# Induced Representations of Hopf Algebras: Applications to Quantum Groups at Roots of 1

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**Introduction** This paper serves three purposes. The first is to build up a connection between the representation theories of the quantum enveloping algebra $U$ of a semisimple Lie algebra and the quantum groups defined in [15]. The second one is to study the structure of the derived functors of the induction functor, defined in [1]. At each integral weight, these derived functors give rise to certain $U$-modules resembling the cohomology groups of line bundles over the flag varieties of the corresponding algebraic groups. Finally we would like to generalize the representation theory of the hyperalgebras of algebraic groups to Hopf algebras.

Since Lusztig first studied the quantum deformations of certain simple representations of semisimple Lie algebras in [10], he has established certain connections between the representations of $U$ and the representations of an algebraic group in prime characteristic in [11, 13, 14]. It has been a hope that some of the problems in modular representation theory of algebraic groups could be solved through the representations of quantum enveloping algebras. The representation theory of $U$ resembling that of algebraic groups is studied in [1] by Andersen, Polo and Wen (APW for short). Following another line of defining quantum groups, Parshall and Wang [15] studied the representation theory of quantum groups in terms of comodules of the quantum coordinate algebra. They have studied the linear quantum groups in detail and many of the classical theories have been extended in this special case. Out of curiosity, one would like to see how these two theories are related. In the appendix to [1], Polo uses the representation theory developed in that paper and shows that the coordinate algebra (defined by Lusztig in [14]) of $U$ is isomorphic to the quantum coordinate algebra of $SL_q(n)$ (with $v \mapsto q^{-1}$) if the Lie algebra has root system $\mathrm{A}_{n-1}$. In this paper we further establish a connection between the induction functors and the associated cohomology theories through the isomorphism of quantum coordinate algebras.

This paper is organized as follows. In Section 1, we give a general setting on the coordinate algebra, which is called the *Hopf dual*, of a certain representation category,

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which is called *admissible category*, of a Hopf algebra over a Dedekind domain. The idea goes back to Kostant [8], where the coordinate algebra of the simply connected semisimple group over $\mathbb{Z}$ is constructed from the category of rational representations of the Hopf algebra — the so called Kostant $\mathbb{Z}$-form. It is proved that the admissible category is naturally equivalent to the category of comodules of the Hopf dual. In Section 2, we start from a Hopf algebra $H$ and construct its distribution algebra $D(H)$, which is a Hopf algebra again if $H$ is *infinitesimally flat* as in the classical situation. Under certain completeness condition, the category of all $D(H)$-modules arising from $H$-comodules form an admissible category for $D(H)$ and its Hopf dual with respect to this admissible category is isomorphic to $H$ in a natural way.

We define, in Section 3, the induction functor from a Hopf subalgebra of $H$ with respect to the admissible category inherited from the admissible category for $H$. The induction functor satisfies the Frobenius reciprocity. The idea is based on Andersen's definition of induction for a quantum enveloping algebra. Donkin [5] define the induction in terms of comodules. Then we show that the two induction functors are naturally isomorphic via the equivalence between the admissible category and the comodule category of the Hopf dual. As an example, Polo's isomorphism for $A_{n-1}$ induces an isomorphism (see 5.2) of the induction functors defined in [1] and [15].

The normal Hopf subalgebras and their representations are studied in Section 4. This generalizes the theory of normal subgroup schemes and their associated hyperalgebras as in [18]. This is a dual version of what is called quantum normal subgroups of a quantum group in [15]. The analogous theorem about the commutativity of derived functors of the induction functor with the "Frobenius twists" and the analogous tensor product theorem for Hopf algebras are proved in 4.7 and 4.8.

Starting from Section 5 we concentrate on the quantum enveloping algebra $U$ and its subalgebras. We first show that the finite dimensional Hopf algebra $u$ constructed by Lusztig [13] is a normal Hopf subalgebra of $U$ at an $l$th primitive root of 1. By using a basis argument we prove that the induction from $u$ to some interesting subalgebras are exact. In Section 6, we use the results of Nanhua Xi and APW [1] to show that the Steinberg type module is injective when restricted to $u$. As a consequence, the induction functor $\mathrm{Ind}_u^U$ is exact and injective $u$-modules extend to $U$-modules. Using the established results we are able to get some results about the socle and radical structure of cohomology modules in 6.7. This resembles the author's earlier results on structure of cohomology groups of line bundles on flag varieties [9].

It should be mentioned that several conquences in Section 6 have been proved in [2] by APW, in which the inductions in the *infinitesimal level* and the $U$-structure of the injective hulls of simple $u$-modules are also studied.

# 1 Hopf duals and admissible categories

**1.1.** Let $A$ be a Dedekind domain. All tensor products will be taken over $A$ throughout this paper unless otherwise indicated. Let $H$ be a Hopf algebra over $A$ with comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to A$, antipode $\gamma : H \to H$, multiplication $m : H \otimes H \to H$, and unit $u : A \to H$. We further assume that $H$ is projective as $A$-module. Let $\mathcal{I}$ be a set of cofinite two-sided ideals of $H$. We call $\mathcal{I}$ admissible if it satisfies the following conditions:

(i) for each $I \in \mathcal{I}$, there is $J \in \mathcal{I}$ such that $J \subseteq \gamma^{-1}(I)$,

(ii) there is an $I \in \mathcal{I}$ such that $I \subseteq \ker(\varepsilon)$,

(iii) for any $I_1, I_2 \in \mathcal{I}$, there is $J \in \mathcal{I}$ such that $J \subseteq I_1 \cap I_2$,

(iv) for any $I_1, I_2 \in \mathcal{I}$, there is a $J \in \mathcal{I}$ such that $J \subseteq I_1 \wedge I_2$. Here $I_1 \wedge I_2$ is the kernel of the composition map $H \xrightarrow{\Delta} H \otimes H \to H/I_1 \otimes H/I_2$.

We define $H^0_{\mathcal{I}} = \{f \in \mathrm{Hom}_A(H, A) \mid f(I) = 0 \text{ for some } I \in \mathcal{I}\}$. It follows from (i) that $f \circ \gamma \in H^0_{\mathcal{I}}$ if $f \in H^0_{\mathcal{I}}$, from (ii) that $\varepsilon \in H^0_{\mathcal{I}}$, and from (iii) and (iv) that $H^0_{\mathcal{I}}$ is an $A$-algebra with the multiplication $m^0 : H^0_{\mathcal{I}} \otimes H^0_{\mathcal{I}} \to H^0_{\mathcal{I}}$ defined by $m^0(f \otimes g) = (f \overline{\otimes} g) \circ \Delta$ and identity $\varepsilon$.

**1.2.** Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two admissible sets of ideals in $H$. We say $\mathcal{I}_1 \leq \mathcal{I}_2$ if for each $I_1 \in \mathcal{I}_1$ there is $I_2 \in \mathcal{I}_2$ such that $I_1 \supseteq I_2$. We say $\mathcal{I}_1$ and $\mathcal{I}_2$ are equivalent, denoted by $\mathcal{I}_1 \sim \mathcal{I}_2$, if $\mathcal{I}_1 \leq \mathcal{I}_2$ and $\mathcal{I}_2 \leq \mathcal{I}_1$.

**Lemma.** (i) *$H^0_{\mathcal{I}_1} \subseteq H^0_{\mathcal{I}_2}$ if $\mathcal{I}_1 \leq \mathcal{I}_2$,*

(ii) *$H^0_{\mathcal{I}_1} = H^0_{\mathcal{I}_2}$ if $\mathcal{I}_1 \sim \mathcal{I}_2$.* ∎

Let $\mathcal{I}$ be an admissible set of ideals of $H$. We call a subset $\mathcal{I}_0$ of $\mathcal{I}$ a basis (of $\mathcal{I}$) if for each $I \in \mathcal{I}$ there is $J \in \mathcal{I}_0$ such that $J \subseteq I$. It follows from the definition that $\mathcal{I}_0$ also is admissible and $\mathcal{I}_0 \sim \mathcal{I}$.

We call an $H$-module $M$ locally finite if for each $x \in M$ there is an $H$-submodule $N$ of $M$ such that $x \in N$ and $N$ is finitely generated as $A$-module.

It follows from the cofiniteness of the ideals in $\mathcal{I}$ that $H^0_{\mathcal{I}}$ is a locally finite $H$-module via the *right regular* (or *left regular*) action $(hf)(x) = f(xh)$ (or $(hf)(x) = f(\gamma(h)x)$ ) for all $h, x \in H$ and $f \in H^0_{\mathcal{I}}$. (In this paper all $H$-modules are left $H$-modules).

**1.3.** We call an admissible set $\mathcal{I}$ flat if it has a basis $\mathcal{I}_0$ such that $H/I$ is projective over $A$ for each $I \in \mathcal{I}_0$.

**Lemma.** *Let $\mathcal{I}_1, \mathcal{I}_2$ be two flat admissible set of ideals of $H$. Then $H^0_{\mathcal{I}_1} = H^0_{\mathcal{I}_2}$ if and only if $\mathcal{I}_1 \sim \mathcal{I}_2$.*

*Proof:* We may assume that $H/I$ is $A$-projective for each $I \in \mathcal{I}_1$ or $I \in \mathcal{I}_2$. Let $I \in \mathcal{I}_1$. Then $N = \mathrm{Hom}_A(H/I, A) \subseteq H^0_{\mathcal{I}_1}$ as an $H$-submodule (with right regular action). Projectivity of $H/I$ over $A$ implies $\mathrm{Ann}_H(N) = I$. Let $\{f_1, \cdots, f_s\}$ be a finite generating set of $N$ over $A$. Since $f_i \in H^0_{\mathcal{I}_1} = H^0_{\mathcal{I}_2}$, there exists $I_i \in \mathcal{I}_2$ such that $f_i(I_i) = 0$ for each $i$. Thus there exists $J \in \mathcal{I}_2$ such that $J \subseteq \cap_{i=1}^s I_i \subseteq I$ and $\mathcal{I}_1 \leq \mathcal{I}_2$. Similarly, we have $\mathcal{I}_2 \leq \mathcal{I}_1$. ∎

**1.4. Proposition.** *If $\mathcal{I}$ is a flat admissible set of ideals of $H$, then $H^0_{\mathcal{I}}$ is a Hopf algebra.*

*Proof:* Use a similar argument as in [3]. ∎

**1.5.** For a fixed flat admissible set $\mathcal{I}$ of ideals of $H$, let $\mathcal{C}_{\mathcal{I}}$ be the full subcategory of those $H$-modules such that for each $x \in M$ there is an $H$-submodule $N$ of $M$ with $x \in N$ and $\mathrm{Ann}_H(N) \supseteq I$ for some $I \in \mathcal{I}$. We simply write $H^0$ (or $\mathcal{C}$) for $H^0_{\mathcal{I}}$ (or $\mathcal{C}_{\mathcal{I}}$) if $\mathcal{I}$ is understood.

An $H$-module $M$ is called trivial if $hm = \varepsilon(h)m$ for all $m \in M$ and $h \in H$. The tensor product $M \otimes N$ of two $H$-modules $M$ and $N$ is also an $H$-module with the $H$-action defined by $h(m \otimes n) = \sum h_{(1)}m \otimes h_{(2)}n$. Here $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ (using the $\sum$-notation in Sweedler's book [16]). The dual module $M^*$ of an $H$-module $M$ is an $H$-module with the $H$-action defined by $(hf)(m) = f(\gamma(h)m)$ for all $h \in H$, $f \in M^*$ and $m \in M$. The following properties can be easily verified.

(i) Each module in $\mathcal{C}$ is locally finite in $\mathcal{C}$, i.e., for each module $M$ in $\mathcal{C}$ and each $m \in M$, there is an $A$-finite $H$-submodule $N$ of $M$ in $\mathcal{C}$ such that $x \in N$;

(ii) $\mathcal{C}$ is closed under direct limit;

(iii) $\mathcal{C}$ is closed under tensor product;

(iv) $M^*$ is also in $\mathcal{C}$ if $M \in \mathrm{Ob}(\mathcal{C})$ is finitely generated and projective over $A$;

(v) The trivial module $A$ is in $\mathcal{C}$;

(vi) For each exact sequence $0 \to K \to F \to M \to 0$ of $H$-modules, $F$ is in $\mathcal{C}$ if and only if both $K$ and $M$ are in $\mathcal{C}$;

(vii) Each $A$-finite module $M$ in $\mathcal{C}$ has a projective $A$-finite cover $F$ in $\mathcal{C}$, i.e., there is a surjective $H$-map $F \to M$ in $\mathcal{C}$ such that $F$ is finitely generated and projective over $A$.

