_*(M) = C_H\mathrm{Hom}_G(H,M)$. The Frobenius reciprocity [L3] says that $\sigma_*$ is right adjoint to the restriction functor $\sigma^*$. Thus $\sigma$ sends injective objects to injective objects. On the other hand, the map $\sigma$ induces a homomorphism $\sigma^0 : H^0 \to G^0$ of Hopf algebras and $H^0$ can be regarded as a left $G^0$-comodule with the structure map $\delta = (\sigma^0 \otimes 1)\Delta : H^0 \to G^0 \otimes H^0$. Note that $\mathcal{C}_H$ and $\mathcal{C}_G$ are equivalent to the categories of (right) $H^0$-comodules and $G^0$-comodules respectively. The following proposition was proved in [L3, 3.6] under the assumption that the antipode of $G$ is bijective due to the assumption in [APW1, 2.9]. Now this assumption can be removed by using Lemma 1.2. However the proof given here is more direct.

**Proposition.** *For each* $M$ *in* $\mathcal{C}_G$*, the map* $\varphi : M \otimes H^0 \to \mathrm{Hom}_A(H,M)$ *induces an isomorphism* $M\square_{G^0}H^0 \cong \sigma_*(M)$ *of* $H$*-modules. Therefore, we have an isomorphism of functors* $-\square_{G^0}H^0 \cong \sigma_*(-)$.

*Proof:* Let $M$ be a rational $G$-module. If $\rho_M : M \to M \otimes G^0$ is the $G^0$-comodule structure map and $\rho_M(m) = \sum m_{(0)} \otimes y_{(1)}$, then $gm = \sum y_{(1)}(g)m_{(0)}$ for all $g \in G$. Since $\mathrm{Ind}_G^H M \subseteq C_H \mathrm{Hom}_A(H, M) = M \otimes H^0 \subseteq \mathrm{Hom}_A(H, M)$ by [L3, 1.6], we only need to show $\mathrm{Ind}_G^H M = M \square_{G^0} H^0$ as subsets of $\mathrm{Hom}_A(H, M)$. On the other hand, we have the inclusion $M \otimes G^0 \otimes H^0 \subseteq \mathrm{Hom}_A(G \otimes H, M)$ given by $(m \otimes y \otimes f)(g \otimes h) = f(h)y(g)m$ for $y \in G^0$ and $f \in H^0$. Thus an element $\sum m_i \otimes f_i$ of $M \otimes H^0$ is in $M \square_{G^0} H^0$ if and only if

$$\sum m_i \otimes \sigma^0(f_{i,(1)}) \otimes f_{i,(2)} = \sum m_{i,(0)} \otimes y_{i,(1)} \otimes f_i$$

as elements in $\mathrm{Hom}_A(G \otimes H, M)$. But this is equivalent to

$$\sum f_{i,(1)}(\sigma(g)) f_{i,(2)}(h) m_i = \sum y_{i,(1)}(g) f_i(h) m_{i,(0)} \tag{1}$$

for all $g \in G$ and $h \in H$. However the left hand side of (1) is

$$\sum f_i(\sigma(g)h)m_i = (\sum m_i \otimes f_i)(\sigma(g)h)$$

and the right hand side of (1) actually is

$$\sum f_i(h) y_{i,(1)}(g) m_{i,(0)} = \sum f_i(h) g m_i = g(\sum m_i \otimes f_i)(h).$$

Thus an element $\sum m_i \otimes f_i$ of $M \otimes H^0$ is in $M \square_{G^0} H^0$ if and only if it is in $\mathrm{Hom}_G(H, M)$. Note that $M \otimes H^0 = C_H(\mathrm{Hom}_A(H, M))$ [L3, 1.6]. We thus have $M \square_{G^0} H^0 = (M \otimes H^0) \cap \mathrm{Hom}_G(H, M) = \mathrm{Ind}_G^H M$. $\square$

If $G$ is a Hopf subalgebra of $H$, we can take the category $\mathcal{C}_G$ to be the one corresponding to the flat admissible set $\mathcal{I}_G$ of ideals $I \cap G$ ($I \in \mathcal{I}_H$) of $G$ which defines the induced topology on $G$. In this case we will write $\sigma_* = \mathrm{Ind}_G^H$. Since each admissible category has enough injective objects [L3, 1.8], we can consider the derived functor $R^i\mathrm{Ind}_G^H$ of $\mathrm{Ind}_G^H$ at a rational $G$-module.

*Remark.* the functors $\sigma_* : \mathcal{C}_G \to \mathcal{C}_H$ depends only on the categories $\mathcal{C}_G$ and $\mathcal{C}_H$ not on the admissible sets of ideals defining the categories $\mathcal{C}_G$ and $\mathcal{C}_H$.

**1.4.** The following is a test for the exactness of the induction functors.

**Lemma.** *Suppose that $A$ is a field and we have compatible homomorphisms of Hopf algebras $H_1 \xrightarrow{\phi} H \xrightarrow{\psi} H_1$ such that $\psi \circ \phi = 1_{H_1}$. Then $\phi_*$ is exact.*

*Proof:* Let $M$ be a rational $H_1$-module. Then $M = \phi^*\psi^* M$. Here $\psi^* M$ is a rational $H$-module. Since $A$ is a field (this is the only place that requires $A$ to be a field for the results in Theorem 2.4), by the tensor identity [PW, 2.8.2], we have $\phi_* M \cong \psi^* M \otimes \phi_* A$ and the exactness follows. $\square$

**1.5.** The maps $\phi_r : H \otimes H \to H$ (or $\phi_l : H \otimes H \to H$), defined by $\phi_r(x \otimes y) = \sum \gamma(y_{(1)})xy_{(2)}$ (or $\phi_l(x \otimes y) = \sum x_{(1)}y\gamma(x_{(2)})$) for all $x, y \in H$, make $H$ into a right (or left) $H$-module, which is called the right (or left) *adjoint* representation of $H$. An $A$-submodule $D$ of $H$ is called right (or left) *normal* in $H$ if $\phi_r(D \otimes H) \subseteq D$ (or $\phi_l(H \otimes D) \subseteq D$). $D$ is called normal if it is both left and right normal. If $D$ is a normal Hopf subalgebra of $H$ and $D_+ = \mathrm{Ker}(\varepsilon|_D)$ the augmentation ideal of $D$, then $HD_+ = D_+H$ is a Hopf ideal and $H/HD_+$ is called the quotient Hopf algebra, denoted by $H/D$ (by abuse of notations). Assuming that the topology of $H/D$ is the quotient topology of $H$, all modules in $\mathcal{C}_H$ on which $D$ acts trivially have the form $\pi^*N$ for some module $N$ in $\mathcal{C}_{H/D}$. Here $\pi : H \to H/D$ is the canonical map and $\mathcal{C}_{H/D}$ is the admissible category for $H/D$. For each $H$-module $M$ in $\mathcal{C}$, the $D$-fixed point set $M^D$ is an $H$-submodule and $M^D = \pi^*N$ for some $H/D$-module $N$ in $\mathcal{C}_{H/D}$ [L3, 4.4]. We will denote $N$ by $M^D$ if the $H/D$-module structure on $M^D$ is clear from the context. Furthermore, when we want to emphasize the $D$-action on an $H$-module, we will denote it by $\sigma^*M$ with $\sigma : D \to H$ being the inclusion map.

**Lemma.** *For each $M$ in $\mathcal{C}_H$, $M^D$ is an $H$-submodule of $M$ and has a natural $H/D$-module structure. Thus it defines a functor $M \mapsto M^D$ from $\mathcal{C}_H$ to $\mathcal{C}_{H/D}$, which is right adjoint to the exact functor $\pi^*$ and therefore sends injective objects to injective objects.* □

The proof of this lemma is straight forward keeping in mind that the topology on $H/D$ is generated by the basis $\{\pi(I) \mid I \in \mathcal{I}\}$.

**Proposition.** *Suppose $D$ is a normal Hopf subalgebra of $H$ and the induction functor $\mathrm{Ind}_D^H$ is exact. If $E$ is in $\mathcal{C}_H$, finitely generated over $A$, then the $i$th right derived functor of the functor $V \mapsto \mathrm{Hom}_D(\sigma^*E, \sigma^*V)$ from $\mathcal{C}_H$ to $\mathcal{C}_{H/D}$ can be identified with $V \mapsto \mathrm{Ext}^i_D(\sigma^*E, \sigma^*V)$. Furthermore, there exists a spectral sequence with the $E_2$ terms*

$$E_2^{i,j} = \mathrm{Ext}^i_{H/D}(M, \mathrm{Ext}^j_D(\sigma^*E, \sigma^*V)) \Rightarrow \mathrm{Ext}^{i+j}_H(\pi^*M \otimes E, V) \tag{2}$$

*for each $M$ in $\mathcal{C}_{H/D}$.*

*Proof:* First we show that $V \mapsto \mathrm{Hom}_D(\sigma^*E, \sigma^*V)$ defines a functor from $\mathcal{C}_H$ to $\mathcal{C}_{H/D}$. In fact, by Lemma 1.2 and the above lemma, $\mathrm{Hom}_D(\sigma^*E, \sigma^*V) = \mathrm{Hom}_A(E, V)^D$ is an $H$-submodule of $\mathrm{Hom}_A(E, V)$ and is $D$-trivial. Since $E$ is finitely generated over $A$, there exists a rational $H$-module $P$ which is finitely generated and projective over $A$ such that $P \to E \to 0$ is exact as $H$-modules. Thus, as $H/D$-modules, we have $\mathrm{Hom}_D(\sigma^*E, \sigma^*V) \subseteq \mathrm{Hom}_D(\sigma^*P, \sigma^*V) = \mathrm{Hom}_A(P, V)^D$ by Lemma 1.2. Since $P$ is finitely generated and projective over $A$, the natural $A$-isomorphism $\mathrm{Hom}_A(P, V) \cong V \otimes P^*$ is actually an isomorphism of $H$-modules and the latter is in $\mathcal{C}_H$. Thus $\mathrm{Hom}_D(\sigma^*E, \sigma^*V)$ is in $\mathcal{C}_{H/D}$ by [L3, 4.4]. One can verify that any $H$-homomorphism

$V \to V'$ induces an $H/D$-homomorphism $\mathrm{Hom}_D(E,V) \to \mathrm{Hom}_D(E,V')$. Thus we have an functor $\mathrm{Hom}_D(\sigma^*(E), \sigma^*(-)) : \mathcal{C}_H \to \mathcal{C}_{H/D}$.

Since $\mathrm{Ind}_D^H$ is exact, $\sigma^* I$ is injective in $\mathcal{C}_D$ for each injective module $Q$ in $\mathcal{C}_H$ (see [L3, 3.7], [PW, 2.9.1]). If $0 \to V \to I^{\cdot}$ is an injective resolution in $\mathcal{C}_H$, $0 \to \sigma^* V \to \sigma^*(I^{\cdot})$ is an injective resolution in $\mathcal{C}_D$. By applying the functor $\mathrm{Hom}_D(\sigma^*(E), -)$ to this resolution, we have $R^i\mathrm{Hom}_D(\sigma^*(E), \sigma^*(-))(V) = \mathrm{Ext}_D^i(\sigma^*(E), \sigma^*(V))$ as $A$-modules. This shows that $\mathrm{Ext}_D^i(\sigma^*(E), \sigma^*(V))$ has an $H/D$-module structure and is in $\mathcal{C}_{H/D}$ for each $V$ in $\mathcal{C}_H$.

For any rational $H/D$-module $M$, we have

$$\begin{aligned} \mathrm{Hom}_H(\pi^*(M) \otimes E, V) &= \mathrm{Hom}_A(M \otimes E, V)^H = \mathrm{Hom}_A(M, \mathrm{Hom}_A(E,V))^H \\ &= (\mathrm{Hom}_A(M, \mathrm{Hom}_A(E,V))^D)^{H/D} \end{aligned}$$

by the lemma above. Since $M$ is trivial as $D$-module we have

$$\begin{aligned} \mathrm{Hom}_A(M, \mathrm{Hom}_A(E,V))^D &= \mathrm{Hom}_D(M, \mathrm{Hom}_A(E,V)) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(E,V)^D) \\ &= \mathrm{Hom}_A(M, \mathrm{Hom}_D(E,V)). \end{aligned}$$

Thus we have an isomorphism $\varphi : \mathrm{Hom}_H(\pi^* M \otimes E, V) \cong \mathrm{Hom}_{H/D}(M, \mathrm{Hom}_D(E,V))$, which is natural in both $M$ in $\mathcal{C}_{H/D}$ and $V$ in $\mathcal{C}_H$. The spectral sequence (2) follows from the Grothendieck spectral sequence for the composition of functors $\mathrm{Hom}_H(\pi^* M \otimes E, -) = \mathrm{Hom}_{H/D}(M, -) \circ \mathrm{Hom}_D(\sigma^* E, \sigma^* -)$. $\square$

*Remark.* The spectral sequence (2) is a (type of) Hochschild-Serre spectral sequence. The special case of (2) when $M = A$ is a field is proved in [PW, 2.11.1]. If, in addition, $E$ is projective over $A$, one can follow the remark in [J, I, 6.4] to establish the isomorphism $\varphi$.

**1.6.** If $H_1$ is a Hopf subalgebra of $H$, then $G_1 = \pi(H_1)$ is a Hopf subalgebra of $G$ and $DH_1 = H_1 D$ is a Hopf subalgebra of $H$ as well. Denote by $\pi_1 : DH_1 \to G_1$ the restriction of $\pi$ to $DH_1$. Let $E$ be a rational $H_1$-module, which is finitely generated and projective over $A$. *We assume that the $DH_1$-module $\mathrm{Ind}_{H_1}^{DH_1} E$ extends to a rational $H$-module* in this subsection.

Define the functors $F_1, F_2$ and $F_3$ from $\mathcal{C}_{G_1}$ to $\mathcal{C}_H$ as follows. For each $M$ in $\mathcal{C}_{G_1}$,

$$\begin{aligned} F_1(M) &= \pi^*(\mathrm{Ind}_{G_1}^G M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E, \\ F_2(M) &= \mathrm{Ind}_{H_1}^H(\pi_1^* M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E \cong \mathrm{Ind}_{H_1}^H(\pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E), \\ F_3(M) &= \mathrm{Ind}_{H_1}^H(\pi_1^* M \otimes E). \end{aligned}$$

There are natural transformations $\varphi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ defined as follows. The map $\rho : \pi_1^* M \to \mathrm{Ind}_{H_1}^{DH_1}(\pi_1^* M)$ resulted from Frobenius reciprocity is natural in $M$. Using [L3, 4.6], we have

$$F_1(M) \cong \mathrm{Ind}_{DH_1}^H(\pi_1^* M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E.$$

We can define $\varphi(M) = \mathrm{Ind}_{DH_1}^{H}(\rho) \otimes id$ since $\mathrm{Ind}_{H_1}^{H}(\pi_1^* M) = \mathrm{Ind}_{DH_1}^{H}\mathrm{Ind}_{H_1}^{DH_1}(\pi_1^* M)$. Let $ev : \mathrm{Ind}_{H_1}^{DH_1} E \to E$ be the evaluation map, define $\psi(M) = \mathrm{Ind}_{H_1}^{H}(id \otimes ev)$. The naturality of $\varphi$ and $\psi$ can be easily verified.

**Lemma.** *If $\pi_1^* Q \otimes E$ and $\pi_1^* Q$ are $\mathrm{Ind}_{H_1}^{H}$-acyclic for each $G_1$-injective module $Q$, then for any $G_1$-module $M$ and $i \geq 0$, we have*

$$\begin{aligned} R^i F_1(M) &\cong \pi^* R^i \mathrm{Ind}_{G_1}^{G} M \otimes \mathrm{Ind}_{H_1}^{DH_1} E \\ R^i F_2(M) &\cong R^i \mathrm{Ind}_{H_1}^{H}(\pi_1^* M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E \\ R^i F_3(M) &\cong R^i \mathrm{Ind}_{H_1}^{H}(\pi_1^* M \otimes E) \end{aligned}$$

*Proof:* First of all, $\pi^*$ is always exact and $\mathrm{Ind}_{H_1}^{DH_1}(E)$ is flat over $A$ since $E$ is flat over $A$ ($E$ is torsion free implies that $\mathrm{Ind}_{H_1}^{DH_1}(E)$ is torsion free and therefore flat since $A$ is a Dedekind domain). Since the induction functor sends injectives to injectives, the first isomorphism follows from a Grothendieck spectral sequence. The assumptions of the lemma also implies that there are Grothendieck spectral sequences which will yield the last two isomorphisms. □

**Proposition.** *Suppose $\mathrm{Ind}_{H_1}^{DH_1}$ is exact and $\pi_1^* Q \otimes E$ and $\pi_1^* Q$ are $\mathrm{Ind}_{H_1}^{H}$-acyclic for each $G_1$-injective module $Q$. Then for any $G_1$-module $M$ and all $i \geq 0$, we have*

(i) *$R^i(\psi \circ \phi)(M)$ is an isomorphism for $i \geq 0$ if $M$ is $A$-flat ,*

(ii) *if $M$ is $A$-flat, there are natural isomorphisms of $H$-modules*

$$\begin{aligned} R^i \mathrm{Ind}_{H_1}^{H}(\pi_1^*(M) \otimes E) &\cong \pi^*(R^i \mathrm{Ind}_{G_1}^{G} M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E, \\ R^i \mathrm{Ind}_{H_1}^{H}(E \otimes \pi_1^*(M)) &\cong \mathrm{Ind}_{H_1}^{DH_1}(E) \otimes \pi^*(R^i \mathrm{Ind}_{G_1}^{G} M). \end{aligned}$$

*Proof:* Note that for any $G_1$-module $M$ we have

$$\begin{aligned} F_1(M) &\cong \mathrm{Ind}_{DH_1}^{H}(\pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E); \\ F_1(M) &\cong \mathrm{Ind}_{DH_1}^{H}\mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E); \\ F_1(M) &\cong \mathrm{Ind}_{DH_1}^{H}\mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(M) \otimes E). \end{aligned}$$

Here we have used [L3, 4.7,3.7] and the tensor identity for the first isomorphism, and the transitivity of induction functors for the second and third isomorphism. Now we consider the following commutative diagram:

$$\begin{array}{ccc} \pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E & \xrightarrow{\phi_1(M)} & \mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E) \\ \theta_1(M) \downarrow & & \downarrow \psi_1(M) \\ \mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(M) \otimes E) & \xleftarrow{\theta_1(M)} & \pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E. \end{array}$$

Here $\psi_1$ is the evaluation map, $\phi_1$ is the map resulted from Frobenius reciprocity such that $\psi_1 \circ \phi_1 = id$, and $\theta_1$ is obtained via Frobenius reciprocity corresponding to the map $id \otimes ev : \pi_1^*(M) \otimes \mathrm{Ind}_{H_1}^{DH_1} E \to \pi_1^*(M) \otimes E$. By applying the functor $\mathrm{Ind}_{DH_1}^{H}$ to the above commutative diagram we have $\phi = \mathrm{Ind}_{DH_1}^{H}(\phi_1)$ and $\psi = \mathrm{Ind}_{DH_1}^{H}(\theta_1 \circ \psi_1)$. Thus $\psi \circ \phi = \mathrm{Ind}_{DH_1}^{H}(\theta_1)$.

Since $\mathrm{Ind}_{H_1}^{DH_1}$ is exact, then $\mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(-) \otimes E)$ and $\mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(-) \otimes \mathrm{Ind}_{H_1}^{DH_1} E) \cong \mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(-)) \otimes \mathrm{Ind}_{H_1}^{DH_1} E$ send injective $G_1$-modules to $\mathrm{Ind}_{DH_1}^{H}$-acyclic modules by the assumption and a generalized tensor identity. The functor $\pi_1^*(-) \otimes \mathrm{Ind}_{H_1}^{DH_1} E$ sends injective $G_1$-modules to $\mathrm{Ind}_{DH_1}^{H}$-acyclic modules follows from [L3, 4.7] and generalized tensor identity. Since $\psi_1$, $\phi_1$, and $\theta_1$ are natural in $M$, we can apply the functor $R^i\mathrm{Ind}_{DH_1}^{H}$ to the above commutative diagram and use three Grothendieck spectral sequences to get

$$R^i\mathrm{Ind}_{DH_1}^{H}(\phi_1) = R^i\phi, \quad R^i\mathrm{Ind}_{DH_1}^{H}(\theta_1 \circ \psi_1) = R^i\psi, \text{ and that}$$

$$R^i\mathrm{Ind}_{DH_1}^{H}(\theta_1) : R^i\mathrm{Ind}_{DH_1}^{H}(\pi_1^*(-) \otimes \mathrm{Ind}_{H_1}^{DH_1} E) \to R^i\mathrm{Ind}_{DH_1}^{H}(\mathrm{Ind}_{H_1}^{DH_1}(\pi_1^*(-) \otimes E))$$

is a natural transformation. Therefore $R^i\psi \circ R^i\phi = R^i(\psi \circ \phi) = R^i\mathrm{Ind}_{DH_1}^{H}(\theta_1)$, Note that $\theta_1(M)$ is an isomorphism if $M$ is $A$-flat (the isomorphism in the tensor identity, see the proof in [PW, 2.7.1]). Thus we have $R^i\mathrm{Ind}_{DH_1}^{H}(\theta_1)(M) = R^i\mathrm{Ind}_{DH_1}^{H}(\theta_1(M))$ is an isomorphism. This shows (i) and the first isomorphism. The second isomorphism can be shown using a similar argument by tensoring $E$ and $\mathrm{Ind}_{H_1}^{DH_1} E$ on the left side and using the right generalized tensor identity [PW, 2.7.1, 2.8.2]. □

*Remark.* Let $H$ be the distribution algebra of a semisimple algebraic group, $D$ the distribution algebra of the $r$th Frobenius kernel and $H_1$ the distribution algebra of the Borel subgroup (corresponding to negative roots for a chosen root system). If $E = (p^r - 1)\rho$ ($\rho$ is the half sum of all positive roots), the isomorphism given here is the Andersen-Haboush identity [A2, H].

**1.7.** Let $P$ be a Hopf subalgebra of $H$ and $U$ an $A$-submodule of $H$. For any $H$-module $M$ in $\mathcal{C}_H$, we set $M^U = \{m \in M \mid um = \varepsilon(u)m, \ \forall u \in U\}$.

**Proposition.** *Suppose the multiplication map $P \otimes U \to H$ is onto. Then for each rational $P$-module $M$, the restriction of the evaluation map $ev : \mathrm{Ind}_P^H M \to M$ to $(\mathrm{Ind}_P^H M)^U$ is injective.*

*Proof:* Recall that $\mathrm{Ind}_P^H M \subseteq \mathrm{Hom}_P(H, M)$. For each $f \in \mathrm{Hom}_P(H, M)^U$, $uf = \varepsilon(u)f$ means $f(xu) = (uf)(x) = \varepsilon(u)f(x)$ for all $x \in H$ and $u \in U$. If $ev(f) = f(1) = 0$, then $f(p) = pf(1) = 0$ for all $p \in P$. By the assumption, each $x \in H$ can be written as $x = \sum p_i u_i$ and $f(x) = \sum p_i f(u_i) = \sum p_i \varepsilon(u_i) f(1) = 0$. Thus $f = 0$. □

**1.8.** We say a Hopf subalgebra $L$ *normalizes* $U$ (an $A$-submodule of $H$) from right (or left) if $\phi_r(U \otimes L) \subseteq U$ (or $\phi_l(L \otimes U) \subseteq U$).

**Lemma.** *If $L$ normalizes $U$ from right, then $M^U$ is a rational $L$-module for each rational $H$-module $M$.*

*Proof:* Since $M|_L$ is a rational $L$-module, we only need to show that $M^U$ is an $L$-submodule of $M|_L$. For $m \in M^U$ and $x \in L$, we have, by [L3, 4.2],

$$u(xm) = (ux)m = \sum x_{(1)}\phi_r(u \otimes x_{(2)})m = \sum x_{(1)}\varepsilon(u)\varepsilon(x_{(2)})m = \varepsilon(u)xm$$

for all $u \in U$. Thus $xm \in M^U$. $\square$

**1.9.** We state a special Mackey decomposition theorem with conditions given in terms of admissible sets of the ideals. Let $G$ and $H_1$ be two Hopf subalgebras of $H$ with induced topologies. Then $G_1 = G \cap H_1$ is also a Hopf subalgebra of $H$ (see [Sw]).

**Proposition.** *Suppose $H_1 \otimes_{G_1} G \cong H$ as $A$-modules. If for each $I' \in \mathcal{I}_{H_1}$ and $J \in \mathcal{I}_G$, there exists $I \in \mathcal{I}_H$ such that $I \subseteq I'G + H_1J$, then $\mathrm{Res}^H_G \mathrm{Ind}^H_{H_1} M \cong \mathrm{Ind}^G_{G_1} \mathrm{Res}^{H_1}_{G_1} M$ for each rational $H_1$-module $M$. Furthermore, if $\mathrm{Ind}^{H_1}_{G_1}$ is exact, then $\mathrm{Res}^H_G R^i\mathrm{Ind}^H_{H_1} M \cong R^i\mathrm{Ind}^G_{G_1}\mathrm{Res}^{H_1}_{G_1} M$ for all $i \geq 0$.*

*Proof:* By the assumption, we have the isomorphism

$$\mathrm{Hom}_{H_1}(H, M) \to \mathrm{Hom}_{G_1}(G, M)$$

of $G$-modules given by $f \mapsto f|_G$ for all $f \in \mathrm{Hom}_{H_1}(H, M)$.