(vii) can be verified as follows; let $\{m_i\}_{i=1}^s$ be a finite generating set of $M$ as $A$-module. There is $I_i \in \mathcal{I}$ such that $H/I_i$ is $A$-projective and $I_i \subseteq \mathrm{Ann}_H(Hm_i)$ for each $i$. The surjective maps $H/I_i \to Hm_i$ of $H$-modules $i =, \ldots, s$ induce a surjective $H$-map $\oplus H/I_i \to M$.

If $M$ is an $A$-module, we define

$$H(M) = \{f \in \mathrm{Hom}_A(H, M) \mid f(I) = 0 \text{ for some } I \in \mathcal{I}\}.$$

In particular, $H^0_{\mathcal{I}} = H(A)$. It can be verified that $H(M)$ is an $H$-module in $\mathcal{C}$ with either the right regular action $(hf)(x) = f(xh)$ or the left regular action $(hf)(x) = f(\gamma(h)x)$.

**Proposition.** *$H(-)$ defines an exact functor from the category of $A$-modules to $\mathcal{C}$.*

*Proof:* Since $\mathcal{C}$ is a full subcategory of the category $H$-Mod, we denote by $C$: $H$-Mod $\to \mathcal{C}$ the right adjoint functor of the embedding functor. Then $C(M) = \{m \in M \mid \mathrm{Ann}_H(Hm) \supseteq I \text{ for some } I \in \mathcal{I}\}$ is the maximal submodule of $M$ in $\mathcal{C}$ for each $H$-module $M$. Therefore we have $H(M) = C(\mathrm{Hom}_A(H, M))$ for each $A$-module $M$. Hence the left exactness of $H$ follows. The exactness can be easily verified by using the fact that both $H$ and $H/I$ are projective over $A$ for $I \in \mathcal{I}$. ∎

**1.6. Proposition.** *Let $M$ be an $A$-module. The natural map $\phi_M : M \otimes H^0_{\mathcal{I}} \to H(M)$, defined by $\phi_M(m \otimes f)(x) = f(x)m$, is an isomorphism of $H$-modules when $H^0_{\mathcal{I}}$ and $H(M)$ are both considered as $H$-modules through the right (or left) regular actions and $M$ is considered as a trivial $H$-module.*

*Proof:* It is clear that the map $\phi_M$ is an $H$-map when $H^0_{\mathcal{I}}$ and $H(M)$ are both considered as $H$-modules through right (or left) multiplications. one can verify that $\phi : ? \otimes H^0_{\mathcal{I}} \to H(?)$ is a natural transformation of functors. To see that $\phi$ is an isomorphism we need to show that $\phi_M$ is bijective for each $M$. This follows from the arguments of [1] (1.30). ∎

**1.7.** A full subcategory $\mathcal{C}$ of $H$-Mod is called admissible if it satisfies (i)–(vii) in 1.5. For an admissible category $\mathcal{C}$, we define

$$\mathcal{I}_{\mathcal{C}} = \{\mathrm{Ann}_H N \mid N \in \mathrm{Ob}(\mathcal{C}) \text{ is finitely generated and projective over } A\}.$$

**Proposition.** (i) *$\mathcal{I}_{\mathcal{C}}$ is a flat admissible set of ideals of $H$.*

(ii) *$\mathcal{C}_{\mathcal{I}_{\mathcal{C}}} = \mathcal{C}$.*

(iii) *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two flat admissible set of ideals. Then $\mathcal{I}_1 \sim \mathcal{I}_2$ if and only if $\mathcal{C}_{\mathcal{I}_1} = \mathcal{C}_{\mathcal{I}_2}$.*

*Proof:* It is easily verified that $\mathcal{I}_{\mathcal{C}}$ is admissible (cf.1.1 and 1.5). We only show that $\mathcal{I}_{\mathcal{C}}$ is flat. If $I = \mathrm{Ann}_H(N)$ with $N$ finitely generated and projective over $A$, then $H/I \subseteq \mathrm{Hom}_A(N, N)$ shows that $H/I$ is finitely generated and projective over $A$ as well.

In (ii), $\mathcal{C} \subseteq \mathcal{C}_{\mathcal{I}_\mathcal{C}}$ follows from the facts that $N$ in $\mathcal{C}$, with $N$ being finitely generated and projective over $A$, is in $\mathcal{C}_{\mathcal{I}_\mathcal{C}}$ and each module in $\mathcal{C}$ is a union of submodules isomorphic to the quotients of certain $N$. To show $\mathcal{C} \supseteq \mathcal{C}_{\mathcal{I}_\mathcal{C}}$, it suffices to show that $H/I$ is in $\mathcal{C}$ for each $I \in \mathcal{I}_\mathcal{C}$. Since $I = \mathrm{Ann}_H(N)$ with $N$ in $\mathcal{C}$ and finitely generated and projective over $A$, then $\mathrm{Hom}_A(N,N) \cong N \otimes N^*$ as $H$-modules with $H$ acting on $\mathrm{Hom}_A(N,N)$ by $(hf)(n) = h(f(n))$ and on $N^*$ trivially. Furthermore the $A$-linear embedding $H/I \to \mathrm{Hom}_A(N,N)$ is actually an $H$-map. Therefore, $H/I$ is in $\mathcal{C}$.

It is clear that $\mathcal{I}_1 \sim \mathcal{I}_2$ implies $\mathcal{C}_{\mathcal{I}_1} = \mathcal{C}_{\mathcal{I}_2}$ by the definition in 1.5. We show $\mathcal{I}_1 \leq \mathcal{I}_2$ if $\mathcal{C}_{\mathcal{I}_1} = \mathcal{C}_{\mathcal{I}_2}$. Let $I \in \mathcal{I}_1$, then $H/I$ is in $\mathcal{C}_{\mathcal{I}_1} = \mathcal{C}_{\mathcal{I}_2}$. Let $\{x_i\}$ be a finite set of generators of $H/I$ over $A$. There is $J_i \in \mathcal{I}_2$ for each $i$ such that $J_i \subseteq \mathrm{Ann}_H(Hx_i)$. Therefore there is $J \in \mathcal{I}_2$ such that $J \subseteq \cap J_i \subseteq \cap \mathrm{Ann}_H(Hx_i) \subseteq I$. ∎

*Remark.* By (iii) and 1.2(ii), the Hopf algebra $H^0_\mathcal{I}$ depends only on the admissible category $\mathcal{C}$.

**1.8.** For each $H$-module $M$, there is a natural map $\rho_M : M \to \mathrm{Hom}_A(H,M)$ defined by $\rho_M(m)(h) = hm$. Since $H(M) \subseteq \mathrm{Hom}_A(H,M)$, we can identify $M \otimes H^0_{\mathcal{I}_\mathcal{C}}$ with $H(M)$ via the isomorphism $\phi_M$ in 1.6. Using the same argument as in ([16] 2.1.1–2.1.2) one can show that $M$ is an $H^0_{\mathcal{I}_\mathcal{C}}$-comodule if and only if $\rho_M(M) \subseteq H(M)$. If $M$ is in $\mathcal{C}$, we have $\rho_M(M) \subseteq H(M)$ and, therefore, $M$ is an $H^0_{\mathcal{I}_\mathcal{C}}$-comodule.

**Theorem.** *The above correspondence gives a category equivalence between the admissible category $\mathcal{C}$ and the category of all $H^0_{\mathcal{I}_\mathcal{C}}$-comodules.*

*Proof:* If $\rho_M : M \to M \otimes H^0_{\mathcal{I}_\mathcal{C}}$ is an $H^0_{\mathcal{I}_\mathcal{C}}$-comodule structure, we define $m_M : H \otimes M \to M$ as the composition of the following maps:

$$H \otimes M \overset{id\otimes\rho_M}{\to} H \otimes M \otimes H^0_{\mathcal{I}_\mathcal{C}} \overset{T_{1,2}\otimes id}{\to} M \otimes H \otimes H^0_{\mathcal{I}_\mathcal{C}} \overset{id\bar{\otimes}v}{\to} M.$$

Here $T_{1,2}$ is the twist map and $v(h \otimes f) = f(h)$ for all $h \in H$ and $f \in H^0_{\mathcal{I}_\mathcal{C}}$. Once again one can argue as in ([16] 2.1.1–2.1.2) and show that $m_M$ does give an $H$-module structure on $M$. To see $M$ is in $\mathcal{C}$ as $H$-module, let $m \in M$ and $\rho_M(m) = \sum m_i \otimes f_i$ (a finite sum). For any $h \in H$, $hm = \sum f_i(h)m_i = \rho_M(m)(h)$ (regarding $M \otimes H^0_{\mathcal{I}_\mathcal{C}}$ as a subset of $\mathrm{Hom}_A(H,M)$). Since $f_i \in H^0_{\mathcal{I}_\mathcal{C}}$, there are $I_i \in \mathcal{I}_\mathcal{C}$ such that $f_i(I_i) = 0$. There exists $I \in \mathcal{I}_\mathcal{C}$ such that $I \subseteq \cap I_i$. Thus $I \subseteq \mathrm{Ann}_H(Hm)$. Furthermore, one can check that the $H^0_{\mathcal{I}_\mathcal{C}}$-comodule structure on $M$ determined by the $H$-module structure on $M$ is the same as the original comodule structure. $\mathrm{Hom}_H(M,N) = \mathrm{Hom}_{H^0_{\mathcal{I}_\mathcal{C}}}(M,N)$ and the naturality can be checked routinely. ∎

*Remark.* The right regular action of $H$ on $H^0_{\mathcal{I}_\mathcal{C}}$ corresponds to the comodule $\Delta^0 : H^0_{\mathcal{I}_\mathcal{C}} \to H^0_{\mathcal{I}_\mathcal{C}} \otimes H^0_{\mathcal{I}_\mathcal{C}}$ and the left regular action of $H$ on $H^0_{\mathcal{I}_\mathcal{C}}$ corresponds to the $H^0_{\mathcal{I}_\mathcal{C}}$-comodule

$$H^0_{\mathcal{I}_\mathcal{C}} \overset{\Delta^0}{\to} H^0_{\mathcal{I}_\mathcal{C}} \otimes H^0_{\mathcal{I}_\mathcal{C}} \overset{T(\gamma\otimes 1)}{\to} H^0_{\mathcal{I}_\mathcal{C}} \otimes H^0_{\mathcal{I}_\mathcal{C}}.$$

The properties of the category of all comodules for a coalgebra over a field has been studied by Green in [6]. One can modify the arguments to get the following:

**Corollary.** (i) *Each admissible category has enough injectives.*

(ii) *Each injective module in $\mathcal{C}$ is a direct summand of $Q \otimes H^0_{\mathcal{I}_\mathcal{C}}$ for some injective $A$-module $Q$.* ∎

**1.9.** Let $\phi : G \to H$ be a homomorphism of Hopf algebras. Suppose $\mathcal{I}_H$ and $\mathcal{I}_G$ are flat admissible sets of ideals in $H$ and $G$ respectively. We say $\phi$ is *compatible* if for each $I \in \mathcal{I}_H$ there is $I' \in \mathcal{I}_G$ such that $I' \subseteq \phi^{-1}(I)$. For example, if $\mathcal{I}_H$ is a given flat admissible set of ideals in $H$, one can define $\mathcal{I}_G = \{\phi^{-1}(I) \mid I \in \mathcal{I}_H\}$. Then $\mathcal{I}_G$ is a flat admissible set of ideals in $G$. Such a set $\mathcal{I}_G$ is called the *inherited* set from $\mathcal{I}_H$ through the map $\phi$. We will also say the admissible category corresponding to $\mathcal{I}_G$ is inherited from the admissible category corresponding to $\mathcal{I}_H$. In this case, $\phi$ is clearly compatible.