Since the topology on $G$ is induced from the topology on $H$, $C_H\mathrm{Hom}_{H_1}(H, M) \subseteq C_G\mathrm{Hom}_{G_1}(G, M)$ is clear. If $f \in \mathrm{Hom}_{H_1}(H, M)$ is such that $f|_G \in C_G\mathrm{Hom}_{G_1}(G, M)$, then $f(J) = 0$ for some $J \in \mathcal{I}_G$. Thus $f(G)$ is a finitely generated $A$-submodule of $M$ and $N = H_1 f(G)$ finitely generated over $A$ since $M$ is a rational $H_1$-module. There exists $I' \in \mathcal{I}_{H_1}$ such that $I'N = 0$. By the assumption, there exists $I \in \mathcal{I}_H$ such that $I \subseteq I'G + H_1J$ and $f(I) \subseteq f(I'G + H_1J) = I'f(G) + H_1f(J) = 0$. Hence $f \in C_H\mathrm{Hom}_{H_1}(H, M)$ and the first part of the proposition follows. Since the exactness of $\mathrm{Ind}^{H_1}_{G_1}$ implies that $\mathrm{Res}^{H_1}_{G_1} Q$ is injective in $\mathcal{C}_{G_1}$ for each injective module $Q$ in $\mathcal{C}_{H_1}$ [PW, 2.9.1], the second part of the proposition follows from a Grothendieck spectral sequence. $\square$

**Corollary.** *If $H = H_1 \otimes G$ as Hopf algebras such that $H$ has the tensor product topology, then $\mathrm{Ind}^H_{H_1}$ is exact.*

*Proof:* Note that $G_1 = H_1 \cap G = A$. Thus both $\mathrm{Ind}^G_{G_1}$ and $\mathrm{Ind}^{H_1}_{G_1}$ are exact by [L3, 1.6, 3.5]. The conclusion follows from the above proposition. $\square$

**1.10.** Before we end this section, we discuss the situation when the admissible category $\mathcal{C}_H$ is defined by a set of left ideals. Let $\mathcal{L}$ be a collection of left ideals of $H$. For each $H$-module $M$, we define $C_{\mathcal{L}}(M) = \{m \in M \mid \mathrm{Ann}_H(m) \supseteq I \text{ for some } I \in \mathcal{L}\}$. Let $\mathcal{C}_{\mathcal{L}}$ be the category of $H$-modules $M$ such that $C_{\mathcal{L}}(M) = M$. Suppose that $\mathcal{C}_{\mathcal{L}}$ is an admissible category and $\mathcal{I}$ is an admissible set of the ideals of $H$ defining $\mathcal{C}_{\mathcal{L}}$ [L3, 1.7]. Then that relation between $\mathcal{I}$ and $\mathcal{L}$ can be stated as follows: Each $I$ in $\mathcal{I}$ contains a left ideal in $\mathcal{L}$ and each left ideal in $\mathcal{L}$ contains an ideal in $\mathcal{I}$. Such an admissible category $\mathcal{C}_{\mathcal{L}}$ is said to be defined by $\mathcal{L}$. For an admissible set $\mathcal{I}$ of ideals and a collection $\mathcal{L}$ of left ideals, the above relation between $\mathcal{I}$ and $\mathcal{L}$ is actually equivalent to $\mathcal{C}_{\mathcal{I}} = \mathcal{C}_{\mathcal{L}}$.

Let $\sigma : G \to H$ be a homomorphism of Hopf algebras with admissible categories $\mathcal{C}_G$ and $\mathcal{C}_H$ defined by collections $\mathcal{L}_G$ and $\mathcal{L}_H$ of the left ideals respectively. Then $\sigma$ is compatible if and only if for each left ideal $I \in \mathcal{L}_H$, $\sigma^{-1}(I)$ contains a left ideal in $\mathcal{L}_G$. Furthermore, if $G$ is a Hopf subalgebra of $H$, the induced admissible category $\mathcal{C}_G$ from $\mathcal{C}_H$ is defined by $\mathcal{L}_G = \{I \cap G \mid I \in \mathcal{L}_H\}$.

Suppose the admissible category $\mathcal{C}_H$ is defined by a collection $\mathcal{L}_H$ of left ideals. Then for the subalgebras $G$, $H_1$ as in Proposition 1.9, the induced admissible categories $\mathcal{C}_G$ and $\mathcal{C}_{H_1}$ are also defined by collections $\mathcal{L}_G$ and $\mathcal{L}_{H_1}$ of left ideals. By the relation between $\mathcal{I}$ and $\mathcal{L}$ we mentioned above, the condition on ideals in Proposition 1.9 is equivalent to the following: for each $I' \in \mathcal{L}_{H_1}$ and $J \in \mathcal{L}_G$, there exists $I \in \mathcal{L}_H$ such that $I \subseteq I'G + H_1 J$. This condition will have the advantage in application to quantum groups since the categories for quantum groups are defined by certain collections of left ideals (see [APW1]).

# 2 A Mackey decomposition theorem for quantum groups

**2.1.** We still assume that $A$ is a Dedekind domain through 2.2. Fix a ring homomorphism $\mathbb{Z}[v, v^{-1}] \to A$ and denote the image of $v$ in $A$ by $v$ as well. Let $U = U_{\mathbb{Z}[v,v^{-1}]} \otimes A$. Here $U_{\mathbb{Z}[v,v^{-1}]}$ is the $\mathbb{Z}[v, v^{-1}]$-algebra defined by Lusztig in [Lu4] with generators $\{E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1} | 1 \le i \le n, m \ge 0\}$ and relations related to the Cartan matrix $(a_{ij})$ of a semisimple complex Lie algebra with root system $R$ of rank $n$. $U$ has a Hopf algebra structure with a bijective antipode. Lusztig has proved that $U = U^- \otimes U^0 \otimes U^+$ as $A$-modules. Here $U^-$ (or $U^+$) is the subalgebra of $U$ generated by $\{F_i^{(m)} \mid 1 \le i \le n,\ m \ge 0\}$ (or $\{E_i^{(m)} \mid 1 \le i \le n,\ m \ge 0\}$) and $U^0$ is generated by $\{K^{\pm 1}, \begin{bmatrix} K_i \\ m \end{bmatrix} \mid 1 \le i \le n, m \ge 0\}$. For more details about the algebra $U_{\mathbb{Z}[v,v^{-1}]}$ and notations, one is referred to [APW1, Lu2, Lu3, Lu4].

Let $X$ be the lattice of integral weights of the root system $R$ corresponding to the Cartan matrix $(a_{ij})$ and $(d_1, \ldots, d_n)$ a sequence of (smallest) positive integers such that $(d_i a_{ij})$ is a symmetric matrix. Denote by $X_+$ the set of dominant weights with

respect to the set $S$ of simple roots, which correspond to the $E_i$'s. Each $\lambda \in X$ defines an homomorphism $\lambda : U^0 \to A$ of $A$-algebras as in [APW1, 1.1] (with all $\sigma_i = 1$). Let $\mathcal{L}_U$ be the collection of left ideals of $U$ consisting of those left ideals generated by $\mathrm{Ker}\lambda, E_i^{(r)}, F_i^{(s)}$ (all $r \geq m_i, s \geq n_i$ and all $i$) and their finite intersections. The category $\mathcal{C}_U$ of integrable module of type **1** (defined similarly as in [APW1, 1.6]) is defined by $\mathcal{L}_U$. It is shown in [L4] that $\mathcal{C}_U$ is an admissible category. Modules in $\mathcal{C}_U$ can also be described as follows: A $U$-module $M$ is in $\mathcal{C}_U$ if it is a direct sum of weight spaces (as $U^0$-module) and locally finite as $U^+$- and $U^-$-modules. Thus all theory we developed in Section 1 can be used for the Hopf algebra $U$ and the admissible category $\mathcal{C}_U$. The induction functors we defined in Section 1 (cf. 1.10) from various Hopf subalgebras coincide with those defined in [APW1, APW2, AW].

**2.2.** Let $I \subseteq \{1, 2, \ldots, n\}$ and $P_I$ be the $A$-subalgebra of $U$ generated by $U^{\leq 0} = U^- U^0$ and $\{E_i^{(m)} \mid i \in I, m \geq 0\}$. Similarly, let $U_I$ (or $S_I$) be the subalgebra of $U$ generated by $U^0$ and $\{F_i^{(m)}, E_i^{(m)} \mid i \in I, m \geq 0\}$ (or $\{K_i^{\pm 1}, \begin{bmatrix} K_i \\ m \end{bmatrix}, F_i^{(m)}, E_i^{(m)} \mid i \in I, m \geq 0\}$). Thus $S_I$ is isomorphic to the quantum enveloping algebra corresponding to the Cartan matrix of the subroot system $R_I$ in $R$ generated by the simple roots $\{\alpha_i \mid i \in I\}$. Let $U_I^{\leq 0}$ and $S_I^{\leq 0}$ be the intersections of $U_I$ and $S_I$ with $U^{\leq 0}$ respectively. Note that all these subalgebras are Hopf subalgebras of $U$ with bijective antipodes. In each of these Hopf subalgebras, the topology is induced from the topology of $U$.

The left ideals in $\mathcal{L}_{P_I}$ are generated by $\{F_i^{(r_i)}, E_j^{(s_j)}, I_0 \mid r_i \geq m_i, s_j \geq n_j, 1 \leq i \leq n, j \in I\}$ for some integers $m_i, n_j \geq 0$ and some finite intersection $I_0$ of $\mathrm{Ker}(\lambda)$'s with $\lambda \in X$. Similarly, the left ideals in $\mathcal{L}_{U_I}$ and $\mathcal{L}_{U^{\leq 0}}$ are generated by similar sets with indices $i, j$'s being limited to $i, j \in I$ and $1 \leq i \leq n$ respectively. The left ideals in $\mathcal{L}_{S_I}$ and $\mathcal{L}_{S_I^{\leq 0}}$ are generated in a similar way, where $I_0$ should be replaces by the intersection of the subalgebra $S_I$ with finite intersections of $\mathrm{Ker}(\lambda)$'s with $\lambda \in X$.

Let $E_I$ be the subalgebra of $U^+$ generated by $\{E_i^{(m)} \mid i \in I, m \geq 0\}$ and $W_I$ the parabolic subgroup generated by $\{s_{\alpha_i} \mid i \in I\}$ in $W$ (the Weyl group of $R$ with $s_\alpha$ being the reflection defined by $\alpha \in R$). By extending a reduced expression of the longest element $w_I$ of $W_I$ to a reduced expression of the longest element $w_0$ of $W$, we can assume that the first $|R_I^+|$ roots in the total ordering of $R^+$ used in the construction of Dyer-Lusztig bases for $U^+$ [Lu4, Appendix] are exactly those roots in $R_I^+$. For each $i \in I$, the braid group element $T_i$ acting on $S_I$ and $U$ commutes with the embedding of $S_I$ into $U$. Using the Dyer-Lusztig bases for $S_I$ and $U$, we see that the subset $\{\prod_{\beta \in R_I^+} E_\beta^{(m_\beta)}\}$ of $U^+$ is a basis for $E_I$. Using these bases, we can get the following isomorphisms of $A$-modules resulted from multiplications:

$$P_I \cong U^{\leq 0} \otimes E_I, \quad U_I \cong U_I^{\leq 0} \otimes E_I, \quad S_I \cong S_I^{\leq 0} \otimes E_I.$$

**Lemma.** (i) *For each $J' \in \mathcal{L}_{U^{\leq 0}}$ and $J'' \in \mathcal{L}_{U_I}$, there exists $J \in \mathcal{L}_{P_I}$ such that $J \subseteq J' U_I + U^{\leq 0} J''$.*

(ii) *For each* $J' \in \mathcal{L}_{U_I^{\leq 0}}$ *and* $J'' \in \mathcal{L}_{S_I}$, *there exists* $J \in \mathcal{L}_{U_I}$ *such that* $J \subseteq J'S_I + U_I^{\leq 0}J''$.

*Proof:* (i) Suppose the left ideal $J'$ of $U^{\leq 0}$ is generated by $\{F_i^{(r_i)}, I_0' \mid r_i \geq m_i', 1 \leq i \leq n\}$ and the left ideal $J''$ of $U_I$ is generated by $\{F_i^{(r_i)}, E_j^{(s_j)}, I_0'' \mid r_i \geq m_i'', s_j \geq n_j'', i, j \in I\}$. Let $J$ be the left ideal of $P_I$ generated by the set $D = \{F_i^{(r_i)}, E_j^{(s_j)}, I_0 \mid r_i \geq m_i, s_j \geq n_j, 1 \leq i \leq n, j \in I\}$ with $n_j = n_j''$ $(j \in I)$, $m_i = \max(m_i'', m_i')$ (for $i \in I$), $m_i = m_i'$ for $i \notin I$, and $I_0 = I_0' \cap I_0''$. Thus $J \in \mathcal{L}_{P_I}$. Note that $D = D_1 \cup D_2$ with $D_1 = \{F_i^{(r_i)} \mid r_i \geq m_i, i \notin I\}$ and $D_2 = \{F_i^{(r_i)}, E_i^{(s_i)}, I_0 \mid r_i \geq n_i, s_i \geq m_i, i \in I\}$. Each element of $E_I$ commutes with all elements of $D_1$. Thus we have $E_I D_1 \subseteq J'U_I$. Since $D_2 \subseteq J''$, we have $E_I D_2 \subseteq J''$. Therefore, we have

$$J = P_I D = U^{\leq 0}(E_I D_1 + E_I D_2) \subseteq U^{\leq 0}(J'U_I + J'') \subseteq J'U_I + U^{\leq 0}J''.$$

(ii) Note that for each $\lambda \in X$, we have $E_i^{(r)}\mathrm{Ker}(\lambda) = \mathrm{Ker}(\lambda + r\alpha_i)E_i^{(r)}$ by using the generating relations of the algebra $U$ and considering the generators of the ideal $\mathrm{Ker}(\lambda) \subseteq U^0$ [APW1, 1.1]. Suppose $J'$ is the left ideal of $U_I^{\leq 0}$ generated by $\{F_i^{(r_i)}, I_0' \mid r_i \geq m_i', i \in I\}$ and $J''$ the left ideal of $S_I$ generated by $\{F_i^{(r_i)}, E_i^{(s_i)}I_0'' \mid r_i \geq m_i'', s_i \geq n_i'', i \in I\}$ for a set of integers $m_i', m_i'', n_i'' \geq 0$ and $I_0' = \cap_{i=1}^t \mathrm{Ker}(\lambda^i)$ for some $\lambda^1, \ldots, \lambda^t$ in $X$. Let $J_I$ be the left ideal of $E_I$ generated by $\{E_i^{(s_i)} \mid s_i \geq n_i'', i \in I\}$, then $J_I$ is a $U^0$-submodule of $E_I$ under the adjoint action and $J_I \subseteq J''$. It can be proved that $E_I/J_I$ is a finitely generated free $A$-module using Lusztig's canonical basis [Lu5] for $E_I$ (with respect to the Cartan matrix $(a_{ij})_{i,j\in I}$). Let $\Pi$ be the set of $U^0$-weights of $E_I/J_I$. Set $I_0 = \bigcap_{i=1}^t \bigcap_{\beta\in\Pi} \mathrm{Ker}(\lambda^i - \beta)$. If $E$ is a finite product of $E_i^{(r_i)}$'s $(r_i \geq 0, i \in I)$ with weight $\gamma$ under the adjoint action of $U^0$ on $E_I$, the above argument shows that

$$EI_0 = \bigcap_{i=1}^{t} \bigcap_{\beta\in\Pi} \mathrm{Ker}(\lambda^i - \beta + \gamma)E \subseteq \begin{cases} I_0'S_I & \text{if} \quad \gamma \in \Pi, \\ U_I^{\leq 0}J_I & \text{if} \quad \gamma \notin \Pi. \end{cases}$$

Here we have used the fact that $E \in J_I$ if $\gamma \notin \Pi$ since the image of $E$ in $E_I/J_I$ is a $\gamma$-weight vector. Let $m_i = \min(m_i', m_i'')$ for $i \in I$ and $J$ be the left ideal of $U_I$ generated by $D = \{F_i^{(r_i)}, E_i^{(s_i)}, I_0' \mid r_i \geq m_i, s_i \geq n_i'', i \in I\}$. Since $E_I\{F_i^{(r_i)}, E_i^{(s_i)}\} \subseteq J''$ for all $r_i \geq m_i, s_i \geq n_i''$ and $i \in I$, we have $E_I D \subseteq J'S_I + U_I^{\leq 0}J''$ and

$$J = U_I D = U_I^{\leq 0}E_I D \subseteq U_I^{\leq 0}(J'S_I + U_I^{\leq 0}J'') = J'S_I + U_I^{\leq 0}J''. \ \square$$

**Proposition.** *We have isomorphisms of functors*

$$\mathrm{Res}_{U_I}^{P_I}\mathrm{Ind}_{U^{\leq 0}}^{P_I} \cong \mathrm{Ind}_{U_I^{\leq 0}}^{U_I}\mathrm{Res}_{U_I^{\leq 0}}^{U^{\leq 0}}, \quad \textit{and} \quad \mathrm{Res}_{S_I}^{S_I}\mathrm{Ind}_{U_I^{\leq 0}}^{U_I} \cong \mathrm{Ind}_{S_I^{\leq 0}}^{S_I}\mathrm{Res}_{S_I^{\leq 0}}^{U_I^{\leq 0}}.$$

*Proof:* Use the above lemma, Proposition 1.9, and 1.10. □

**2.3.** From now on, we assume that $A$ is a field unless it is specified. Let $V = W \oplus W'$ as vector spaces. Identify both $T(W)$ and $T(W')$ with subalgebras of the tensor algebra $T(V)$. Let $\langle W' \rangle$ be the ideal of $T(V)$ generated by $W'$. The maps of vector spaces $W \to V \to W$, with the last map being the projection, induce homomorphisms of algebras $T(W) \xrightarrow{\phi} T(V) \xrightarrow{\psi} T(W)$ with $\psi \circ \phi = 1_{T(W)}$.

**Lemma.** *If $J$ is an ideal of $T(V)$ generated by a set $X = X_0 \cup X_1$ such that $X_0 \subseteq T(W)$ and $X_1 \subseteq \langle W' \rangle$, then the maps $\phi$ and $\psi$ induce homomorphisms of algebras $T(W)/\langle X_0 \rangle \xrightarrow{\bar{\phi}} T(V)/J \xrightarrow{\bar{\psi}} T(W)/\langle X_0 \rangle$ such that $\bar{\phi} \circ \bar{\psi} = 1_{T(W)/\langle X_0 \rangle}$. Here $\langle X_0 \rangle$ is the ideal of $T(W)$ generated by $X_0$.*

*Proof:* Since $T(V) = T(W) \oplus \langle W' \rangle$ and $\psi$ is the projection with respect to this direct sum decomposition, $J = T(V)XT(V) \subseteq \langle W' \rangle + \langle X_0 \rangle$. Therefore $\langle X_0 \rangle = J \cap T(W)$ and $\psi(J) \subseteq \langle X_0 \rangle$. This shows that $\bar{\phi}$ and $\bar{\psi}$ are well defined. $\bar{\psi} \circ \bar{\psi} = 1$ follows from $\psi \circ \phi = 1$. □

**Proposition.** *The functors $\mathrm{Ind}_{U_I^{\leq 0}}^{U^{\leq 0}}$ and $\mathrm{Ind}_{S_I^{\leq 0}}^{U_I^{\leq 0}}$ are exact.*

*Proof:* To show the exactness of $\mathrm{Ind}_{S_I^{\leq 0}}^{U_I^{\leq 0}}$, we consider $S_I^0 = S_I^{\leq 0} \cap U^0$ and $U_I^{\leq 0} = S_I^{\leq 0} \otimes_{S_I^0} U^0$. For each left ideal $J' \in \mathcal{L}_{S_I^{\leq 0}}$ and each left ideal $J'' \in \mathcal{L}_{U^0}$, $J'U^0 + S_I^{\leq 0}J''$ is already a left ideal of $U_I^{\leq 0}$ and contains a left ideal $J \in \mathcal{L}_{U_I^{\leq 0}}$ (consider the generators). Thus by Proposition 1.9 and 1.10, we have isomorphism of functors $\mathrm{Res}_{U^0}^{U_I^{\leq 0}} \mathrm{Ind}_{S_I^{\leq 0}}^{U_I^{\leq 0}} \cong \mathrm{Ind}_{S_I^0}^{U^0} \mathrm{Res}_{S_I^0}^{S_I^{\leq 0}}$. So we only need to show the exactness of $\mathrm{Ind}_{S_I^0}^{U^0}$. Let $(U_I^0)'$ be the subalgebra of $U^0$ generated by $K_i^{\pm}, \begin{bmatrix} K_i \\ m \end{bmatrix}$ with $i \notin I$ and $m \geq 0$, then $U^0 \cong S_I^0 \otimes (U_I^0)'$ as algebras. By the commutativity of all three algebras, one can easily verify that the topology of $U^0$ is generated by the set of ideals $\{J_1 \otimes (U_I^0)' + S_I^0 \otimes J_2 \mid J_1 \in \mathcal{I}_{S_I^0}, J_2 \in \mathcal{I}_{(U_I^0)'}\}$. The exactness of $\mathrm{Ind}_{S_I^0}^{U^0}$ follows from Corollary 1.9.

To show the exactness of $\mathrm{Ind}_{U_I^{\leq 0}}^{U^{\leq 0}}$, we first consider the quantum enveloping algebra $U_{\mathbb{Q}(v)}$ and its subalgebras over $\mathbb{Q}(v)$ (the field of rational functions). The subalgebra $U_{\mathbb{Q}(v)}^{\leq 0}$ is generated by the set $\{F_i, K_i, K_i^{-1} \mid i = 1, \ldots, n\}$. Let $V$ be the vector space with basis $\{f_i, k_i, k_i^{-1} \mid i = 1, \ldots, n\}$ (of dimension $3n$), $W'$ and $W$ be the subspaces of $V$ with bases $\{f_i \mid i \notin I\}$ and $\{f_i \mid i \in I\} \cup \{k_i, k_i^{-1} \mid i = 1, \ldots, n\}$ respectively. Examining the generating relations of the algebra $U_{\mathbb{Q}(v)}^{\leq 0}$ (see [Lu4, 1.1]), one finds out that $U_{\mathbb{Q}(v)}^{\leq 0} \cong T(V)/D$ such that the ideal $D$ has a generating set $X$ with each element

being either in $\langle W' \rangle$ or in $T(W)$. Using the above lemma, we get the homomorphisms of algebras $(U^{\leq 0}_{\mathbb{Q}(v)})_I \stackrel{\phi}{\to} U^{\leq 0}_{\mathbb{Q}(v)} \stackrel{\psi}{\to} (U^{\leq 0}_{\mathbb{Q}(v)})_I$ such that $\phi$ gives the natural embedding and $\psi \circ \phi = 1$. By considering the Dyer-Lusztig basis of $U^{\leq 0}_{\mathbb{Q}(v)}$ with respect to a reduced expression of $w_0$ extending a reduced expression of $w_I$ as in 2.1, we note that $\psi(F^{(r)}_\beta) = 0$ for all $\beta \in R^+ \backslash R_I$ and $\psi(F^{(r)}_i) = F^{(r)}_i$ for all $i \in I$. Thus one can verify that the maps $\phi$ and $\psi$ preserve the $\mathbb{Z}[v, v^{-1}]$-subalgebras generated by $F^{(m)}_i, K^{\pm}_i, \left[ \begin{smallmatrix} K_i \\ m \end{smallmatrix} \right]$ and the like using the above mentioned bases. Tensoring $A$ to the induced homomorphisms over $\mathbb{Z}[v, v^{-1}]$, we can get homomorphisms $U^{\leq 0}_I \stackrel{\phi}{\to} U^{\leq 0} \stackrel{\psi}{\to} U^{\leq 0}_I$ such that $\psi \circ \phi = 1$. It is easily verified that the both homomorphisms are compatible with the induced topologies (consider the generators of the left ideals). Now the proposition follows from Lemma 1.4. $\square$

**2.4. Theorem.** *If $A$ is a field and $j \geq 0$, then* $\mathrm{Res}^{P_I}_{U_I} R^j \mathrm{Ind}^{P_I}_{U^{\leq 0}} \cong R^j \mathrm{Ind}^{U_I}_{U^{\leq 0}_I} \mathrm{Res}^{U^{\leq 0}}_{U^{\leq 0}_I}$ *and* $\mathrm{Res}^{U_I}_{S_I} R^j \mathrm{Ind}^{U_I}_{U^{\leq 0}_I} \cong R^j \mathrm{Ind}^{S_I}_{S^{\leq 0}_I} \mathrm{Res}^{U^{\leq 0}_I}_{S^{\leq 0}_I}$.

*Proof:* By 2.3, $\mathrm{Res}^{U^{\leq 0}}_{U^{\leq 0}_I}$ and $\mathrm{Res}^{U^{\leq 0}_I}_{S^{\leq 0}_I}$ send injective objects to injective objects. The isomorphisms follow from 2.2 by Grothendieck spectral sequences. $\square$

For each $\lambda \in X$, we denote by $A_\lambda$ (or simply $\lambda$ if there is no confusion) the rank 1 $U^{\leq 0}$-module, on which $U^0$ acts via the character $\lambda$ and $U^-$ acts trivially. For $1 \leq i \leq n$, let $I = \{i\}$. Then $S_I$ is isomorphic to the quantum enveloping algebra of type $\mathrm{A}_1$. The following result is fundamental to several important results (over fields) in [APW1]. Let $\rho$ be the half sum of all positive roots in $R$.

**Corollary.** *Let $A$ be a field. If $I = \{i\}$, $\lambda \in X$, then $R^j \mathrm{Ind}^{P_I}_{U^{\leq 0}}(\lambda) \neq 0$ if and only if either $j = 1$ with $\langle \lambda + \rho, \alpha^\vee_i \rangle < 0$ or $j = 0$ with $\langle \lambda + \rho, \alpha^\vee_i \rangle > 0$.*

*Proof:* Since $A$ is a field, the corollary follows from the theorem and [APW1, Sect. 4]. $\square$

**2.5.** Using Kempf's vanishing theorem for the quantum enveloping algebra $S_I$ (see [APW1, AW]), the followings are the consequences of Theorem 2.4 and certain spectral sequences. Here $A$ remains to be a field.

(i) $R^j \mathrm{Ind}^{P_I}_{U^{\leq 0}} A = \begin{cases} A & \text{for } j = 0, \\ 0 & \text{for } j > 0. \end{cases}$

(ii) $R^j \mathrm{Ind}^U_{U^{\leq 0}} M \cong R^j \mathrm{Ind}^U_{P_I} M$ for any $P_I$-module $M$.

(iii) $R^j \mathrm{Ind}^U_{P_I} \lambda = 0$ for all $j > 0$ and $\lambda \in X_+$ with $\langle \lambda, \alpha^\vee_i \rangle = 0$ for all $i \in I$.

The proof of the following proposition is just an application of the above facts and the generalized tensor identity to get a degenerate spectral sequence from the composition

of functors $\mathrm{Hom}_U(V', \mathrm{Ind}_{P_I}^U -) \cong \mathrm{Hom}_{P_I}(V', -)$. One can also take $V' = A$ to get the $U$- and $P_I$-cohomologies. A special case will be that $I = \varnothing$ and $P_I = U^{\leq 0}$. This resembles the "transfer theorem" in [CPSK]. See [J, II 4.7] for proof.

**Proposition.** *Let $V, V'$ be rational $U$-modules and $M$ a rational $P_I$-module such that $R^j\mathrm{Ind}_{P_I}^U M = 0$ for $j > 0$. Then,*

(i) $\mathrm{Ext}_U^j(V', V \otimes \mathrm{Ind}_{U^{\leq 0}}^U M) \cong \mathrm{Ext}_{P_I}^j(V', V \otimes M)$,

(ii) $\mathrm{Ext}_U^j(V', V) \cong \mathrm{Ext}_{P_I}^j(V', V)$,

(iii) $H^j(U, V) = H^j(P_I, V)$,

(iv) $H^j(P_I, M) = H^j(U^{\leq 0}, M)$.