**Proposition.** *If $\phi$ is compatible, then the dual map $\phi^* : H^* \to G^*$ induces a homomorphism of Hopf algebras $\phi^0 : H^0_{\mathcal{I}_H} \to G^0_{\mathcal{I}_G}$.*

*Proof:* We first show that $\phi^*(H^0_{\mathcal{I}_H}) \subseteq G^0_{\mathcal{I}_G}$. In fact, if $f \in H^0_{\mathcal{I}_H}$, there is $I \in \mathcal{I}_H$ such that $f(I) = 0$. There exists $I' \in \mathcal{I}_G$ such that $\phi(I') \subseteq I$ by the compatibility of $\phi$. Then $\phi^*(f)(I') = f(\phi(I')) = 0$ and $\phi^*(f) \in G^0_{\mathcal{I}_G}$. Other properties can be checked from the definitions. For example,

$$\begin{aligned} \Delta^0_G(\phi^0(f))(x \otimes y) &= f(\phi(xy)) = f(\phi(x)\phi(y)) \\ &= \Delta^0_H(f)(\phi(x) \otimes \phi(y)) = (\phi^0 \otimes \phi^0)\Delta^0_H(x \otimes y). \end{aligned}$$ ∎

**Corollary.** *Suppose $\mathcal{I}_G$ is inherited from $\mathcal{I}_H$ through $\phi$ and $A$ is a field. If $\phi$ is injective, then $\phi^0$ is surjective.*

*Proof:* If $f \in G^0_{\mathcal{I}_G}$, then $f(\phi^{-1}(I)) = 0$ for some $I \in \mathcal{I}_H$ and $f$ factors through $G/\phi^{-1}(I) \subseteq H/I$. Since $A$ is a field and any linear map on a subspace extends to a linear map of the whole space, we can extend $f$ from $G/\phi^{-1}(I)$ to $H/I$. Thus there is an $A$-linear map on $H$ which extends $f$ and vanishes at $I$. This shows the surjectivity of $\phi^0$. ∎

*Remark.* If the flat admissible set $\mathcal{I}_G$ in $G$ is given first and $\phi$ is surjective, then the set $\mathcal{I}_H = \{\phi(I) \mid I \in \mathcal{I}_G\}$ is an admissible set of ideals in $H$. In general it is not flat if $A$ is not a field. However if $A$ is a field, all admissible set of ideals are flat. In this case $\phi$ is compatible and $\phi^0$ is an injective homomorphism of Hopf algebras. So we can identify $H^0_{\mathcal{I}_H}$ as a Hopf subalgebra of $G^0_{\mathcal{I}_G}$.

**1.10.** Let $\phi : G \to H$ be a compatible homomorphism of Hopf algebras with respect to the given admissible sets $\mathcal{I}_H$ and $\mathcal{I}_G$ of ideals in $H$ and $G$ respectively. For each $H^0_{\mathcal{I}_H}$-comodule $M$, $M \to M \otimes H^0_{\mathcal{I}_H} \overset{1\otimes\phi^0}{\to} M \otimes G^0_{\mathcal{I}_G}$ defines a $G^0_{\mathcal{I}_G}$-comodule structure on $M$, denoted by $\phi^0(M)$. On the other hand, for each $H$-module $M$, $G\otimes M \overset{\phi\otimes 1}{\to} H\otimes M \to M$ defines a $G$-module structure on $M$, denoted by $\phi^*(M)$. Let $\tau_H$ denote the category equivalence from $\mathcal{C}_H$ to the category of $H^0_{\mathcal{I}_H}$-comodules as in 1.8. Similarly, $\tau_G$ denotes the category equivalence from $\mathcal{C}_G$ to the category of $G^0_{\mathcal{I}_G}$-comodules.

**Proposition.** *Suppose $\phi$ is compatible. If $M$ is an $H$-module in $\mathcal{C}_{\mathcal{I}_H}$ and $N$ is an $H^0_{\mathcal{I}_H}$-comodule, then*

(i) *$\phi^*(M)$ is in $\mathcal{C}_{\mathcal{I}_G}$,*

(ii) *the identity map $\tau_G\phi^*(M) \to \phi^0\tau_H(M)$ is an isomorphism of $G^0_{\mathcal{I}_G}$-comodules,*

(iii) *the identity map $\phi^*\tau_H^{-1}(N) \to \tau_G^{-1}\phi^0(N)$ is an isomorphism of $G$-modules in $\mathcal{C}_{\mathcal{I}_G}$.*

*Proof:* Since $M$ is in $\mathcal{C}_{\mathcal{I}_H}$, there exists $I \in \mathcal{I}_H$ for each $m \in M$ such that $I \subseteq \mathrm{Ann}_H(Hm)$. Since $Gm = \phi(G)m \subseteq Hm$, one has $\mathrm{Ann}_G(Gm) \supseteq \phi^{-1}(\mathrm{Ann}_H(Hm) \supseteq \phi^{-1}(I) \supseteq I'$ for some $I' \in \mathcal{I}_G$ by the compatibility of $\phi$. This shows that $\phi^*(M)$ is in $\mathcal{C}_{\mathcal{I}_G}$. To show (ii), let $\rho_M : M \to M \otimes H^0_{\mathcal{I}_H}$ be the $H^0_{\mathcal{I}_H}$-comodule structure on $\tau_H(M)$ induced from the $H$-module structure on $M$ as in 1.8. Then $\rho'_M = (1 \otimes \phi^0)\rho_M$ is the $G^0_{\mathcal{I}_G}$-comodule structure on $\phi^0\tau_H(M)$. Denote by $\rho''_M : M \to M \otimes G^0_{\mathcal{I}_G}$ the comodule structure on $\tau_G\phi^*(M)$. Consider the diagram:

$$\begin{array}{ccccc} M & \overset{\rho''_M}{\to} & M \otimes G^0_{\mathcal{I}_G} & \subseteq & \mathrm{Hom}_A(G, M) \\ \| & & \uparrow (1 \otimes \phi^0) & & \uparrow \phi^* \\ M & \overset{\rho_M}{\to} & M \otimes H^0_{\mathcal{I}_H} & \subseteq & \mathrm{Hom}_A(H, M) \end{array} .$$

For each $m \in M$, $f \in H^0_{\mathcal{I}_H}$ and $g \in G$

$$\rho''_M(m)(g) = gm = \phi(g)m = \rho_M(m)(\phi(g)) = (\phi^*(\rho_M(m)))(g).$$

$$\phi^*(m \otimes f)(g) = (m \otimes f)(\phi(g)) = f(\phi(g))m = \phi^0(f)(g)m = (1 \otimes \phi^0)(m \otimes f)(g).$$

Thus the above diagram commutes and $\rho'_M = \rho''_M$. This proves (ii). (iii) follows from (ii) and 1.8. ∎

**1.11.** If $G$ and $H$ are two Hopf algebras and $\mathcal{I}_G, \mathcal{I}_H$ are flat admissible sets of ideals in $G$ and $H$ respectively, then $\mathcal{I}_{G\otimes H} = \{I \otimes H + G \otimes J \mid I \in \mathcal{I}_G, J \in \mathcal{I}_H\}$ defines a flat admissible set of ideals in $G \otimes H$ and $G^0_{\mathcal{I}_G} \otimes H^0_{\mathcal{I}_H} = (G \otimes H)^0_{\mathcal{I}_{G\otimes H}}$ (see the proof Lemma 3 in [3]).

If $A \to B$ is a homomorphism of rings, $H_B = H \otimes B$ is a Hopf algebra over $B$ and $\mathcal{I}_B = \{I \otimes B \mid I \in \mathcal{I}\}$ is a flat admissible set of ideals in $H_B$ if $\mathcal{I}$ is so. Furthermore, $(H \otimes B)^0_{\mathcal{I}_B} \cong H^0_{\mathcal{I}} \otimes B$ as Hopf algebras.

# 2 Distribution algebras of Hopf algebras

**2.1.** As in Section 1, we assume once again that $A$ is a Dedekind domain and $H$ an $A$-projective Hopf algebra over $A$. Let $I = \mathrm{Ker}\varepsilon$ be the augmentation ideal of $H$. We define $D(H) = \{f \in H^* \mid f(I^n) = 0 \text{ for some } n \geq 0\}$. Since $H^*$ is already an algebra, $D(H)$ is a subalgebra of $H^*$. In fact the set $\mathcal{I} = \{I^n \mid n \geq 0\}$ is an admissible set of ideals in $H$ and $D(H) = H^0_{\mathcal{I}}$. This can be easily checked using $\Delta(I^n) \subseteq \sum_{s=0}^{n}(I^s \otimes I^{n-s})$. We call $D(H)$ the distribution algebra of $H$.

We call $H$ *infinitesimally flat* if $H/I^n$ is finitely generated and projective over $A$ for each $n$, i.e., $\mathcal{I}$ is a flat admissible set of ideals in $H$. In this case, $D(H)$ is a Hopf algebra (cf. 1.4).

**Proposition.** *If $H$ is infinitesimally flat, then $D(H)$ is $A$-projective.*

*Proof:* By the definition we have $D(H) = \bigcup_{n\geq 0}(H/I^n)^*$. The projectivity of $H/I^n$ for all $n$ and the exact sequence

$$0 \to I^n/I^{n+1} \to H/I^{n+1} \to H/I^n \to 0$$

show that $I^n/I^{n+1}$ is finitely generated and projective over $A$. Thus $(I^n/I^{n+1})^*$ is $A$-projective and the proposition follows. ∎

*Remark.* The above proof is an imitation of the proof of APW [1](1.31). For a given flat admissible set $\mathcal{I}$ ideals, if $\mathcal{I}$ has a basis $\{I_i\}_{i\in\mathbf{N}}$ such that $I_i \supseteq I_{i+1}$, the proposition remains true for the projectivity of $H^0_{\mathcal{I}}$. For such $\mathcal{I}$, all the results in the rest of this section can be stated by replacing $D(H)$ for $H^0_{\mathcal{I}}$ under the assumption $\cap_{I\in\mathcal{I}} I = \{0\}$.

**2.2. Proposition.** *Let $H$ be infinitesimally flat and $\cap_{n\geq 0} I^n = \{0\}$. For an $A$-module $M$ and an $A$-linear map $\omega : M \to M \otimes H$, we define an $A$-linear map $\psi_\omega$ as the composition of maps*

$$D(H) \otimes M \overset{id\otimes\omega}{\to} D(H) \otimes M \otimes H \overset{T_{1,2}\otimes 1}{\to} M \otimes D(H) \otimes H \to M,$$

*i.e., $\psi_\omega(f \otimes m) = \sum f(h_i)m_i$ if $\omega(m) = \sum m_i \otimes h_i$. Then $\omega$ defines an $H$-comodule structure on $M$ if and only if $\psi_\omega$ defines a $D(H)$-module structure on $M$.*

*Proof:* We can essentially use the same argument as in Sweedler's book ([16] 2.1.1) together with the following lemma. ∎

**Lemma.** *Under the assumption of the proposition, $D(H)$ is dense in $H^*$, i.e., if $x \in H$ such that $f(x) = 0$ for all $f \in D(H)$, then $x = 0$. Furthermore, $D(H) \otimes D(H)$ is dense in $(H \otimes H)^*$.*

*Proof:* The first assertion follows from $\cap_{n>0} I^n = \{0\}$ and the projectivity of $H/I^n$. The second assertion follows from $D(H) \otimes D(H) = \oplus_{n,m}(I^n/I^{n+1})^* \otimes (I^n/I^{n+1})^*$. The latter is dense in $(H \otimes H)^*$. ∎

**2.3.** We assume the condition in Proposition 2.2 on $H$ in the rest of this section. Then $H = \oplus_{n\geq 0}(I^n/I^{n+1})$ and $D(H) = \oplus_{n\geq 0}(I^n/I^{n+1})^*$ as $A$-modules. For an $A$-module $M$, one has $M \otimes (I^n/I^{n+1}) \cong \mathrm{Hom}_A((I^n/I^{n+1})^*, M)$ and

$$\oplus_{n\geq 0}\mathrm{Hom}_A((I^n/I^{n+1})^*, M) \subseteq \mathrm{Hom}_A(\oplus_{n\geq 0}(I^n/I^{n+1})^*, M) = \mathrm{Hom}_A(D(H), M).$$

Thus the canonical map $\tau_M : M \otimes H \to \mathrm{Hom}_A(D(H), M)$, defined by $(m \otimes h)(f) = f(h)m$ for all $h \in H$, $m \in M$ and $f \in D(H)$, is injective. So we can see $M \otimes H$ as a subset of $\mathrm{Hom}_A(D(H), M)$. If $M$ is a $D(H)$-module, we have the map, as in 1.8,

$$\rho : M \to \mathrm{Hom}_A(D(H), M) \text{ by } \rho(m)(x) = xm, \forall x \in D(H), \forall m \in M.$$

Using the terminology in ([16] 2.1.1-2.1.2), a $D(H)$-module $M$ is called *rational* if $\rho(M) \subseteq M \otimes H$. Following a similar argument as in ([16] 2.1.2) we have

**Lemma.** *$M$ is a rational $D(H)$-module if and only if the $D(H)$-module structure on $M$ is the one arising from an $H$-comodule structure as in Proposition 2.2.* ∎

Using this lemma, the following properties of the category of all rational $D(H)$-modules can be easily verified.