*Proof:* (i) follows from the property (ii) and the above mentioned Grothendieck spectral sequence. (ii) and (iii) are special cases of (i) by taking $M = A$ and $V' = A$ respectively. To show (iv), we consider $\mathrm{Hom}_{P_I}(A, \mathrm{Ind}_{U^{\leq 0}}^{P_I} -) \cong \mathrm{Hom}_{U^{\leq 0}}(A, -)$. Using the property (i) and generalized tensor identity, we have $R^j\mathrm{Ind}_{U^{\leq 0}}^{P_I}(M) = 0$ if $j > 0$. Now (iv) is a consequence of a degenerate spectral sequence. □

# 3 The structure of $R^1\mathrm{Ind}_{U^{\leq 0}}^U \lambda$

**3.1.** We will keep the same notations as in Section 2 and assume that $A$ is a field. Let $E_I'$ be the subspace of $U^+$ with basis $\prod_{\beta \in R^+ \setminus R_I} E_\beta^{(m_\beta)}$ ($m_\beta \geq 0$). Thus we have $E_I \otimes E_I' \cong U^+$ as vector spaces with the isomorphism given by multiplication. On the other hand, following Lusztig [Lu4, 1.6, 8.8], we have $U^+ = \sum_\lambda U_\lambda^+$ as vector spaces. Here the sum is taken over $\mathbb{Z}R^+$ and $U_\lambda^+$ is the $A$-span of all monomials of the form $E_{i_1}^{(m_{i_1})} \cdots E_{i_s}^{(m_{i_s})}$ with $i_1, \ldots, i_s \in \{1, \ldots, n\}$ and $\lambda = \sum_{t=1}^s m_{i_t} \alpha_{i_t}$. It follows from the definition that $U_\lambda^+ U_\mu^+ \subseteq U_{\lambda+\mu}^+$. Note that the elements $\prod_{\beta \in R^+} E_\beta^{(m_\beta)} \in U_\lambda^+$, with $\lambda = \sum_{\beta \in R^+} m_\beta \beta$, of the Dyer-Lusztig basis in $U^+$ form a basis for $U_\lambda^+$. Recall that the ordering in $R^+$, over which the product is taken, is given by a reduced expression of $w_0$ extending a reduced expression of $w_I$ in $W_I$ such that all factors $E_\beta^{(m_\beta)}$ with $\beta \in R_I^+$ appear on the left hand side.

**Proposition.** *For each rational $U$-module $M$, $M^{E_I'}$ is a $U_I$-submodule of $M$.*

*Proof:* Since $M$ is a rational $U_I$-module via restriction, we only need to show that $M^{E_I'}$ is $U_I$-invariant. Note that $M^{E_I'} = \{v \in M \mid E_\beta^{(m)} v = 0, \forall \beta \in R^+ \setminus R_I, m > 0\}$. Using the defining relations of the algebra $U$ and the definition of $E_\beta^{(m)}$ via the braid group action, which leaves the subalgebra $U^0$ invariant, we can show that for each $h \in U^0$, $\beta \in R^+$ and $m > 0$, $E_\beta^{(m)} h = h' E_\beta^{(m)}$ for some $h' \in U^0$. Thus $U^0 M^{E_I'} \subseteq M^{E_I'}$.

We first show $E_I M^{E_I'} \subseteq M^{E_I'}$. $U_\lambda^+$ has an $A$-basis $\{\prod_{\beta\in R^+} E_\beta^{(m_\beta)} \mid \sum_{\beta\in R^+} m_\beta\beta = \lambda\}$. With the above order in $R^+$ as we mentioned prior to the proposition, one can see that $U_\lambda^+ v = 0$ for all $v \in M^{E_I'}$ if $\lambda = \sum_{i=1}^n n_i\alpha_i$ with $n_i \neq 0$ for some $i \notin I$. If $\beta \in R^+ \backslash R_I$ and $m > 0$, then for any $i \in I$ and $k \geq 0$, one has $E_\beta^{(m)} E_i^{(k)} \in U^+_{m\beta+k\alpha_i}$ and $E_i^{(k)} M^{E_I'} \subseteq M^{E_I'}$.

Finally, we show that $F_i^{(n)} M^{E_I'} \subseteq M^{E_I'}$ for all $i \in I$ and $n \geq 0$. Let $\beta \in R^+ \setminus R_I$ and $m > 0$. Since $E_\beta^{(m)} \in U^+_{m\beta}$,

$$E_\beta^{(m)} = \sum a_{(n)} E_{i_1}^{(n_{i_1})} \cdots E_{i_s}^{(n_{i_s})}.$$

Here the sum is taken over $\{i_1, \ldots, i_s\} \in \{1, \cdots, n\}^s$ for with $s \geq 0$ and $\{n_{i_1}, \ldots, n_{i_s}\} \in \mathbb{N}^s$ such that $\sum_{t=1}^s n_{i_t}\alpha_{i_t} = m\beta$. For each $i \in I$ and $t = 1, \ldots, s$, we have

$$E_{i_t}^{(n_{i_t})} F_i^{(n)} = \begin{cases} F_i^{(n)} E_{i_t}^{(n_{i_t})} & \text{if} \quad i \neq i_t, \\ \sum F_i^{(n-k)} h_k E_{i_t}^{(n_{i_t}-k)} & \text{if} \quad i = i_t. \end{cases}$$

By a direct calculation we can write

$$E_\beta^{(m)} F_i^{(n)} = \sum_{k=0}^{n} F_i^{(k)} (\sum_{(n)} h_{(n)}(k) e_{(n)}(k)).$$

Here, $h_{(n)}(k) \in U^0$ and $e_{(n)}(k) = \prod_{\alpha\in R^+} E_\alpha^{(n_\alpha)}$ with the above fixed order in $R^+$ and $\sum_{\alpha\in R^+} n_\alpha\alpha = m\beta - k\alpha_i$. Since $\beta \notin R_I$, there exists $\alpha \in R^+ \setminus R_I$ such that $n_\alpha > 0$. Then $e_{(n)}(k)v = 0$ for all $v \in M^{E_I'}$. Therefore $E_\beta^{(m)} F_i^{(n)} M^{E_i'} = 0$ and $F_i^{(n)} M^{E_I'} \subseteq M^{E_I'}$. Now the proposition follows from the fact that $U_I$ is generated as algebra by $F_i^{(k)}$, $E_i^{(k)}$ ($i \in I$ and $k \geq 0$), and $U^0$. $\square$

**3.2.** Recall that for $\lambda, \mu \in X(T)$, $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{N}R^+$. For each $U^0$-module $M$, denote by $M_\lambda$ the $\lambda$-weight space of $M$ for $\lambda \in X$.

**Proposition.** (i) *If $M$ is a finite-dimensional $U$-module such that $M^{U^+} = M_\lambda$, then $\mu \leq \lambda$ for all $\mu \in X$ with $M_\mu \neq 0$ and $M^{E_I'} = \oplus_{\nu\in\mathbb{N}R_I^+} M_{\lambda-\nu}$.*

(ii) *If $L(\lambda)$ is the simple $U$-module of highest weight $\lambda$, then $L(\lambda)^{E_I'}$ is a simple $U_I$-module.*

(iii) *If $N$ is a rational $P_I$-module, then the restriction of the evaluation map $ev$ : $\mathrm{Ind}_{P_I}^U N \to N$ to $(\mathrm{Ind}_{P_I}^U N)^{E_I'}$ is injective.*

*Proof:* (i). If $0 \neq v \in M_\mu$ and $\mu \neq \lambda$, by the assumption, there exist $i$ and $m > 0$ such that $0 \neq E_i^{(m)} v \in M_{\mu+m\alpha_i}$. Thus $\lambda$ is the only maximal weight of $M$ and $\mu \leq \lambda$. (If $\chi$ is a maximal weight of $N = U^+v$, then $N_\chi \subseteq M^{U^+}$). Let $\beta \in R^+ \setminus R_I$ and $m > 0$. For $\nu \in \mathbb{N}R_I^+$, $E_\beta^{(m)} M_{\lambda-\nu} \subseteq M_{\lambda-\nu+m\beta} = 0$ since $\lambda - \nu + m\beta \not\leq \lambda$. Thus $\oplus_{\nu\in\mathbb{N}R_I^+} M_{\lambda-\nu} \subseteq M^{E_I'}$.

On the other hand, $M^{E_I'}$ is a rational $U_I$-module. If $\mu \in X$ such that $(M^{E_I'})_\mu \neq 0$, then for each $0 \neq v \in (M^{E_I'})_\mu$, $V = U_I v \subseteq M^{E_I'}$ is finitely dimensional and has a maximal weight $\mu' = \mu + \nu$ for some $\nu \in \mathbb{N}R_I^+$. For $0 \neq v' \in V_{\mu'}$, we have $E_\beta^{(m)} v' = 0$ for all $\beta \in R^+$ and $m > 0$. Hence $\mu'$ is a maximal weight of $M$ and $\mu' = \lambda$ by the assumption. Thus $v \in M_{\lambda-\nu}$ and $M^{E_I'} \subseteq \oplus_{\nu \in \mathbb{N}R_I^+} M_{\lambda-\nu}$.

(ii). Suppose that $M = L(\lambda)$, then $L(\lambda)^{E_I'}$ is a $U_I$-module with simple socle since $(L(\lambda)^{E_I'})^{E_I} = L(\lambda)_\lambda$. Using the Dyer-Lusztig basis for $U^-$ such that the factors $F_\beta^{(m)}$ with $\beta \in R_I^+$ are on the right side in the products, one can show that $\oplus_{\nu \in \mathbb{N}R_I^+} L(\lambda)_{\lambda-\nu} = U_I^- v^+$ which is, therefore, simple. Here $v^+ \in L(\lambda)$ is a highest weight vector.

(iii). It follows from the definition of $E_I$ and $E_I'$ that $P_I \otimes E_I' \cong U$ as vector spaces given by the multiplication. Thus (iii) is a consequence of Proposition 1.7. □

**3.3.** The isomorphism classes of simple $U_I$-modules in $\mathcal{C}_{U_I}$ are parameterized by the set of weights $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I$. We denote by $L_I(\lambda)$ the simple $U_I$-module with highest weight $\lambda$.

**Corollary.** *For $\lambda, \mu \in X_+$, $[\mathrm{Ind}_{U^{\leq 0}}^U \lambda : L(\mu)] = [\mathrm{Ind}_{U^{\leq 0}}^{P_I} \lambda : L_I(\mu)]$.*

*Proof:* If $v$ is not a root of 1 and the ground field is of characteristic 0, then $\mathrm{Ind}_{U^{\leq 0}}^U \lambda$ is $U$-irreducible. Therefore $\mathrm{Ind}_{U^{\leq 0}}^{P_I} \lambda$ is irreducible as $U_I$-module by Proposition 2.2. Thus we have $\sum_{\nu \in \mathbb{N}R_I^+} (\mathrm{Ind}_{U^{\leq 0}}^U \lambda)_{\lambda-\nu} = \mathrm{Ind}_{U^{\leq 0}}^{P_I} \lambda$ in this case by 3.2(ii). However, since both $\mathrm{Ind}_{U^{\leq 0}}^U \lambda$ and $\mathrm{Ind}_{U^{\leq 0}}^{P_I} \lambda$ (use Proposition 2.2 ) are finitely generated free modules over $\mathbb{Z}[v, v^{-1}]$ by Lusztig's canonical basis [Lu5], one can use the base change relation as in [APW1, Section 3] over $\mathbb{Z}[v, v^{-1}]$, which has global dimension 2, to show the above equality for other fields $A$. Then one can follow a similar argument as in [J, 5.21]. □

**3.4. Lemma.** *Let $M$ be a $P_I$-module.*

(i) *If $\mathrm{Soc}_{U_I} M = L_I(\lambda)$ and $\mathrm{Ind}_{P_I}^U M \neq 0$, then $\mathrm{Soc}_U \mathrm{Ind}_{P_I}^U M = L(\lambda)$ and $\lambda \in X_+$.*

(ii) *If $\lambda$ is a maximal weight of $\mathrm{Ind}_{P_I}^U M$, then $\lambda \in X_+$ and $(M^{E_I})_\lambda \neq 0$.*

(iii) *Suppose $F_j^{(m)} M = 0$ for all $j \notin I$ and $m > 0$. If $\lambda \in X_+$ such that $(M^{E_I})_\lambda \neq 0$, then $\lambda$ is a weight of $\mathrm{Ind}_{P_I}^U M$.*

*Proof:* One can copy the proof of Lemma 5.14 of [J, II] using Proposition 3.2, Theorem 2.4, and the weights of $R^1 \mathrm{Ind}_{U^{\leq 0}}^{P_{\{i\}}} \mu$ given as in [APW1, 4.4]. The subspace $E_I'$ should be in the position of $U(\bar{P})$ as in the proof of [J, II 5.14]. □

**3.5.** In the rest of Section 3, we assume that $A$ is a field of characteristic 0 and $v$ is a primitive $l$th root of 1 with $l$ odd and $3 \nmid l$ if $R$ has a component of type $\mathrm{G}_2$ (i.e., we can choose $d_i$ such that $d_i \nmid l$ whenever $d_i \neq 1$). The dot action of

the Weyl group $W$ on $X$ is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for all $w \in W$ and $\lambda \in X$. Here $\rho$ is the half sum of positive roots. Let $i \in \{1, \dots, n\}$, $I = \{i\}$ and $\lambda \in X_+$ such that $0 \le \langle \lambda, \alpha_i^\vee \rangle = r = r_1 l + r_0$ with $0 \le r_0 < l$. If $\mu = s_{\alpha_i} \cdot \lambda$, then $N = R^1\mathrm{Ind}_{U^{\le 0}}^{P_I}\mu \cong R^1\mathrm{Ind}_{S_I^{\le 0}}^{S_I}\mu$ has a basis $\{f_0, \dots, f_r\}$ such that $f_j \in N_{s_{\alpha_i}(\lambda)+j\alpha_i}$ and the actions of $E_i^{(m)}$ and $F_i^{(m)}$ are given in [APW1, 4.4].

**Lemma.** *$F_j^{(m)} N = 0$ for all $j \ne i$ and $m > 0$. Furthermore,*

$$\mathrm{Soc}_{U_I} N = \begin{cases} L_I(\lambda) & \text{if } r_0 = l - 1, \\ L_I(s_{\alpha_i} \cdot \lambda + r_1 l \alpha_i) & \text{if } 0 \le r_0 < l - 1. \end{cases}$$

*Proof:* The first statement is a consequence of the weight space decomposition of $N$. If $L_I(\mu)$ is a simple submodule of $N$, then $L_I(\mu)^{E_I} \subseteq N^{E_I}$. For $0 \le j \le r$, $f_j \in N^{E_I}$ if and only if $E_i^{(m)} f_j = \begin{bmatrix} m+j \\ j \end{bmatrix}_{d_i} f_{j+m} = 0$ for all $m > 0$. This is equivalent to $\begin{bmatrix} m \\ j \end{bmatrix}_{d_i} = 0$ for $m = j+1, \dots, r$. Write $j = j_1 l + j_0$ with $0 \le j_0 < l$. It is clear that $f_r \in N^{E_I}$. We assume $j < r$. If $r_0 = l-1$, then $\begin{bmatrix} r \\ j \end{bmatrix}_{d_i} = \binom{r_1}{j_1} \begin{bmatrix} l-1 \\ j_0 \end{bmatrix}_{d_i} \ne 0$. This shows that $N^{E_I} = A f_r$ and $\mathrm{Soc}_{U_I} N = L_I(\lambda)$. If $r_0 < l - 1$, one can see that $\begin{bmatrix} m \\ r_1 l - 1 \end{bmatrix}_{d_i} = 0$ for $r_1 l \le m \le r$ and $f_{r_1 l - 1} \in N^{E_I}$. However, one has $\begin{bmatrix} r_1 l - 1 \\ j \end{bmatrix}_{d_i} \ne 0$ for $j < r_1 l - 1$ and $\begin{bmatrix} r \\ j \end{bmatrix}_{d_i} \ne 0$ for $r > j > r_1 l - 1$. Thus $N^{E_I} = A f_r + A f_{r_1 l - 1}$. Since $f_r$ generates the whole module $N$ (as $F_i^{(m)} f_r = f_{r-m}$ for $m = 0, \dots, r$), we see that $\mathrm{Soc}_{U_I} N$ is generated by $f_{r_1 l - 1}$, which is a proper submodule. The lemma follows from the fact that $f_{r_1 l - 1}$ has the weight $s_{\alpha_i}(\lambda) + (r_1 l - 1)\alpha_i = s_{\alpha_i} \cdot \lambda + r_1 l \alpha_i$. $\square$

**3.6. Proposition.** *For $\lambda \in X$ and $\alpha \in S$, we write $\langle \lambda, \alpha^\vee \rangle = r_1 l + r_0$ with $0 \le r_0 < l$.*

(i) *If $r_0 = l - 1$, then $R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda) \ne 0$ if and only if $\lambda \in X_+$, and if so,*

$$\mathrm{Soc}_U R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda) = L(\lambda).$$

(ii) *If $r_0 < l - 1$, then $R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda) \ne 0$ if and only if $s_\alpha \cdot \lambda + r_1 l \alpha \in X_+$, and if so,*

$$\mathrm{Soc}_U R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda) = L(s_\alpha \cdot \lambda + r_1 l \alpha).$$

(iii) *If $R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda) \ne 0$, then $\lambda$ is a maximal weight of $R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda)$ if $\lambda \in X_+$ and $s_\alpha \cdot \lambda + r_1 l \alpha$ is a maximal weight of $R^1\mathrm{Ind}_{U^{\le 0}}^{U}(s_\alpha \cdot \lambda)$ if $\lambda \notin X_+$.*

*Proof:* Let $\alpha = \alpha_i$ and $I = \{i\}$. If $r_0 = l - 1$, there is an $P_I$-module isomorphism $\phi : R^1\mathrm{Ind}_{U^{\le 0}}^{P_I}(s_\alpha \cdot \lambda) \to \mathrm{Ind}_{U^{\le 0}}^{P_I}\lambda$ as $S_I$-modules by [APW1, 4.5-4.6]. However, by

examining the $P_I$-actions on these two modules, one can see that $\phi$ is also a $P_I$-map. Thus (i) follows.

To show (ii), let $N = R^1\mathrm{Ind}_{U^{\leq 0}}^{P_I}(s_\alpha \cdot \lambda)$. If $R^1\mathrm{Ind}_{U^{\leq 0}}^{U}(s_\alpha \cdot \lambda) = \mathrm{Ind}_{P_I}^{U} N \neq 0$, then $s_\alpha \cdot \lambda + r_1 l\alpha \in X_+$ and $\mathrm{Soc}_U R^1\mathrm{Ind}_{U^{\leq 0}}^{U}(s_\alpha \cdot \lambda) = L(s_\alpha \cdot \lambda + r_1 l\alpha)$ by 3.4 , 3.5. Conversely, one knows from 3.5 that $(N^{E_I})_{s_\alpha \cdot \lambda + r_1 l\alpha} \neq 0$ and $F_j^{(m)} N = 0$ for all $j \neq i$ and $m > 0$. If $s_\alpha \cdot \lambda + r_1 l\alpha \in X_+$, then by 3.4, it is a weight of $\mathrm{Ind}_{P_I}^{U} N$. Thus $R^1\mathrm{Ind}_{U^{\leq 0}}^{U} s_\alpha \cdot \lambda \neq 0$.

If $\mu$ is a maximal weight of $R^1\mathrm{Ind}_{U^{\leq 0}}^{U} s_\alpha \cdot \lambda$, then it has to be a weight of $N^{E_I}$ and in $X_+$ by 3.4 (ii). However, if $\lambda \in X_+$, $\lambda$ is a weight of $R^1\mathrm{Ind}_{U^{\leq 0}}^{U} s_\alpha \cdot \lambda$ and $\lambda \geq s_\alpha \cdot \lambda + r_1 l\alpha$. Now (iii) follows from (i) and (ii). □

**3.7.** Let $\alpha \in R^+$, we define the reflection $s_\alpha^* \cdot$ in the *upper $\alpha$-hyper-plane* by setting $s_\alpha^* \cdot \lambda = s_\alpha \cdot \lambda + c_\lambda(\alpha) l\alpha$ if $\lambda \in X$ and $(c_\lambda(\alpha) - 1)l < \langle \lambda + \rho, \alpha \rangle \leq c_\lambda(\alpha) l$ for some $c_\lambda(\alpha) \in \mathbb{Z}$. The proof of the following is similar to those as in [A3, 2.3,2.4,2.9] using the spectral sequence in 1.5.

**Corollary.** (i) *For $\mu \in X$, if $R^1\mathrm{Ind}_{U^{\leq 0}}^{U} \mu \neq 0$, then it has a simple socle.*

(ii) *For $\lambda \in X_+$, $\mathrm{Soc}_U R^1\mathrm{Ind}_{U^{\leq 0}}^{U} \mu = L(\lambda)$ if and only if $\mu = s_\alpha \cdot (s_\alpha^* \cdot \lambda)$ for some $\alpha \in S$.*

(iii) *If $\lambda \in X_+$ and $\mu \in X \setminus X_+$, then*

$$\mathrm{Ext}^1_{U^{\leq 0}}(L(\lambda), \mu) = \begin{cases} A & \text{if } \mu = s_\alpha \cdot (s_\alpha^* \cdot \lambda) \text{ for some } \alpha \in S, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *$H^1(U^{\leq 0}, \mu) = A$ if $\mu = -l\alpha$ for some $\alpha \in S$ and 0 otherwise.*

(v) *If $H^2(U^{\leq 0}, \mu) \neq 0$ then $\mu = -(i\alpha + l\beta)$ for some $\alpha, \beta \in S$, $i > 0$ with $i + l\langle \beta, \alpha^\vee \rangle > 0$, and $l\mu + (l-1)\rho$ is not a weight of $St$.* □

# 4 Non-vanishing of certain $R^i\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda$

**4.1.** We remain assume that $A$ is a field of characteristic 0 in this section. If $v$ is a primitive $l$-th root of 1 with $l$ odd, let $Z$ be the subalgebra of $U^0$ generated by $K_i^l$ $(i = 1, \ldots, n)$. Then $Z$ acts trivially on each module in $\mathcal{C}_U$. $Z$ is a normal Hopf subalgebra of $U$ ($Z$ is contained in the center of $U$). Let $\tilde{U} = U/Z$ be the quotient Hopf algebra. Any module in $\mathcal{C}_U$ is also a $\tilde{U}$-module. For any Hopf subalgebra $H_1$ containing $Z$, write $\tilde{H}_1 = H_1/Z$, which is a Hopf subalgebra of $\tilde{U}$. By [L3, 4.7], we have $R^i\mathrm{Ind}_{H_1}^{U}\pi_1^* \cong \pi^* R^i\mathrm{Ind}_{\tilde{H}_1}^{\tilde{U}}$. Here $\pi$ and $\pi_1$ are the quotient maps. All the theory about the induction on $\tilde{}$ level proved in [L3] can be stated for $U$. So in this paper we use the induction theory for the subalgebras of $U$ instead of $\tilde{U}$.

In [Lu3], Lusztig constructed a finite-dimensional subalgebra $u$ in $U$, which is generated as a subalgebra of $U$ by $E_i, F_i, K_i^{\pm 1}$ $(i = 1, \ldots, n)$. It is proved in [L3, 5.3] that $u$ is

a normal Hopf subalgebra of $U$. Since all of the generators of $Z$ are in $u$, then $Z \subseteq u$ and $U/u = \tilde{U}/\tilde{u} = G$. Here $G$ is isomorphic to the enveloping algebra of a split semisimple Lie algebra over $A$ [Lu4, 8.14]. Let $\pi^* : U \to G$ be the quotient map. Corresponding to the decomposition $U = U^-U^0U^+$, we can define similarly the following subalgebras $u^-$, $u^0$, $u^+$, $u^{\leq 0}$, and $u^{\geq 0}$ in $u$ and $G^-$, $G^0$, $G^+$, $G^{\leq 0}$, and $G^{\geq 0}$ in $G$. Note that all of the above mentioned subalgebras, except $u^-$, $u^+$, $U^-$, $U^+$ are actually Hopf subalgebras of $U$ (or $G$) with bijective antipodes. Let $X_l = \{\lambda \in X_+ \mid \langle \lambda, \alpha^\vee \rangle < l,\ \forall \alpha \in S\}$ be the set of $l$-restricted weights.

**Lemma.** *There is a natural isomorphism of $U^0u^-$-modules* $\mathrm{Res}^{u^+U^0u^-}_{U^0u^-}\mathrm{Ind}^{u^+U^0u^-}_{u^+U^0}M \cong \mathrm{Ind}^{U^0u^-}_{U^0}\mathrm{Res}^{u^+U^0}_{U^0}M$ *for each rational $u^+U^0$-module $M$.*

*Proof:* It is clear that $U^0 = u^+U^0 \cap U^0u^-$ and $u^+U^0u^- = u^+U^0 \otimes_{U^0} U^0u^-$ as vector spaces over $A$ (consider the PBW basis constructed by Lusztig in [Lu3, Lu4]). Following the proof of [APW2, 1.1-1.2], we have $\mathrm{Ind}^{u^+U^0u^-}_{u^+U^0}M = \mathrm{Hom}_A(u^+, M) = \mathrm{Ind}^{U^0u^-}_{U^0}M$ as vector spaces over $A$. Now the isomorphism follows from 1.9, 1.10, and a similar argument on the set of left ideals as in 2.2. $\square$

The Steinberg module $St$ is the irreducible $U$-module in $\mathcal{C}_U$ with highest weight $(l-1)\rho$. It is proved in [APW1] that $St = \mathrm{Ind}^U_{U^{\leq 0}}(l-1)\rho$. It is further proved in [L3, 6.3] that $St$ is isomorphic to $\mathrm{Ind}^{uU^{\leq 0}}_{U^{\leq 0}}(l-1)\rho$ as $uU^{\leq 0}$-modules and to $\mathrm{Ind}^{uU^0}_{u^{\leq 0}U^0}(l-1)\rho$ as $uU^0$-modules. In both cases, $St$ remains simple when restricted to the subalgebras $uU^0$ and $uU^{\leq 0}$ respectively. As $uU^0$-module, we can embed $St$ into $\mathrm{Ind}^{uU^0}_{U^0u^+}(-(l-1)\rho$ via Frobenius reciprocity since $\mathrm{Hom}_{u^+U^0}(St, A_{-(l-1)\rho}) \neq 0$. However, $\mathrm{Ind}^{uU^0}_{U^0u^+}(-(l-1)\rho) \subseteq \mathrm{Hom}_A(u^-, A)$ as vector spaces. By comparing the dimensions (see. [Lu3, Lu4]), we have $St \cong \mathrm{Ind}^{uU^0}_{U^0u^+}(-(l-1)\rho)$. Thus applying the Lemma to the $u^+U^0$-module $-(l-1)\rho$, we have

**Corollary.** *When restricted to $U^0u^-$, $St$ is isomorphic to* $\mathrm{Ind}^{U^0u^-}_{U^0}(-(l-1)\rho)$. $\square$

**4.2.** Note that the admissible category for $U^0$ is a semisimple category, i.e., each module is a direct sum of simple modules, and therefore every $U^0$-module is injective in $\mathcal{C}_{U_0}$. For any Hopf subalgebra $U_1$ of $U$ containing $U^0$, $\mathrm{Ind}^{U_1}_{U^0}M$ is injective in the category $\mathcal{C}_{U_1}$. Furthermore, every injective module in $\mathcal{C}_{U_1}$ is a direct summand of $\mathrm{Ind}^{U_1}_{U^0}M$ for some $U^0$-module $M$. Indeed, if $Q$ is an injective $U_1$-module, by Frobenius reciprocity, $\mathrm{Hom}_{U^0}(Q, Q) = \mathrm{Hom}_{U_1}(Q, \mathrm{Ind}^{U_1}_{U^0}Q)$. The identity map $Q \to Q$ gives an embedding of $Q$ into $\mathrm{Ind}^{U_1}_{U^0}Q$. The injectivity of $Q$ implies that $Q$ is a direct summand of $\mathrm{Ind}^{U_1}_{U^0}Q$.