**Proposition.** *Let $L, M, F$ be $D(H)$-modules with $M, F$ rational. Then*

(i) *any $D(H)$-submodule $N$ of $M$ is rational,*

(ii) *every cyclic $D(H)$-submodule of $M$ is $A$ finite,*

(iii) *any $D(H)$-quotient module of $M$ is rational,*

(iv) *$L$ has a unique maximal rational submodule,*

(v) *$f : M \to F$ is a $D(H)$-map if and only if it is an $H$-comodule map,*

(vi) *$M \otimes F$ is rational,*

(vii) *$M^*$ is rational if $M$ is finitely generated and projective over $A$.*

(viii) *The category of all rational $D(H)$-modules are closed under direct limit.* ∎

**2.4.** In this subsection we show that all rational $D(H)$-modules form an admissible category for $D(H)$ (cf. 1.6). If $A$ is a field, this follows from the proposition in 2.3. But for $A$ being a Dedekind domain, we need to verify (i) and (vii) in 1.5.

**Proposition.** (i) *Every $H$-comodule if locally finite,*

(ii) *For every $A$-finite $H$-comodule $M$, there is a surjective $H$-comodule map $F \to M$ such that $F$ is finitely generated and projective over $A$.*

*Proof:* (i) can be proved for all $A$-projective $H$-modules by studying its coefficient space as in [6]. In general, for each $H$-comodule $M$, the comodule structure map $\rho_M : M \to M \otimes H$ is an injective $H$-comodule map regarding $M$ as a trivial $H$-comodule in the tensor product $M \otimes H$. If $P$ is the projective cover of $M$ as $A$-module, the induced surjective map $\phi : P \otimes H \to M \otimes H$ is a homomorphism of $H$-comodules. For each finite subset $S$ of $M$, choose a finite subset $T$ of $P \otimes H$ such that $\phi(T) = \rho_M(S)$. Thus the $H$-subcomodule $< T >$ of $P \otimes H$ generated by $T$ is finitely generated over $A$ and, therefore, $\phi(< T >) =< S >$ is also finitely generated over $A$.

To show (ii), let be a generating set of $M$ over $A$. $< T >$ is $A$-projective since $P \otimes H$ is projective over $A$. ∎

**2.5.** It follows from 2.4 that the category $\mathcal{C}$ of all rational $D(H)$-modules is an admissible category for $D(H)$. Following 1.7, we denote by $\hat{H}$ the Hopf algebra $D(H)^0_{\mathcal{I}_\mathcal{C}}$. By 1.8 and 2.3, there is a category equivalence between the category of all $H$-comodules and the category of all $\hat{H}$-comodules. Since $D(H)$ is dense in $H^*$ the natural map $\phi : H \to D(H)^*$, defined by $\phi(h)(f) = f(h)$ for all $h \in H$ and $f \in D(H)$, is injective

**Theorem.** *$\phi$ defines an isomorphism $H \to \hat{H}$ of Hopf algebras*

*Proof:* For each $h \in H$, let $M$ be an $H$-subcomodule of $H$ (with the right regular action) with $h \in M$ such that $M$ is finitely generated and projective over $A$. Then $M$ is a rational $D(H)$-module with the module structure defined as in 2.2. Let $J = \mathrm{Ann}_{D(H)}(M)$. Then $J \in \mathcal{I}_\mathcal{C}$. If $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and $f \in J$, then $0 = fh = \sum f(h_{(2)})h_{(1)}$ and

$$\phi(h)(f) = f(h) = f(\sum \varepsilon(h_{(1)})h_{(2)}) = \sum \varepsilon(h_{(1)})f(h_{(2)}) = \varepsilon(\sum f(h_{(2)})h_{(1)}) = 0.$$

Hence $\phi(h)(J) = 0$ and $\phi(h) \in \hat{H}$. This shows $\phi(H) \subseteq \hat{H}$.

Following the embedding $\hat{H} \otimes \hat{H} \subseteq (D(H) \otimes D(H))^*$, we have

$$\begin{aligned} \Delta_{\hat{H}}(\phi(h))(f \otimes g) &= \phi(h)(fg) = (fg)(h) = \sum f(h_{(1)})g(h_{(2)}) \\ &= \sum \phi(h_{(1)})(f)\phi(h_{(2)})(g) = \sum(\phi(h_{(1)}) \otimes \phi(h_{(2)}))(f \otimes g). \end{aligned}$$

Hence $\Delta_{\hat{H}}\phi = (\phi \otimes \phi)\Delta_H$. The counitary condition follows from $1_{D(H)} = \varepsilon_H$. Thus $\phi$ is a map of Hopf algebras.

Since $\hat{H}$ is a rational $D(H)$-comodule, there is an $H$-comodule structure $\Delta'_{\hat{H}} : \hat{H} \to \hat{H} \otimes H$ such that $f\hat{h} = \sum \hat{h}_{(2)}(f)\hat{h}_{(1)}$ for all $\hat{h} \in \hat{H}$ and $f \in D(H)$. The natural

embedding $\mathrm{Hom}_A(D(H), H) \to \mathrm{Hom}_A(D(H), \hat{H})$ induced by $\phi$ yields the embedding $\hat{H} \otimes H \stackrel{1\otimes\phi}{\to} \hat{H} \otimes \hat{H}$. Thus we have a commutative diagram

$$\begin{array}{ccc} \hat{H} & \stackrel{\Delta'_{\hat{H}}}{\longrightarrow} & \hat{H} \otimes H \\ id \downarrow & & \downarrow 1 \otimes \phi \\ \hat{H} & \stackrel{\Delta_{\hat{H}}}{\longrightarrow} & \hat{H} \otimes \hat{H} \end{array} .$$

It follows that, for each $h \in \hat{H}$, $\Delta_{\hat{H}}(h) = \sum(h_{(1)} \otimes h_{(2)})$ with $h_{(2)} \in \phi(H)$. Thus $h = \sum \varepsilon(h_{(1)})h_{(2)} \in \phi(H)$ and $\phi$ is onto. ∎

# 3 Induced representations

**3.1.** Let $A$ be a fixed Dedekind domain. All Hopf algebras in this section are assumed to be projective over $A$ with bijective antipodes. Let $H$ be a Hopf algebra over $A$ with a fixed flat admissible set $\mathcal{I}$ of ideals (admissible category $\mathcal{C}$) for $H$. Denote by $C$ the functor from the category of all $H$-modules to $\mathcal{C}$, which is right adjoint to the embedding functor from $\mathcal{C}$ to the category of all $H$-modules. $H^0_{\mathcal{I}}$ is the Hopf dual of $H$ corresponding to $\mathcal{C}$ or $\mathcal{I}$.

Let $\sigma : G \to H$ be a homomorphism of Hopf algebras. By adding $'$ we denote by $\mathcal{I}'$, $\mathcal{C}'$, and $C'$ a flat admissible set of ideals, the admissible category and the corresponding functor for $G$ respectively. We will assume throughout this section that $\sigma$ is compatible (cf. 1.9) with respect to the admissible categories $\mathcal{C}$ and $\mathcal{C}'$. Equivalently, each module in $\mathcal{C}$ yields a $G$-module in $\mathcal{C}'$ through $\sigma$, i.e., $\sigma^* : \mathcal{C} \to \mathcal{C}'$ defines a functor (cf. 1.10).

Following 1.9, $\sigma$ induces a homomorphism of Hopf algebras $\sigma^0 : H^0 \to G^0$, which also defines an exact functor (which is called the *restriction functor*) $\sigma^0 : \mathcal{C} \to \mathcal{C}'$. Via the category equivalence in 1.8, the two functors $\sigma^0$ and $\sigma^*$ are isomorphic (cf. 1.10).

**3.2.** Let $M \in \mathrm{Ob}(\mathcal{C}')$. $\mathrm{Hom}_A(H, M)$ is a $G - H$-bimodule with the $G$-module structure defined by $(gf)(x) = \sum g_{(1)} f(\gamma_H(\sigma(g_{(2)}))x)$ and the $H$-module structure defined by $(hf)(x) = f(xh)$ for all $g \in G$, $f \in \mathrm{Hom}_A(H, M)$ and $x, h \in H$. So $g(hf) = h(gf)$ for all $f \in \mathrm{Hom}_A(H, M)$, $g \in G$ and $h \in H$.

For each $G$-module $M$, $M^G = \{m \in M \mid gm = \varepsilon_G(g)m, \forall g \in G\}$ is the $G$-fixed point set of $M$. $\mathrm{Hom}_A(H, M)^G$ is an $H$-submodule of $\mathrm{Hom}_A(H, M)$ by the commutativity of the two actions on $\mathrm{Hom}_A(H, M)$. Setting $\sigma_*(M) = C(\mathrm{Hom}_A(H, M)^G)$ defines a left exact functor $\sigma_* : \mathcal{C}' \to \mathcal{C}$, which is called the induction functor.

**3.3.** For each $M \in \mathrm{Ob}(\mathcal{C}')$, the evaluation map $ev : \sigma^*(\sigma_*(M)) \to M$ is defined by $ev(f) = f(1)$ for all $f \in \sigma_*(M)$. $ev$ is a $G$-module map since $\mathrm{Hom}_A(H, M)^G = \mathrm{Hom}_G(H, M)$ by the bijectivity of $\gamma_G$ (cf. [1] 2.9). Here we regard $H$ as a left $G$-module through the algebra homomorphism $\sigma$.

**Proposition. (Frobenius Reciprocity)** *Let $E \in \mathrm{Ob}(\mathcal{C})$ and $M \in \mathrm{Ob}(\mathcal{C}')$. The map $\tau : \mathrm{Hom}_H(E, \sigma_*(M)) \to \mathrm{Hom}_G(\sigma^*(E), M)$, defined by $\tau(f) = ev \circ f$, is an isomorphism of $A$-modules.*

*Proof:* $\tau^{-1}$ is constructed as follows: For each $G$-module map $f : \sigma^*(E) \to M$, define $\tilde{f} : E \to \mathrm{Hom}_A(H, M)$ by $\tilde{f}(x) : H \to M$ with $\tilde{f}(x)(h) = f(hx)$. $\tilde{f}$ is an $H$-map. For each $g \in G$,

$$\begin{aligned}(g(\tilde{f}(x)))(h) &= \sum g_{(1)}(\tilde{f}(x)(\gamma_H\sigma(g_{(2)})h)) = \sum g_{(1)}f(\sigma(\gamma_G(g_{(2)}))hx) \\ &= \sum g_{(1)}(\gamma_G(g_{(2)})f(hx)) = \sum(g_{(1)}\gamma_G(g_{(2)}))f(hx) = \varepsilon_G(g)\tilde{f}(x)(h)\end{aligned}$$

for all $h \in H$. Therefore $\tilde{f}(x) \in \mathrm{Hom}_A(H, M)^G$ and $\tilde{f}(E) \subseteq \sigma_*(M)$ since $E$ is in $\mathcal{C}$. One can verify that $\tau^{-1}(f) = \tilde{f}$ and $\tau(\tilde{f}) = f$ easily. ∎

*Remark.* The Frobenius reciprocity shows that $\sigma_*$ is right adjoint to the restriction functor $\sigma^*$. Furthermore it shows that the module $\sigma_*(M)$ is determined uniquely from the universal property that, for each $H$-module $E$ and any $G$-module map $f : \sigma^*(E) \to M$, there is a unique $H$-module map $\tilde{f} : E \to \sigma_*(M)$ such that $f = ev \circ \tilde{f}$.