If we choose $U_1$ to be one of the algebras $U^{\leq 0}$, $U^{\geq 0}$, $U^0u^{\leq 0}$ and $U^0u^{\geq 0}$, then all irreducible $U_1$-modules in $\mathcal{C}_{U_1}$ are 1-dimensional.

**Proposition.** *For $U_1$ being one of the subalgebras mentioned above, an indecomposable module $M$ in $\mathcal{C}_{U_1}$ is injective if and only if $M \cong \mathrm{Ind}_{U^0}^{U_1}\lambda$ for some $\lambda \in X$.*

*Proof:* First of all, $\mathrm{Ind}_{U^0}^{U_1}\lambda$ is injective. By Frobenius reciprocity, one obtains that $\lambda$ is the only simple submodule of $\mathrm{Ind}_{U^0}^{U_1}\lambda$. Conversely, if $Q$ is an indecomposable injective $U_1$-module, let $\lambda$ be a simple $U_1$-submodule of $Q$. Thus $\mathrm{Ind}_{U^0}^{U_1}\lambda$ is a direct summand of $Q$ using the fact that $\mathrm{Ind}_{U^0}^{U_1}\lambda$ is injective with simple socle $\lambda$. Hence $Q \cong \mathrm{Ind}_{U^0}^{U_1}\lambda$. □

**4.3.** Recall that $\pi : U \to G$ is the quotient map and $\pi_1 : U^{\leq 0} \to G^{\leq 0}$ is the restriction of $\pi$. Now we are ready to prove the following result which resembles [A2, Prop. 1.4].

**Proposition.** *For each injective module $Q$ in $\mathcal{C}_{G^{\leq 0}}$ and any $U^{\leq 0}$-module $E$, both $E \otimes \pi_1^* Q$ and $\pi_1^*(Q) \otimes E$ are $\mathrm{Ind}_{U^{\leq 0}}^{U}$-acyclic. In particular, $\pi_1^*(Q)$ is $\mathrm{Ind}_{U^{\leq 0}}^{U}$-acyclic.*

*Proof:* Using a result analogous to Proposition 4.2 for subalgebras of $G$, we can assume that $Q = \mathrm{Ind}_{G^0}^{G^{\leq 0}}\lambda$ for some $\lambda \in X$. Considering both $\pi_1^*\mathrm{Ind}_{G^0}^{G^{\leq 0}}\lambda$ and $St$ as $U^{\leq 0}$-modules via restriction, we can use transitivity of induction functors, tensor identity, [L3, 4.7] and Corollary 4.1 to get

$$\begin{aligned}(\pi_1^*\mathrm{Ind}_{G^0}^{G^{\leq 0}}\lambda) \otimes St &\cong (\mathrm{Ind}_{u^-U^0}^{U^{\leq 0}}\pi_1^*\lambda) \otimes St \\ &\cong \mathrm{Ind}_{u^-U^0}^{U^{\leq 0}}(\pi_1^*\lambda \otimes \mathrm{Ind}_{U^0}^{u^-U^0}(-(l-1)\rho)) \\ &\cong \mathrm{Ind}_{U^0}^{U^{\leq 0}}(\pi_1^*\lambda \otimes (-(l-1)\rho)).\end{aligned}$$

By Proposition 4.2, the last term is injective in $\mathcal{C}_{U^{\leq 0}}$. This shows that $\pi_1^*Q \otimes St$ is injective and therefore $E \otimes \pi_1^*Q \otimes St$ is also injective for any $U^{\leq 0}$-module $E$. Since $St$ is a $U$-module, we can use generalized tensor identity to get

$$0 = R^i\mathrm{Ind}_{U^{\leq 0}}^{U}(E \otimes \pi_1^*Q \otimes St) \cong R^i\mathrm{Ind}_{U^{\leq 0}}^{U}(E \otimes \pi_1^*Q) \otimes St$$

for $i > 0$. Thus $R^i\mathrm{Ind}_{U^{\leq 0}}^{U}(E \otimes \pi_1^*Q) = 0$ for all $i > 0$ and $E \otimes \pi_1^*Q$ is $\mathrm{Ind}_{U^{\leq 0}}^{U}$-acyclic. The conclusion for $\pi_1^*Q \otimes E$ follows similarly by using the other generalized tensor identity (see [PW, 2.8.2]). Take $E = A$ to be the trivial module to get the last statement. □

For each $G^{\leq 0}$-module $M$, let

$$\begin{aligned}\varphi &: (\pi^*\mathrm{Ind}_{G^{\leq 0}}^{G}M) \otimes St \to \mathrm{Ind}_{U^{\leq 0}}^{U}(\pi_1^*(M)) \otimes St \\ \psi &: \mathrm{Ind}_{U^{\leq 0}}^{U}(\pi_1^*(M)) \otimes St \to \mathrm{Ind}_{U^{\leq 0}}^{U}(\pi_1^*(M) \otimes (l-1)\rho)\end{aligned}$$

be the homomorphisms defined in 1.6. Proposition 4.3 shows that the hypothesis of Proposition 1.6 is satisfied, and we thus have

**Corollary.** *For each $G^{\leq 0}$-module $M$ and $i \geq 0$, the composition $R^i\varphi \circ R^i\psi$ of the maps*

$$\begin{aligned} R^i\varphi &: (\pi^* R^i\mathrm{Ind}_{G^{\leq 0}}^G M) \otimes St \to R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*(M)) \otimes St, \\ R^i\psi &: R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*(M)) \otimes St \to R^i\mathrm{Ind}_{U^{\leq 0}}^H(\pi_1^*(M) \otimes (l-1)\rho) \end{aligned}$$

*is an isomorphism. Therefore $R^i\varphi$ is injective and $R^i\psi$ is surjective. Furthermore, $\pi^* R^i\mathrm{Ind}_{G^{\leq 0}}^G M \subseteq R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*(M))$.* $\square$

In the next subsection, we will use this result and an analogous argument as in [A2] to study the vanishing behavior of $R^i\mathrm{Ind}_{U^{\leq 0}}^U \lambda$.

**4.4.** Since $U$ is not cocommutative, one cannot expect $M \otimes N \cong N \otimes M$ in general. However, as $U^{\leq 0}$-modules, $A_\lambda \otimes A_\mu \cong A_{\lambda+\mu} \cong A_\mu \otimes A_\lambda$ for all $\lambda, \mu \in X$. So we can identify these three modules. Indeed this is true as $U^0$-modules. We need to see that $U^-$ acts trivially on $A_\lambda \otimes A_\mu$. ($U^-$ is not a Hopf algebra!) Since

$$\Delta(F_i^{(m)}) = \sum_{s=0}^{m} v^{-d_i s(m-s)} F_i^{(s)} \otimes K_i^{-s} F_i^{(m-s)}$$

(see [Lu4, 1.2]), $F_i^{(m)}(1_\lambda \otimes 1_\mu) = \sum_{s=0}^m v^{-d_i s(m-s)} F_i^{(s)} 1_\lambda \otimes K_i^{-s} F_i^{(m-s)} 1_\mu = 0$ for $m > 0$.

To see the vanishing property of $R^i\mathrm{Ind}_{U^{\leq 0}}^{U^0} \lambda$, note that for $\lambda \in X_l$, $\tilde{\lambda} = (l-1)\rho - \lambda$ is also in $X_l$. Thus both $\mathrm{Ind}_{U^{\leq 0}}^U \lambda$ and $\mathrm{Ind}_{U^{\leq 0}}^U \tilde{\lambda}$ are not zero (see [APW1]). Therefore we have the following commutative diagram of $U^{\leq 0}$-modules.

$$\begin{array}{ccc} \mathrm{Ind}_{U^{\leq 0}}^U \lambda \otimes \mathrm{Ind}_{U^{\leq 0}}^U \tilde{\lambda} & \longrightarrow & \lambda \otimes \mathrm{Ind}_{U^{\leq 0}}^U \tilde{\lambda} \\ \downarrow & \searrow & \downarrow \\ St & \longrightarrow & \lambda \otimes \tilde{\lambda} = (l-1)\rho. \end{array} \tag{3}$$

Here the maps are evaluation maps except the vertical map on the left hand side, which is actually a $U$-module map and arises from the diagonal map via Frobenius reciprocity since $St \cong \mathrm{Ind}_{U^{\leq 0}}^U (l-1)\rho$. It is nonzero, hence surjective by the irreducibility of $St$. Let $\chi \in X$ and regard $\chi$ as a $G^{\leq 0}$-module. By tensoring the $U^{\leq 0}$-module $\pi_1^*\chi$ to (3) and then applying the functor $R^i\mathrm{Ind}_{U^{\leq 0}}^U$ (using the generalized tensor identity) we get the following commutative diagram of $U$-modules,

$$\begin{array}{ccc} R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*\chi) \otimes \mathrm{Ind}_{U^{\leq 0}}^U \lambda \otimes \mathrm{Ind}_{U^{\leq 0}}^U \tilde{\lambda} & \longrightarrow & R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*\chi \otimes \lambda) \otimes \mathrm{Ind}_{U^{\leq 0}}^U \tilde{\lambda} \\ \downarrow & \searrow & \downarrow \\ R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*\chi) \otimes St & \longrightarrow & R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*\chi \otimes (l-1)\rho). \end{array} \tag{4}$$

Here the left vertical map and the bottom horizontal map are surjective by Corollary 4.3. Therefore the vertical map on the right hand side is surjective as well. On the other hand, by Corollary 4.3 we have $R^i\mathrm{Ind}_{U^{\leq 0}}^U(\pi_1^*\chi \otimes (l-1)\rho) \cong \pi^* R^i\mathrm{Ind}_{G^{\leq 0}}^G(\chi) \otimes St$.

Note that $G$ and $G^{\leq 0}$ are the distribution algebras of a semisimple algebraic group and its Borel subgroup. The categories of rational $G$- and $G^{\leq 0}$-modules are actually the categories of rational modules of the corresponding algebraic groups [J, II 1.20]. The induction we defined in Section 1 for $G$ and $G^{\leq 0}$ as in 2.1 is the same as the induction defined in terms of algebraic groups. Thus we can use the Borel-Weil-Bott Theorem ($A$ is a field of characteristic 0). Hence $\pi^* R^i \mathrm{Ind}_{G^{\leq 0}}^{G}(\chi) \neq 0$ if and only if there exists $w \in W$ such that $w \cdot \chi \in X_+$ and $i = l(w)$. Examining the homomorphism $U \to G$, one can see the $U^{\leq 0}$-module $\pi_1^*(\chi)$ is isomorphic to $l\chi$. Therefore we have proved the following theorem, from the commutative diagram (4),

**Theorem.** *For each $\lambda \in X_l$, $\chi \in X_+$, and $w \in W$, $R^{l(w)}\mathrm{Ind}_{U^{\leq 0}}^{U}(l(w \cdot \chi) + \lambda) \neq 0$.* $\square$

Applying the Serre duality and using a similar argument as in [W], we have

**Corollary.** *If $\chi \in X_+$, then for any $w \in W$, $R^{l(w)}\mathrm{Ind}_{U^{\leq 0}}^{U}(lw \cdot \chi \pm \lambda) \neq 0$ for all $\lambda \in X_l$.* $\square$

# 5 Good filtrations and composition factors

**5.1.** Although several results in this section hold over any field, we still assume that the field $A$ is of characteristic 0 and $v$ is a primitive $l$-th root of 1. The arguments in 5.1–5.2 are similar to those in [J, II, 4.8–4.18]. As in 4.2, $Y_\lambda = \mathrm{Ind}_{U^0}^{U^{\leq 0}}\lambda$ is the injective hull of the simple $U^{\leq 0}$-module $A_\lambda$. Since $U^-$ is a direct sum of weight spaces as $U^0$-module under the left adjoint action and $Y_\lambda \subseteq \mathrm{Hom}_{U^0}(U^{\leq 0}, A_\lambda)$, the following can be easily verified (cf. [J, II 4.8–4.9]).

(i) All weight spaces of $Y_\lambda$ are finite dimensional.

(ii) All weights of $Y_\lambda$ are in $\lambda + \mathbb{N}R^+$.

(iii) There is an injective resolution of $U^{\leq 0}$-modules $0 \to A \to I_0 \to I_1 \to \cdots$, such that all weight spaces of $I_j$ are finite dimensional and any weight $\mu$ of $I_j$ is in $\mathbb{N}R^+$ and $\mathrm{ht}(\mu) \geq j$. Here each $\mu \in \mathbb{N}R^+$ is written as $\mu = \sum_{i=1}^n n_i\alpha_i$ and $\mathrm{ht}(\mu) = \sum_{i=1}^n n_i$ is the *height* of $\mu$.