**3.4.** Let $\sigma$ be as in 3.3 and $\rho : K \to G$ be another compatible homomorphism of Hopf algebras with respect to a fixed admissible category $\mathcal{C}''$ for $K$ and the same $\mathcal{C}'$ for $G$ as in 3.3. The composition $\sigma \circ \rho : K \to H$ is also a compatible map of Hopf algebras with respect to the admissible categories $\mathcal{C}$ and $\mathcal{C}''$ for $H$ and $K$ respectively. Then we have the isomorphism of functors $\sigma_* \circ \rho_* \cong (\sigma \circ \rho)_*$ following 3.3 and the uniqueness of adjoint functors since $\sigma^* \circ \rho^* \cong (\sigma \circ \rho)^*$. We make the isomorphism explicit as follows:

**Lemma.** *For each $K$-module $M$ in $\mathcal{C}''$, the map $\xi_M : (\sigma \circ \rho)_*(M) \to \sigma_*(\rho_*(M))$, defined by (for $f \in (\sigma \circ \rho)_*(M)$ ) $\xi_M(f) = \tilde{f} : H \to \rho_*(M)$ with $\tilde{f}(h) : G \to M$ for each $h \in H$ such that $\tilde{f}(h)(g) = f(\sigma(g)h)$ for all $g \in G$, is an isomorphism of $H$-modules.*

*Proof:* Since $f \in (\sigma \circ \rho)_*(M)$ implies that $f(\sigma\rho(k)h) = kf(h)$ for all $k \in K$, $h \in H$,

$$\tilde{f}(h)(\rho(k)g) = f(\sigma(\rho(k)g)h) = k(f(\sigma(g)h)) = k(\tilde{f}(h)(g))$$

shows that $\tilde{f}(h) \in \mathrm{Hom}_A(G, M)^K$. To see $\tilde{f}(h)$ is in $\rho_*(M)$, take $I \in \mathcal{I}$ such that $f(I) = 0$. The compatibility of $\sigma$ shows that there exists $I' \in \mathcal{I}'$ such that $I' \subseteq \sigma^{-1}(I)$. Thus we have $I'(\tilde{f}(h)) = f(\sigma(I')h) = 0$ and $\tilde{f}(h) \in \rho_*(M)$. One can show that $\tilde{f} \in \sigma_*(\rho_*(M))$ in a similar way. ∎

**3.5.** The unit map $\sigma = u : A \to H$ is a compatible homomorphism of Hopf algebras with the category of all $A$ modules as the admissible category for $A$. Therefore for each $A$-module $M$, we have $\mathrm{Hom}_A(H, M)^A = \mathrm{Hom}_A(H, M)$. It follows from 1.5 that $\sigma_*(M) = H(M) = M \otimes H^0_{\mathcal{I}}$ as $H$-modules. In particular, $\sigma_*(A) = H^0_{\mathcal{I}}$.

Following 1.8, each $H$-module $M$ in $\mathcal{C}$ has an $H^0_{\mathcal{I}}$-comodule structure $\Delta_M$. One knows that the fixed point set of the $H^0_{\mathcal{I}}$-comodule $M$ is $M^{H^0_{\mathcal{I}}} = \{m \in M \mid \Delta_M(m) = m \otimes 1\}$.

**Lemma.** *For each $M$ in $\mathcal{C}$, $M^H = M^{H^0_{\mathcal{I}}}$.*

*Proof:* Recall that $hm = \sum f_i(h)m_i$ for all $h \in H$ if $\Delta_M(m) = \sum m_i \otimes f_i$ for $m \in M$. It is clear that $\Delta_M(m) = m \otimes 1$ implies $hm = \varepsilon(h)m$. On the other hand, if $hm = \sum f_i(h)m_i = \varepsilon(h)m$ for all $h \in H$, then $\sum m_i \otimes f_i = m \otimes 1$ in $\mathrm{Hom}_A(H, M)$ under the embedding $M \otimes H^0_{\mathcal{I}} \subseteq \mathrm{Hom}_A(H, M)$. Thus we have $\Delta_M(m) = m \otimes 1$. ∎

**3.6.** Let $\sigma : G \to H$ be as in 3.3. For each $G$-module $M$ in $\mathcal{C}'$ with the $G^0_{\mathcal{I}'}$-comodule structure $\rho_M : M \to M \otimes G^0_{\mathcal{I}'}$ , the natural map $\phi_M : M \otimes H^0_{\mathcal{I}} \to \mathrm{Hom}_A(H, M)$ is a $G - H$-bimodule map. Here $M \otimes H^0_{\mathcal{I}}$ is in $\mathcal{C}$ as $H$-module with the $H^0_{\mathcal{I}}$-comodule structure given by $1 \otimes \Delta^0_H : M \otimes H^0_{\mathcal{I}} \to M \otimes H^0_{\mathcal{I}} \otimes H^0_{\mathcal{I}}$. As a $G$-module, $M \otimes H^0_{\mathcal{I}}$ is in $\mathcal{C}'$ with the $G^0_{\mathcal{I}'}$-comodule structure $\delta$ on $M \otimes H^0_{\mathcal{I}}$ defined as the composition of the following maps:

$$M \otimes H^0_{\mathcal{I}} \xrightarrow{\rho_M \otimes \Delta^0_H} M \otimes G^0_{\mathcal{I}'} \otimes H^0_{\mathcal{I}} \otimes H^0_{\mathcal{I}} \xrightarrow{1 \otimes 1 \otimes \gamma^0_G \sigma^0 \otimes 1} M \otimes G^0_{\mathcal{I}'} \otimes G^0_{\mathcal{I}'} \otimes H^0_{\mathcal{I}}$$
$$\xrightarrow{1 \otimes m^0_G \otimes 1} M \otimes G^0_{\mathcal{I}'} \otimes H^0_{\mathcal{I}} \xrightarrow{T_{2,3}} M \otimes H^0_{\mathcal{I}} \otimes G^0_{\mathcal{I}'}.$$

One can easily show that $C(\mathrm{Hom}_A(H, M)^G) = (C\mathrm{Hom}_A(H, M))^G$. Thus $\sigma_*(M) = (M \otimes H^0_{\mathcal{I}})^G = (M \otimes H^0_{\mathcal{I}})^{G^0_{\mathcal{I}'}}$ by Lemma 3.5.

On the other hand, Donkin in [5] defines the induced comodule, denoted by $\sigma_0(M)$, as an $H^0_{\mathcal{I}}$-subcomodule of $M \otimes H^0_{\mathcal{I}}$ by

$$\sigma_0(M) = \{\textstyle\sum m_i \otimes f_i \in M \otimes H^0_{\mathcal{I}} \mid (1 \otimes \sigma^0 \otimes 1)(1 \otimes \Delta^0_H)(\sum m_i \otimes f_i) = (\rho_M \otimes 1)(\sum m_i \otimes f_i)\}.$$

**Proposition.** *$\sigma_0(M) = \sigma_*(M)$ for each $G$-module $M$ in $\mathcal{C}'$.*

*Proof:* Since $\sigma_*(M) = (M \otimes H^0_{\mathcal{I}})^{G^0_{\mathcal{I}'}}$, both $\sigma_0(M)$ and $\sigma_*(M)$ are subsets of $M \otimes H^0_{\mathcal{I}}$. We first show $\sigma_0(M) \subseteq \sigma_*(M)$. If $\sum m_i \otimes f_i \in \sigma_0(M)$, then

$$\begin{aligned}
\delta(\textstyle\sum m_i \otimes f_i) &= (1 \otimes T(m^0_G \otimes 1))(\rho_M \otimes \gamma^0_G \otimes 1)(1 \otimes (\sigma^0 \otimes 1)\Delta^0_H)(\sum m_i \otimes f_i) \\
&= (1 \otimes T(m^0_G \otimes 1))(\rho_M \otimes \gamma^0_G \otimes 1)(\rho_M \otimes 1)(\sum m_i \otimes f_i) \\
&= (1 \otimes T(m^0_G \otimes 1))(1 \otimes 1 \otimes \gamma^0_G \otimes 1)((\rho_M \otimes 1)\rho_M \otimes 1)(\sum m_i \otimes f_i) \\
&= (1 \otimes T(m^0_G \otimes 1))(1 \otimes 1 \otimes \gamma^0_G \otimes 1)((1 \otimes \Delta^0_G)\rho_M \otimes 1)(\sum m_i \otimes f_i) \\
&= (1 \otimes T)(1 \otimes (m^0_G(1 \otimes \gamma^0_G)\Delta^0_G) \otimes 1)(\rho_M \otimes 1)(\sum m_i \otimes f_i) \\
&= (1 \otimes T)(1 \otimes u^0_G \varepsilon^0_G \otimes 1)(\rho_M \otimes 1)(\sum m_i \otimes f_i)
\end{aligned}$$

Here we have used the identity $m_G^0(1\otimes\gamma_G^0)\Delta_G^0 = u_G^0\varepsilon_G^0$ for the Hopf algebra $G_{\mathcal{I}'}^0$. Then by the counitary condition, $(1\otimes\varepsilon_G^0)\rho_M(m) = m\otimes 1$ for all $m\in M$, we have

$$\delta(\sum m_i\otimes f_i) = (\sum m_i\otimes f_i)\otimes 1$$

and $\sum m_i\otimes f_i \in \sigma_*(M)$.

The inclusion map $\iota : \sigma_0(M) \to \sigma_*(M)$ is a homomorphism of $H_{\mathcal{I}}^0$-comodules. The evaluation map $e : \sigma_0(M) \to M$ defined by Donkin is given by $e(\sum m_i\otimes f_i) = \sum f_i(1)m_i = (\sum m_i\otimes f_i)(1)$. Thus we have the commutative diagram

$$\begin{array}{ccc} \sigma_0(M) & \xrightarrow{\iota} & \sigma_*(M) \\ e\downarrow & \swarrow ev & \\ M & & \end{array} .$$

However, both modules $\sigma_0(M)$ and $\sigma_*(M)$ together with the evaluation maps satisfy the Frobenius reciprocity. Therefore $\iota$ has to be an isomorphism and the two sets are the same. ∎

*Remark.* Let $H$ be the distribution algebra of an affine algebraic group and $G$ the distribution algebra of a closed subgroup. If we consider the admissible categories as the categories of rational modules of the algebraic groups, then the induction we defined here is the same as the induction defined for group schemes.

**3.7.** It follows from Corollary 1.8 that the category $\mathcal{C}'$ has enough injective objects. So the right derived functors $R^n\sigma_*$ of $\sigma_*$ can be defined. The following proposition can be found in [15] (2.7-2.9), which is proved over a field there, but it can be easily extended over $A$, which is a Dedekind domain.