For a Hopf algebra $H$ with an admissible category $\mathcal{C}_H$ and any $H$-module $M$ in $\mathcal{C}_H$, recall from 1.2 that the cohomology groups of $H$ with coefficients in $M$ is $H^j(H, M) = \mathrm{Ext}_H^j(A, M)$. Also in analogy to the Weyl modules, we write $V(\lambda) = (\mathrm{Ind}_{U^{\leq 0}}^{U}(-w_0\lambda))^*$ for $\lambda \in X_+$. Here $V(\lambda)$ is a universal highest weight module with unique simple quotient $L(\lambda)$. As a consequence of (iii), one can get an injective resolution of each $U^{\leq 0}$-module $M$ by tensoring $M$ to the above resolution.

**Lemma.** *Let $I \subseteq \{1, 2, \ldots, n\}$ and $M$ be a rational $P_I$-module.*

(i) *If* $\dim M < \infty$, *then* $H^j(P_I, M)$ *is finite dimensional for all* $j$.

(ii) *If* $H^j(P_I, M) \neq 0$, *then there exists a weight* $\nu$ *of* $M$ *such that* $-\nu \in \mathbb{N}R^+$ *and* $\mathrm{ht}(-\nu) \geq j$.

*Proof:* Using 2.5(iv), we can assume $I = \emptyset$ and $P_I = U^{\leq 0}$. Now we can use the above mentioned resolution for $M$ and the lemma follows from $(M \otimes I_j)^{U^{\leq 0}} \subseteq (M \otimes I_j)^{U^0} = \sum_\nu (M_{-\nu} \otimes (I_j)_\nu)$. (cf. [J, II 4.10]) $\square$

The following theorem resembles several results for algebraic groups.

**Theorem.** *Assume that* $A$ *is a field of characteristic 0 and* $v$ *is a primitive* $l$*th root of 1. Let* $\lambda, \mu \in X_+$.

(i) $\mathrm{Ext}^j_U(V(\lambda), \mathrm{Ind}^U_{U^{\leq 0}}\mu) = \begin{cases} A & \text{if } \lambda = \mu \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$

(ii) *If* $\lambda \not> \mu$ *then* $\mathrm{Ext}^j_U(L(\lambda), L(\mu)) = \mathrm{Ext}^{j-1}_U(L(\lambda), (\mathrm{Ind}^U_{U^{\leq 0}}\mu)/L(\mu))$ *for all* $j > 0$.

(iii) *If* $\mathrm{Ext}^j_U(\mathrm{Ind}^U_{U^{\leq 0}}\lambda, \mathrm{Ind}^U_{U^{\leq 0}}\mu) \neq 0$ *or* $\mathrm{Ext}^j_U(L(\lambda), \mathrm{Ind}^U_{U^{\leq 0}}\mu) \neq 0$ *for some* $j > 0$, *then* $\lambda \neq \mu$ *and* $\mu$ *is strongly linked to* $\lambda$ (i.e., $\mu \uparrow \lambda$) [J, II 6.4].

(iv) *Since* $\chi(\lambda) = \mathrm{ch}\mathrm{Ind}^U_{U^{\leq 0}}\lambda$ *for* $\lambda \in X_+$ *is given by Weyl's character formula, for any finite dimensional* $U$*-module* $V$, *we have*

$$\mathrm{ch}V = \sum_{\lambda \in X_+} \left( \sum_i (-1)^i \dim \mathrm{Ext}^i_U(V, \mathrm{Ind}^U_{U^{\leq 0}}\lambda) \right) \chi(\lambda).$$

(v) *There are also analogous results for* $uU^0$*-modules as in* (i)–(iii).

*Proof:* By [APW1, 1.18] and a Grothendieck spectral sequence we have

$$\begin{aligned} & \mathrm{Ext}^i_U(V(\lambda), \mathrm{Ind}^U_{U^{\leq 0}}\mu) \cong H^i(U, \mathrm{Ind}^U_{U^{\leq 0}}(-w_0(\lambda)) \otimes \mathrm{Ind}^U_{U^{\leq 0}}\mu) \\ \cong \quad & H^i(U^{\leq 0}, (-w_0(\lambda)) \otimes \mathrm{Ind}^U_{U^{\leq 0}}\mu) \cong H^i(U^{\leq 0}, \mathrm{Ind}^U_{U^{\leq 0}}(-w_0(\lambda)) \otimes \mu). \end{aligned}$$

Here the last two isomorphisms follow from Proposition 2.5(i). Now (i) follows from the last isomorphism and (ii) of the lemma. Also (ii) of the lemma shows that $H^i(U^{\leq 0}, L(-w_0\lambda) \otimes \mu) = 0$ for all $i > 0$ if $\lambda \not> \mu$. Now using a similar argument as for the proof of (i), (ii) is a consequence the long exact sequence resulted from $L(\mu) \to \mathrm{Ind}^U_{U^{\leq 0}}\mu$. (iii) follows from (ii) and the strong linkage principle [APW1, AW]. To show (iv), one uses the additivity of the Euler characters. Finally for (v) see [J, II ch11] and [L2]. $\square$

*Remark.* For the corresponding results for algebraic groups, one is referred as follows. For (i) and (ii), see [J, II 4.13–4.14], and for (iii)–(vi), see [J, II 6.20–6.21].

**5.2.** A $U$-module is said to have a *good filtration* if it has a filtration such that each subquotient is isomorphic to an $\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda$ for some $\lambda \in X_+$.

**Proposition.** *Let $V$ be a rational $U$-module such that $\dim \mathrm{Hom}_U(V(\lambda), V) < \infty$ for all $\lambda \in X_+$. Then $V$ has a good filtration if and only if $\mathrm{Ext}_U^i(V(\lambda), V) = 0$ for all $\lambda \in X_+$.*

*Proof:* See [J, II 4.15–4.16]. □

Since the injective hull of a simple $U$-module $L(\lambda)$ is finite dimensional ($A$ is of characteristic 0), then it has a good filtration. Furthermore, direct summands of a module with a good filtration also have good filtrations since the functor $\mathrm{Ext}_U^i(V(\lambda), -)$ is additive. If $V$ and $V' \subset V$ both have good filtrations, so does $V/V'$.

**5.3.** For each $\lambda^0 \in X_l$, $Q_1(\lambda^0)$ is the injective hull of the irreducible $u$-module $L_1(\lambda^0)$, which is the restriction of the irreducible $U$-module $L(\lambda^0)$ to $u$ [Lu2]. If we denote by $Q(\lambda)$ the $U$-injective hull of $L(\lambda)$ for $\lambda \in X_+$, then $Q_1(\lambda^0) = Q(\lambda^0)$ for all $\lambda^0 \in X_l$ (see [L3, 6.3] and [APW2, 4.6]). All irreducible $uU^0$-modules (also $uU^{\leq 0}$-modules) are of the form $L_1(\lambda) = l\lambda^1 \otimes L(\lambda^0)$ and the $uU^0$-injective hull of $L_1(\lambda)$ is $Q_1(\lambda) = l\lambda^1 \otimes Q(\lambda^0)$ [L3, 4.8–4.9] [APW2, 0.9–0.10].

**Lemma.** *If $\lambda \in X_+$ and $w \in W$, then $R^{l(w)}\mathrm{Ind}_{uU^{\leq 0}}^{U} Q_1(lw \cdot \lambda^1 + \lambda^0) = \mathrm{Ind}_{uU^{\leq 0}}^{U} Q_1(\lambda) = Q(\lambda)$ is the $U$-injective hull of $L(\lambda)$ and, therefore, has a good filtration.*

*Proof:* Using [L3, 4.7], the generalized tensor identity [PW, 2.8.2], and the Borel-Weil-Bott theorem for $\mathrm{Ind}_{G^{\leq 0}}^{G}$, we have

$$\begin{aligned} &R^{l(w)}\mathrm{Ind}_{uU^{\leq 0}}^{U} Q_1(l(w \cdot \lambda^1) + \lambda^0) \cong R^{l(w)}\mathrm{Ind}_{uU^{\leq 0}}^{U}(l(w \cdot \lambda^1)) \otimes Q(\lambda^0) \\ &\cong \pi^*(R^{l(w)}\mathrm{Ind}_{G^{\leq 0}}^{G}(w \cdot \lambda^1)) \otimes Q(\lambda^0) \cong \pi^*(\mathrm{Ind}_{G^{\leq 0}}^{G}(w \cdot \lambda^1)) \otimes Q(\lambda^0) \\ &\cong \mathrm{Ind}_{uU^{\leq 0}}^{U}(l\lambda^1) \otimes Q(\lambda^0) \cong \mathrm{Ind}_{uU^{\leq 0}}^{U} Q_1(l\lambda^1 + \lambda^0). \end{aligned}$$

On the other hand, since $\mathrm{Ind}_{uU^{\leq 0}}^{U}(Q_1(\lambda))$ is a tensor product of a finite dimensional $U$-module with an injective module, it is injective also by using [APW1, Prop. 1.18]. The second isomorphism follows since the socle of $\mathrm{Ind}_{uU^{\leq 0}}^{U} Q_1(\lambda)$ is $L(\lambda)$. □

**5.4.** The functor $\mathrm{Ind}_{uU^{\leq 0}}^{U}$ defines a group homomorphism $\chi_1$ from the Grothendieck group of $\mathcal{C}_{uU^{\leq 0}}$ to the Grothendieck group of $\mathcal{C}_U$ by $\chi_1(M) = \sum_j (-1)^j \mathrm{ch} R^j \mathrm{Ind}_{uU^{\leq 0}}^{U}(M)$ for each $uU^{\leq 0}$-module $M$. For each $\lambda \in X$, denote by $Z_1(\lambda)$ the induced module $\mathrm{Ind}_{U^{\leq 0}}^{uU^{\leq 0}}(\lambda)$. Note that $\chi(\lambda) = \sum_j (-1)^j R^j \mathrm{Ind}_{U^{\leq 0}}^{U}\lambda$ is given by Weyl's character formula. Since the functor $\mathrm{Ind}_{U^{\leq 0}}^{uU^{\leq 0}}$ is exact [L3, 6.1] [APW2, 1.3], we have $R^i\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda = R^i\mathrm{Ind}_{uU^{\leq 0}}^{U}(Z_1(\lambda))$ for all $i$ and $\lambda$. Thus

$$\begin{aligned} \chi(\lambda) &= \chi_1(Z_1(\lambda)) = \sum_{\mu \in X} [Z_1(\lambda) : L_1(\mu)] \chi_1(L_1(\mu)) \\ &= \sum_{\mu \in X} [Z_1(\lambda) : L_1(\mu)] (\chi(\mu^1))^{(l)} \mathrm{ch} L(\mu^0). \end{aligned}$$

Here, $(l) : \mathbb{Z}[X] \to \mathbb{Z}[X]$ is the ring homomorphism induced by the group homomorphism $X \to X$ given by $\lambda \mapsto l\lambda$ and the last equality follows from the generalized tensor identity [PW, 2.8.2].

In [X], Xi proved that, for each $\mu \in X$, $Q_1(\mu)$ has a filtration with subquotients isomorphic to $Z_1(\lambda)$'s and the number of subquotients isomorphic to $Z_1(\lambda)$ is $[Z_1(\lambda) : L_1(\mu)]$. Now using the additivity of $\chi_1$, we have

$$\begin{aligned}\chi(\mu^1)^{(l)}\mathrm{ch}Q(\mu^0) &= \chi_1(Q_1(\mu)) = \sum_{\lambda\in X}[Z_1(\lambda) : L_1(\mu)]\chi(\lambda)\\ &= \sum_{\lambda\in X_+}\sum_{w\in W}(-1)^{l(w)}[Z_1(w\cdot\lambda), L_1(\mu)]\chi(\lambda)\end{aligned}$$

Note that the Grothendieck groups of $\mathcal{C}_{uU^0}$ and $\mathcal{C}_{uU^{\leq 0}}$ are the same. Using the above character formula, a similar argument as in [DS, 2.5–2.7] will show the following.

**Theorem.** *Let $\lambda = l\lambda^1 + \lambda^0, \mu = l\mu^1 + \mu^0 \in X$. Then*

(i) $[Z_1(\lambda) : L_1(\mu)] \neq 0$ *if and only if* $\mu^0 + l(w\cdot(\mu^1 - \lambda^1) + \lambda^1) \uparrow w\cdot\lambda^0 + l\lambda^1$ *for all* $w \in W$;

(ii) *if* $\lambda, \mu \in X_+$, *then* $[\mathrm{Ind}_{U^{\leq 0}}^{U}\lambda : L(\mu)] = \sum_{w\in W}(-1)^{l(w)}[Z_1(\lambda) : L_1(\mu^0 + lw\cdot\mu^1)]$. □

*Remarks.* (1) Here (i) resembles Ye's theorem [Y] for semisimple algebraic groups which is generalized by Doty and Sullivan [DS].

(2) Suppose $k$ is an algebraically closed field of characteristic $p$ and $G$ the simply connected semisimple algebraic group over $k$ with root system $R$. Let $T$ be a maximal torus of $G$ defining $R$ and $B$ the Borel subgroup corresponding to $R^-$. If $\lambda \in X$ is such that all composition factors of $\mathrm{Ind}_{B_1T}^{G_1T}\lambda$ have multiplicity 1, then a $G_1T$-composition factor of $\mathrm{Ind}_{B_1T}^{G_1T}\lambda$ with highest weight $\mu$ and $L_1(\mu)$ have the same dimension. Therefore the Lusztig conjecture [Lu3, 0.3] holds for all composition factors of $\mathrm{Ind}_{B_1T}^{G_1T}\lambda$.

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