**Proposition.** (A) (Generalized Tensor Identity) *$R^n\sigma_*(\sigma^*(N)\otimes M) \cong N\otimes R^n\sigma_*(M)$ is an isomorphism of $H$-modules in $\mathcal{C}$ for each $A$-flat $H$-module $N$ in $\mathcal{C}$ and $G$-module $M$ in $\mathcal{C}'$.*

(B) *If $\gamma_G$ and $\gamma_H$ are bijective, then followings are equivalent:*

(i) *$\sigma_*$ is exact,*

(ii) *$\sigma^*(H_{\mathcal{I}}^0)$ is injective in $\mathcal{C}'$,*

(iii) *$\sigma^*(Q)$ is injective in $\mathcal{C}'$ for some injective module $Q$ in $\mathcal{C}$,*

(iv) *$\sigma^*(Q)$ is injective in $\mathcal{C}'$ for all injective module $Q$ in $\mathcal{C}$.* ∎

**3.8. Proposition.** *Suppose that $A$ is a field and $H$ is a flat module over $G$. If $\sigma^*(C(N)) = C'(\sigma^*(N)$ for each $H$-module $N$ (not necessarily in $\mathcal{C}$), then $\sigma_*$ is exact.*

*Proof:* Let $M$ be a $G$-module in $\mathcal{C}'$. Then we have the isomorphism of vector spaces

$$\begin{aligned}\mathrm{Hom}_A(H \otimes_G M, A) &\cong \mathrm{Hom}_G(M, \sigma^*(\mathrm{Hom}_A(H, A)) \\ &= \mathrm{Hom}_G(M, C'\sigma^*(\mathrm{Hom}_A(H, A)) \\ &= \mathrm{Hom}_G(M, \sigma^* C(\mathrm{Hom}_A(H, A)) \\ &= \mathrm{Hom}_G(M, \sigma^* H^0_{\mathcal{I}}).\end{aligned}$$

Since $\sigma^* H^0_{\mathcal{I}}$ is injective if and only if $\mathrm{Hom}_G(?, \sigma^* H^0_{\mathcal{I}})$ is exact, we only need to show that the functor $\mathrm{Hom}_A(H \otimes_G ?, A)$ is exact. However $H \otimes_G ?$ is exact since $H$ is flat over $G$. Since $A$ is a field, then $A$ is injective in the category of all vector spaces over $A$. Now the proposition follows from Proposition 3.7. ∎

# 4 Normal Hopf subalgebras

**4.1.** Let $H$ be a Hopf algebra. We define the following operators:

$$\phi_r : H \otimes H \to H \text{ by } \phi_r(x \otimes y) = \sum x_{(1)} y \gamma(x_{(2)}),$$

$$\phi_l : H \otimes H \to H \text{ by } \phi_l(y \otimes x) = \sum \gamma(x_{(1)}) y x_{(2)}.$$

In general $\phi_r$ and $\phi_l$ are not coalgebra homomorphisms (if $H$ is not commutative). However $\phi_r$ and $\phi_l$ do commute with the counit maps. In fact, the counit map $\varepsilon : H \to A$ is a homomorphism of algebras, and

$$\varepsilon(\phi_r(x \otimes y)) = \sum \varepsilon(x_{(1)})\varepsilon(y)\varepsilon(\gamma(x_{(2)})) = \sum \varepsilon(x_{(1)})\varepsilon(\gamma(x_{(2)}))\varepsilon(y) = \varepsilon(x)\varepsilon(y).$$

**Definition.** Let $D \subseteq H$ be a Hopf subalgebra. We call $D$ left (or right) normal if $\phi_l(D \otimes H) \subseteq D$ (or $\phi_r(H \otimes D) \subseteq D$). $D$ is called normal if it is both left and right normal. One can verify that $\phi_l$ (or $\phi_r$) makes $H$ into a right (or left) $H$-module. Thus $D$ is left (or right) normal if and only if $D$ is a submodule of $H$ with the module structure defined by $\phi_l$ (or $\phi_r$).

*Remarks.* (1) The definition given here is motivated by the definitions given by Yanagihara [18] (Section 12) and by Parshall-Wang in [15]. In the former reference, $H$ is assumed to be cocommutative.

(2) If $H$ is cocommutative or commutative, then the left normality and the right normality are equivalent. This is clear from the definition if $H$ is cocommutative. If $H$ is commutative, then $\gamma^2 = id$ and $\gamma$ is an automorphism ([16] 4.0.1). (Compare with [15](1.5.1).)

**Example.** Let $G$ be an algebraic group scheme over an algebraically closed field $A$ and $G_1$ a closed subgroup scheme. Then we have $\mathrm{Dist}(G_1) \subseteq \mathrm{Dist}(G)$. $G_1$ is normal in $G$ if and only if $\mathrm{Dist}(G_1)$ is normal. [18](12.12).

**4.2.** In general, let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. Then $\mathrm{Hom}_A(C, H)$ is an associative $A$-algebra with the multiplication defined by $f*g = m_H(f\otimes g)\Delta_C$ and the identity $u_H \circ \varepsilon_C$. Each element $g \in \mathrm{Hom}_{coalg}(C, H)$ has a multiplicative inverse $\gamma \circ g$ in $\mathrm{Hom}_A(C, H)$, which is an anti-homomorphism of coalgebras. Let $G(C)$ be the multiplicative group generated by $\mathrm{Hom}_{coalg}(C, H)$ in $\mathrm{Hom}_A(C, H)$. (Compare with the definition of quantum groups by PW in [15].) One can easily see that all elements in $G(C)$ commutes with the counits, i.e., $\varepsilon_H(g(c)) = \varepsilon_C(c)$ for all $g \in G(C)$ and $c \in C$.

Take the coalgebra $C = H \otimes H$. Let $p_1$ (or $p_2$) : $C \to H$ be the projection defined by $p_1(h \otimes d) = \varepsilon_H(d)h$ (or $p_2(d \otimes h) = \varepsilon_H(d)h$. Then $p_1$ and $p_2$ are homomorphism of coalgebras. It is easily checked that $\phi_r = p_1 * p_2 * \gamma p_1$ and $\phi_l = \gamma p_2 * p_1 * p_2$.

**Proposition.** *Let $h, d \in H$. Then*

$$hd = \sum \phi_r(h_{(1)} \otimes d)h_{(2)} \ \ and \ \ dh = \sum h_{(1)}\phi_l(d \otimes h_{(2)}).$$

*Proof:* Let $C = H \otimes H$ be as before. $\phi_r * p_1 = p_1 * p_2$ and $p_2 * \phi_l = p_1 * p_2$. For $x \otimes y \in C$, we have

$$\begin{aligned}
p_1 * p_2(x \otimes y) &= m_H(p_1 \otimes p_2)\Delta_C(x \otimes y) \\
&= m_H(\sum(\varepsilon_H(y_{(1)})x_{(1)} \otimes \varepsilon_H(x_{(2)})y_{(2)})) \\
&= m_H(\sum \varepsilon_H(x_{(2)})x_{(1)} \otimes \sum \varepsilon_H(y_{(1)})y_{(2)} = m(x \otimes y) = xy; \\
(p_2 * \phi_l)(x \otimes y) &= m(p_2 \otimes \phi_l)(\sum((x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)}))) \\
&= m(\sum(\varepsilon_H(x_{(1)})y_{(1)} \otimes \phi_l(x_{(2)} \otimes y_{(2)}))) \\
&= m(\sum(y_{(1)} \otimes \phi_l(\sum(\varepsilon_H(x_{(1)})x_{(2)} \otimes y_{(2)})))) \\
&= \sum y_{(1)}\phi_l(x \otimes y_{(2)}).
\end{aligned}$$

Similarly we have $\phi_r * p_1(x \otimes y) = \sum \phi_r(x_{(1)} \otimes y)x_{(2)}$ and the proposition follows. ∎

Let $D$ be a normal Hopf subalgebra of $H$ and $I_D$ be the augmentation ideal of $D$. We denote by $< I_D >$ the two-sided ideal in $H$ generated by $I_D$.

**Corollary.** $< I_D >= I_D H = H I_D$ *is a Hopf ideal in $H$.*

*Proof:* $I_D H = H I_D$ is a two-sided ideal in $H$ by the Proposition. The inclusion relation $I_D \subseteq I_D H \subseteq < I_D >$ gives the equalities in the corollary. That $< I_D >$ is a Hopf ideal follows from the fact that $I_D$ is a Hopf ideal in $D$. ∎

**Definition.** We call the Hopf algebra $H/I_D H$ the quotient Hopf algebra of $H$ by the normal Hopf subalgebra $D$ and denote it by $H/D$.

**4.3. Proposition.** *Let $\sigma : H \to G$ be a homomorphism of Hopf algebras. If $G_1$ is a left (or right) normal Hopf subalgebra of $G$, then $\sigma^{-1}(G_1)$ is also a left (or right) normal Hopf subalgebra of $H$.*

*Proof:* We use $\phi_l^H$ and $\phi_l^G$ to denote the operations as in 4.1 on $H$ and $G$ respectively. The proposition follows from $\sigma \circ \phi_l^H = \phi_l^G \circ (\sigma \otimes \sigma)$, which can be verified easily since $\sigma$ is a map of Hopf algebras. ∎

**Examples.** (1) $A$ is a normal Hopf subalgebra of $H$ for any Hopf algebra $H$.

(2) If $\sigma : H \to G$ is a homomorphism of Hopf algebras, then $\sigma^{-1}(A)$ is a normal Hopf subalgebra of $H$.

(3) If $D$ is a normal Hopf subalgebra of $H$ and $\sigma : H \to H/D$ is the canonical projection, then $\sigma^{-1}(A)$ is a normal Hopf subalgebra of $H$ such that $D \subseteq \sigma^{-1}(A)$ and $H/D = H/\sigma^{-1}(A)$. But in general, $D \neq \sigma^{-1}(A)$.

*Remark.* Unlike the normal subgroups in a group, the normal Hopf subalgebra is not uniquely determined by its quotient. There are two normal Hopf subalgebras with one strictly contained in the other one, but they may have the same quotient Hopf algebra.

**4.4. Proposition.** *Let $D$ be a left (or right) normal Hopf subalgebra of $H$ and $M$ a left (or right) $H$-module. Then the $D$-fixed point set $M^D$ is an $H$-submodule of $M$. If $D$ is normal, then $M^D$ is an $H/D$-module.*

*Proof:* We will show the proposition for left $H$-module $M$ only. If $m \in M^D$, we need to show $hm \in M^D$ for all $h \in H$. Let $d \in D$. By Proposition 4.2 and the left normality of $D$, we have,

$$\begin{aligned} d(hm) &= (dh)m = \sum h_{(1)}\phi_l(d \otimes h_{(2)})m = \sum h_{(1)}\varepsilon_H(\phi_l(d \otimes h_{(2)}))m \\ &= \sum h_{(1)}\varepsilon_H(d)\varepsilon_H(h_{(2)})m = \varepsilon_H(d)hm \end{aligned}$$

for all $d \in D$. Thus $hm \in M^D$.

For each $m \in M^D$ and $\sum h_i d_i \in HI_D$, $\sum h_i d_i m = \sum h_i \varepsilon(d_i)m = 0$. Thus $HI_D$ is in the kernel of the homomorphism $H \to \mathrm{End}_A(M)$. Therefore it induces a homomorphism $H/HI_D \to \mathrm{End}_A(M^D)$. ∎

**4.5.** In the rest of this section, we fix a normal Hopf subalgebra $D$ in $H$. Let $\mathcal{I}'$ (or $\mathcal{I}''$) be a flat admissible set of ideals in $D$ (or $G = H/D$). We assume that the embedding $\sigma : D \to H$ and the projection map $\pi : H \to G$ are compatible. We further assume that $\mathcal{I}''$ is equivalent to $\{\pi(I) \mid I \in \mathcal{I}\}$, which is also an admissible set of ideals in $G$. Denote by $\mathcal{C}'$ and $\mathcal{C}''$ the corresponding admissible categories with respect to $\mathcal{I}'$ and $\mathcal{I}''$ respectively. The restriction functors $\sigma^*$ and $\pi^*$ are defined.

**Lemma.** *$\{\pi(I) \mid I \in \mathcal{I}\}$ and $\mathcal{I}''$ are equivalent if and only if $\mathcal{C}''$ is equivalent to the full subcategory of all $D$-trivial $H$-modules in $\mathcal{C}$ under the functor $\pi^*$.*

*Proof:* It is clear that $\pi^* : \mathcal{C}'' \to \mathcal{C}$ is a faithful functor since $\pi$ is compatible. Suppose the two sets of ideals are equivalent. For each $D$-trivial $H$-module $M$ in $\mathcal{C}$, there exists an $H/D$-module $N$ such that $M = \pi^*(N)$ by 4.4. For each $n \in N$, one has $\pi^{-1}(\mathrm{Ann}_{H/D}((H/D)n)) = \mathrm{Ann}_H(Hn)$. By the assumption, there exists $I'' \in \mathcal{I}''$ such that $I'' \subseteq \mathrm{Ann}_{H/D}((H/D)n)$. Thus $N$ is in $\mathcal{C}''$. The other direction can be checked similarly by considering the modules $H/(I + HI_D)$ for $I \in \mathcal{I}$. ∎

**4.6.** Let $G_1$ be a Hopf subalgebra in $G = H/D$ and $H_1$ be a Hopf subalgebra in $H$ such that $D \subseteq H_1$ and $\pi(H_1) = G_1$. We assume that the admissible categories for $H_1$ and $G_1$ are inherited from the admissible categories for $H$ and $G$ respectively. From now on, we call a module rational if it is in the given admissible category (which will be clear from the context). We denote by $\pi_1 : H_1 \to G_1$ the restriction of $\pi$. Since $\pi$ is compatible, $\pi_1$ is also compatible. For each rational $G_1$-module $N$, $\pi_1^*(N) = \pi^*(N)$ is a rational $H_1$-module, on which $D$ acts trivially.

We denote by $\mathrm{Ind}_{H_1}^H$ and $\mathrm{Ind}_{G_1}^G$ the induction functors (from Hopf subalgebras with the inherited admissible categories) respectively. For simplicity, we still assume that all antipodes are bijective.

**Proposition.** $\pi^* \circ \mathrm{Ind}_{G_1}^G \cong \mathrm{Ind}_{H_1}^H \circ \pi_1^*$.

*Proof:* Let $N$ be a rational $G_1$-module. The natural map $\phi_N : \pi^*\mathrm{Hom}_{G_1}(G, N) \to \mathrm{Hom}_{H_1}(H, \pi_1^*(N))$, defined by $\phi(\tilde{f}) = \tilde{f} \circ \pi : H \to \pi_1^*(N)$ for $\tilde{f} \in \mathrm{Hom}_{G_1}(G, N)$, is an isomorphism of $H$-modules. $\phi_N^{-1}$ is defined as follows: $\mathrm{Hom}_{H_1}(H, \pi_1^*(N))$ is $D$-trivial by 4.2 and the $D$-triviality of $\pi_1^*(N)$. For each $f \in \mathrm{Hom}_{H_1}(H, \pi_1^*(N))$, $f(I_D H) = \varepsilon_H(I_D)f(H) = \{0\}$. Let $\phi^{-1}(f)$ be the induced map $\tilde{f} : G \to N$. It is easily verified that $\tilde{f} \in \mathrm{Hom}_{G_1}(G, N)$. It follows from Lemma 4.5 that $f \in C(\mathrm{Hom}_{H_1}(H, \pi_1^*(N)))$ if and only if $\tilde{f} \in C''(\mathrm{Hom}_{G_1}(G, N))$ for each $\tilde{f} \in \mathrm{Hom}_{G_1}(G, N)$. It is clear that $\phi$ is natural. So the proposition follows. ∎

**4.7.** Note that the functor $\pi^*$ is exact. $\mathrm{Ind}_{G_1}^G$ is left exact and sends injective objects to injective objects. Thus we have an isomorphism $R^n(\pi^* \circ \mathrm{Ind}_{G_1}^G) \cong \pi^* \circ R^n\mathrm{Ind}_{G_1}^G$ of functors arising from a Grothendieck spectral sequence. On the other hand, in order to get a Grothendieck spectral sequence for the composition $\mathrm{Ind}_{H_1}^H \circ \pi_1^*$ of functors, we need $\pi_1^*$ to send injective objects to $\mathrm{Ind}_{H_1}^H$-acyclic objects.

**Proposition.** *If* $\mathrm{Ind}_D^{H_1}$ *and* $\mathrm{Ind}_D^H$ *are both exact, then* $\pi_1^*$ *sends injective objects to* $\mathrm{Ind}_{H_1}^H$*-acyclic objects. Furthermore we have* $R^n\mathrm{Ind}_{H_1}^H \circ \pi_1^* \cong \pi^* \circ R^n\mathrm{Ind}_{G_1}^G$.

*Proof:* It follows from Corollary 1.8 that each injective module in $\mathcal{C}$ is a direct summand of $Q \otimes H_{\mathcal{I}}^0$ for some injective $A$-module $Q$. Keeping in mind that all functors involved are additive and commute with direct sums, one can copy the second part of the proof in ([7] I 6.11). ∎

*Remark.* If $D$ is the distribution algebra of a normal algebraic subgroup of an algebraic group with distribution algebra $H$, the condition in the proposition is always satisfied and the result is classical (cf. [7] I, 6.5).

**4.8.** In this and next two subsections we assume that $A$ is a field. Let $\pi : H \to H/D = G$ be as in 4.5 with certain fixed admissible categories. Then all the simple modules will be finite-dimensional.

**Proposition.** *Suppose each simple $D$-module is the restriction of a simple $H$-module to $D$. Then*

(i) $\mathrm{Soc}_D(M)$ *is an $H$-submodule of $M$ for each rational $H$-module $M$,*

(ii) *each semisimple $H$-module is still semisimple when restricted to $D$,*

(iii) *if* $\mathrm{Hom}_D(L_1, L_1)$ *is 1-dimensional over $A$ for all simple $D$-module $L_1$, then an $H$-module is simple if and only if it has the form $\pi^*(L) \otimes L_1$ for a simple $D$-module $L_1$ and a simple $G$-module $L$.*

*Proof:* Let $L$ be a simple $H$-module such that the restriction $L_1$ of $L$ to $D$ is also simple. Then $\mathrm{Hom}_D(L_1, M)$ is an $H$-module. Similar to [7] (I, 2.14(3)), we consider the injective map $\phi_{L_1} : \mathrm{Hom}_D(L_1, M) \otimes L_1 \to M$ defined by $\phi_{L_1}(f \otimes x) = f(x)$. For each $h \in H$,

$$\begin{aligned} \phi_{L_1}(h(f \otimes x)) &= \phi_{L_1}(\textstyle\sum h_{(1)}f \otimes h_{(2)}x) = \sum (h_{(1)}f)(h_{(2)}x) = \sum h_{(1)}f(\gamma(h_{(2)})h_{(3)}x) \\ &= \textstyle\sum h_{(1)}f(\varepsilon(h_{(2)})x) = \sum h_{(1)}\varepsilon(h_{(2)})f(x) = hf(x) = h\phi_{L_1}(f \otimes x) \end{aligned}$$

shows that $\phi_{L_1}$ is a homomorphism of $H$-modules. Since $\mathrm{Hom}_D(L_1, M)$ is a trivial $D$-module, $\mathrm{Im}(\phi_{L_1})$ is a semisimple $D$-module and $\mathrm{Soc}_D(M) = \sum \mathrm{Im}(\phi_{L_1})$. Here the sum is taken over all simple $D$-modules $L_1$. Therefore the $D$-socle of $M$ is an $H$-module and (i) follows. (ii) is a consequence of (i) by taking $M$ simple.

To show (iii), take $M$ simple in (i). Choose $L_1$ such that $\phi_{L_1} \neq 0$, which has to be an isomorphism. Thus $M$ is a tensor product as in (iii). Conversely, let $M = \pi^*(L) \otimes L_1$. Each $y \in \pi^*(L)$ defines a map $f_y : L_1 \to M$ by $f_y(x) = y \otimes x$. This is a homomorphism of $D$-modules since $f_y(dx) = y \otimes dx = d(y \otimes x)$. This defines an injective $H$-module map $\pi^*(L) \to \mathrm{Hom}_D(L_1, M)$. In fact, for $h \in H$,

$$\begin{aligned} (hf_y)(x) &= \textstyle\sum h_{(1)}(f_y(\gamma(h_{(2)})x)) = \sum h_{(1)}(y \otimes \gamma(h_{(2)})x) = \sum h_{(1)}y \otimes h_{(2)}\gamma(h_{(3)})x \\ &= \textstyle\sum h_{(1)}y \otimes \varepsilon(h_{(2)})x = hy \otimes x. \end{aligned}$$

By tensoring $L_1$ , one can show that the above defined map gives an isomorphism $\pi^*(L) \cong \mathrm{Hom}_D(L_1, M)$ of $H$-modules. $\mathrm{Hom}_D(L_1, M_1) \neq 0$ is an $H$-submodule of $\pi^*(L)$ for any $H$-submodule $M_1 \neq 0$ of $M$. Hence, $\pi^*(L) \otimes L_1 \subseteq \mathrm{Hom}_D(L_1, M_1) \otimes L_1 \subseteq M_1$ and $M$ is simple. ∎

*Remark.* Compare the proof (iii) with the one in [4]. In the case of quantum algebras, (i) and (ii) are proved in [2] 3.3–3.4.

We call a Hopf algebra $H$ together with an admissible category $\mathcal{C}$ *split* over the field $A$ if $\mathrm{Hom}_H(L, L)$ is 1-dimensional over $A$ for every simple object $L$ in $\mathcal{C}$.

**4.9.** We assume that $D$ is split over $A$ and each simple $D$-module is the restriction of a simple $H$-module to $D$.

**Lemma.** *If all finite dimensional rational $G$-modules are semi-simple, then $\mathrm{Soc}_D^i M = \mathrm{Soc}_H^i M$ for all finite dimensional rational $H$-modules $M$ in the admissible category for $H$ and all $i$.*

*Proof:* We show the proposition for $i = 1$ only. It follows from 4.8 that $\mathrm{Soc}_H M \subseteq \mathrm{Soc}_D M$. On the other hand, $\mathrm{Hom}_D(L_1, M)$ is a semisimple $H$-module since all finite dimensional $G$-modules are semisimple. Hence $\mathrm{Im}(\phi_{L_1}) = \mathrm{Hom}_D(L_1, M) \otimes L_1$ and $\mathrm{Soc}_D M$ are semisimple as $H$-modules by 4.8 (iii). ∎

**Proposition.** *Suppose $\mathrm{Ind}_D^H$ is exact. If all finite dimensional rational $G$-modules are semisimple, then the injective hull of a simple $D$-module $L$ in the admissible category for $D$ is the restriction of the injective hull of $L$ in the admissible category for $H$.*

*Proof:* Let $Q$ be the $H$-injective hull of $L$ and $Q_1$ the $D$-injective hull of $L$ (restricted to $D$). Since $\mathrm{Ind}_D^H$ is exact, then $Q$ is injective as $D$-module by 3.7. Since $L$ is a $D$-submodule of $Q$, we can take $Q_1$ as a $D$-submodule of $Q$. Let $M \neq 0$ be any finite dimensional $H$-submodule of $Q$. Then $\mathrm{Soc}_D M = \mathrm{Soc}_H M = L$. Thus $M \subseteq Q_1$ and $Q \subseteq Q_1$ since $Q$ is locally finite as $H$-module. Thus $Q_1 \cong Q$ as $D$-modules. ∎

# 5 Induction theory for Hopf subalgebras of quantum enveloping algebras

**5.1.** Let $v$ be an indeterminate and $\underline{g}$ a complex semisimple Lie algebra with Cartan matrix $(a_{ij})_{1 \leq i,j \leq n}$. The quantum enveloping algebra $U$ is a $\mathbf{Z}[v, v^{-1}]$-algebra with generators $E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1}$ ($i = 1, \ldots, n$, and $m \geq 0$) and some relations related to the Cartan matrix $(a_{ij})$. Here $E_i^{(m)} = E_i^m/[m]_{d_i}^!$, $F_i^{(m)} = F_i^m/[m]_{d_i}^!$ and $\{d_1, \ldots, d_n\}$ is a sequence of positive integers such that $(d_i a_{ij})$ is symmetric. For more detailed definition and notations, one is referred to [1, 13, 14]. Let $A$ be a Dedekind domain and $\mathbf{Z}[v, v^{-1}] \to A$ be a homomorphism of rings. Define $U_A = U \otimes_{\mathbf{Z}[v,v^{-1}]} A$. Then $U_A$ is a Hopf algebra over $A$ which is free over $A$. The following situations are of special interest:

(i) $A$ is a field and the image of $v$ (denoted by $v$ as well) is not a root of 1;

(ii) $A$ is a field of characteristic 0 and $v$ is an $l$-th primitive root of 1 for some odd integer $l$. For example, let $A$ be the $l$-th cyclotomic field over $\mathbf{Q}$;

(iii) Let $l = p^r$ for an odd prime integer $p$ and a positive integer $r$ in (ii). Let $A = \mathbf{Z}[v]$ be the ring of integers in the $l$-th cyclotomic field over $\mathbf{Q}$.

The category of all integrable $U_A$-modules (cf. [10]) is a direct sum of $2^n$ equivalent subcategories. It is proved in [1] that the subcategory $\mathcal{C}$ of integrable $U_A$-modules of type **1** is an admissible category with the corresponding coordinate Hopf algebra $(U_A)^0_{\mathcal{I}_\mathcal{C}}$ (which is denoted by $A[U]$ in [1]). If the root system if type $A_{n-1}$, Polo proves in the Appendix to [1] that $(U_A)^0_{\mathcal{I}_\mathcal{C}}$ is isomorphic to the quantum coordinate algebra $A[SL_q(n)]$ of the quantum special linear group $SL_q(n)$ defined by Parshall and Wang in [15] (with $v = q^{-1}$). (One can modify Polo's proof to get the isomorphism over Dedekind domains.)

**Lemma.** *If the root system is of type $A_{n-1}$, then the category of all integrable $U_A$-modules of type* **1** *is equivalent to the category of all rational modules for the quantum special linear group $SL_q(n)$.* ∎

**5.2.** Let $U_A^{\leq 0}$ be the subalgebra of $U_A$ generated by $\{F_i^{(m)}, \left[\begin{smallmatrix} K_i \\ t \end{smallmatrix}\right], K_i^{\pm}\}$, which is an $A$-free Hopf subalgebra of $U_A$. Let $\mathcal{I}$ be the flat admissible set of ideals in $U_A$ corresponding to the flat admissible category $\mathcal{C}$. Let $\mathcal{I}'$ be the admissible set of ideals in $U_A^{\leq 0}$ inherited from $\mathcal{I}$ and $\mathcal{C}'$ the corresponding admissible category. Then $\mathcal{C}'$ consists of all those locally finite $U_A^{\leq 0}$-modules with weight space decompositions and of type **1** (cf. [1]). Thus the embedding $U_A^{\leq 0} \to U_A$ is compatible with respect to the categories given above. In this situation, the induction functor defined in 3.1 is exactly the same as the one defined by APW in [1]. The map $(U_A)^0_{\mathcal{I}} \to (U_A^{\leq 0})^0_{\mathcal{I}'}$ of 1.9 is given by assigning to each $f \in (U_A)^0_{\mathcal{I}}$ the restriction $f|_{U_A^{\leq 0}}$ of $f$ to $U_A^{\leq 0}$. If the root system is of type $A_{n-1}$, one can check that $X_{ij}|_{U_A^{\leq 0}} = 0$ if $i < j$ in the proof of the isomorphism $(U_A)^0_{\mathcal{I}} \to A[SL_q(n)]$ in ([1] 12.7). So we have an induced homomorphism of Hopf algebras: $A[SL_q(n)]/ < X_{ij} \mid i < j > \to (U_A^{\leq 0})^0_{\mathcal{I}'}$. Here $A[SL_q(n)]/ < X_{ij} \mid i < j >$ is the coordinate algebra $A[B_q]$ of the quantum Borel subgroup $B_q$ of lower triangular matrices as in [15]. We claim that this homomorphism is an isomorphism. By constructing a basis $\{I_s\}_{s\in\mathbf{N}}$ for $\mathcal{I}'$ such that $I_{s+1} \subseteq I_s$ for all $s \in \mathbf{N}$, one can show that $(U_A^{\leq 0})^0_{\mathcal{I}'}$ is $A$-projective. Using the construction of a basis for $A[SL_q(n)]$ in [1]( 12.9–12.11), one can get a basis for $A[B_q]$. Thus $A[SL_q(n)]$ is $A$-free. The isomorphism follows from the argument in [1](12.12).

**Theorem.** *If the root system is of type $A_{n-1}$, then*

(i) $(U_A^{\leq 0})^0_{\mathcal{I}'} \cong A[B_q]$ *(with $v \mapsto q^{-1}$),*

(ii) *the induction functor defined by Andersen in* [1] *and the induction functor* $\mathrm{Ind}_{B_q}^{SL_q(n)}$ *defined by Parshall and Wang in* [15] *are isomorphic through the category equivalence in 1.8 and the isomorphism* $(U_A)_{\mathcal{I}}^0 \to A[SL_q(n)]$ *constructed by Polo.*

*Proof:* (i) follows from the above argument and (ii) follows from 3.6, (i) and 1.8 since Parshall and Wang use the induction functor defined by Donkin. ∎

**5.3.** In the rest of this section, we assume that the image of $v$ in $A$ is an $l$-th primitive root of 1 with a fixed odd integer $l$ and $3 \nmid l$ if the root system has a component $\mathrm{G}_2$. $U_A$ is generated as algebra by $\{K_i^{\pm 1}, E_i, F_i, E_i^{(l)}, F_i^{(l)}\}$. The subalgebra $u$ generated by $\{K_i^{\pm 1}, E_i, F_i\}$ is a Hopf subalgebra, which is $A$-free of finite rank [13, 14].

**Proposition.** *The Hopf subalgebra* $u$ *is a normal Hopf subalgebra of* $U_A$.

*Proof:* It follows from 4.1 that $u$ is left normal if and only if $\phi_l(u \otimes S) \subseteq u$ for a generating set $S$ of the algebra $U_A$. So it suffices to show $\phi_l(u \otimes x) \subseteq u$ and $\phi_l(x \otimes u) \subseteq u$ for $x = F_i^{(l)}, E_i^{(l)}$ with $i = 1, \cdots, n$. It follows from Lusztig [14] that

$$\Delta(E_i^{(l)}) = \sum_{b=0}^{l} v^{d_i b(l-b)} E_i^{(l-b)} K_i^b \otimes E_i^{(b)}.$$

Thus we have, for each $x \in u$,

$$\begin{aligned} \phi_l(x \otimes E_i^{(l)}) &= \sum_{b=0}^{l} v^{d_i b(l-b)} \gamma(E_i^{(l-b)} K_i^b) x E_i^{(b)} \\ &= \gamma(E_i^{(l)}) x + K_i^l x E_i^{(l)} + \sum_{b=1}^{l-1} v^{d_i b(l-b)} \gamma(E_i^{(l-b)} K_i^b) x E_i^{(b)}. \end{aligned}$$

The last term is in $u$. Note that $\gamma(E_i^{(l)}) = (-1)^l v^{d_i l(l-1)} K_i^{-l} E_i^{(l)}$, $v$ is an $l$-th root of 1, $l$ is odd, and $K_i^{2l} = 1$. Thus we have $\gamma(E_i^{(l)}) = -K^l E_i^{(l)}$ and $\phi_l(x \otimes E_i^{(l)}) \in u$ if and only if $[x, E_i^{(l)}] = x E_i^{(l)} - E_i^{(l)} x \in u$ since $K^l \in u$. But this has been proved by Lusztig in [14] (8.5). A similar argument will show $\phi_l(x \otimes F_i^{(l)}) \in u$ by using the formulas

$$\begin{aligned} \Delta(F_i^{(l)}) &= \sum_{a=0}^{l} v^{d_i a(l-a)} F_i^a \otimes K_i^{-a} F_i^{(l-a)}, \\ \gamma(F_i^{(l)}) &= (-1) v^{-d_i l(l-1)} F_i^{(l)} K_i^l, \end{aligned}$$

keeping in mind that $K_i^l$ is central in $U_A$. ∎

**5.4.** For notational simplicity, we drop the subscript $A$ and denote $U = U_A$. Our goal is to prove that the induction functors from $u$ to $U$ and to $uU^{\leq 0}$ are exact. So

we can apply 4.7 in this case. Since the admissible category for $U$ is the category of all integral $U$-modules of type **1**, on which $K_i^l$ acts as identity, the ideal generated by $\{K_i^l - 1\}$ is a Hopf ideal of $U$ and is contained in every ideal in $\mathcal{I}$. Therefore we will consider the Hopf algebra $\tilde{U} = U/ < K_i^l - 1 \mid i = 1, \cdots n >$ and its Hopf subalgebra $\tilde{u}$, which is the image of $u$ in $\tilde{U}$. These algebras have been discussed in [13] and they have the following properties:

(i) $\tilde{u} \cong \tilde{u}^+ \otimes \tilde{u}^0 \otimes \tilde{u}^-$ as $A$-modules, therefore it has basis

$$\prod E_\alpha^{(n_\alpha)} \prod K_{i,t_i} \prod F_\alpha^{(m_\alpha)} \quad (0 \leq t_i, n_\alpha, m_\alpha \leq l-1),$$

and $\tilde{U} \cong \tilde{U}^+ \otimes \tilde{U}^0 \otimes \tilde{U}^-$ as $A$-modules with basis

$$\prod E_\alpha^{(n_\alpha)} \prod K_{i,t_i} \prod F_\alpha^{(m_\alpha)} \quad (0 \leq t_i, n_\alpha, m_\alpha).$$

Here, $K_{i,t} = K_i^{-t\begin{bmatrix} K_i \\ t \end{bmatrix}}$ and $\begin{bmatrix} K_i \\ t \end{bmatrix} = \begin{bmatrix} K_i;0 \\ t \end{bmatrix}$. (See [1, 13, 14] for notations in $U$.)

(ii) $\tilde{u}$ is a normal Hopf subalgebra of $\tilde{U}$ since the quotient map $U \to \tilde{U}$ is a surjective map of Hopf algebras and $\bar{\phi_r}(\bar{h} \otimes \bar{d}) = \overline{\phi_r(h \otimes d)}$ for all $h \in U$ and $d \in u$.

(iii) Let $G = \tilde{U}/\tilde{u}$. The ideal $I_{\tilde{u}}$ of $\tilde{u}$ is generated by $\{E_\alpha, F_\alpha, K_i - 1\}$. Therefore the ideal $\tilde{U}I_{\tilde{u}}$ is the ideal in $\tilde{U}$ generated by $\{E_\alpha, F_\alpha, K_i - 1\}$ as well. It follows from Lusztig [14] that $G$ is isomorphic to the distribution algebra of the simply connected semisimple algebraic group scheme defined over $A$ with the same root system. $G = G^+ \otimes G^0 \otimes G^-$ has a basis $\left\{\prod E_\alpha^{(n_\alpha)} \prod K_{i,t_i} \prod F_\alpha^{(m_\alpha)} \mid 0 \leq t_i, n_\alpha, m_\alpha\right\}$. The quotient map $\pi : \tilde{U} \to G$ is given by $\pi(E_i^{(nl)}) = E_i^{(n)}$, $\pi(K_{i,lt_i}) = K_{i,t_i}$, $\pi(F_i^{(lm)}) = F_i^{(m)}$ and $\pi(E_i^{(n)}) = \pi(F_i^{(n)}) = 0$ if $l \nmid n$. With respect to the basis given above, we can define an $A$-linear map $\psi : G \to \tilde{U}$ such that

$$\psi(\prod E_\alpha^{(n_\alpha)} \prod K_{i,t_i} \prod F_\alpha^{(m_\alpha)}) = \prod E_\alpha^{(ln_\alpha)} \prod K_{i,lt_i} \prod F_\alpha^{(lm_\alpha)}.$$

It is clear that $\pi \circ \psi = 1_G$.

**5.5. Theorem.** *The multiplication map $m : \tilde{u} \otimes \psi(G) \to \tilde{U}$ is an isomorphism of $A$-modules.*

*Proof:* By considering the bases, we have the following identifications of $A$-modules.

$$\begin{aligned} \psi(G) &= \psi(G^-) \otimes \psi(G^0) \otimes \psi(G^+), \\ \tilde{u} &= \tilde{u}^+ \otimes \tilde{u}^0 \otimes \tilde{u}^-, \\ \tilde{U} &= \tilde{U}^- \otimes \tilde{U}^0 \otimes \tilde{U}^+. \end{aligned}$$

Lusztig has shown in [14] (8.8) that the multiplication map gives an isomorphism of $A$-modules $\psi(G^+) \otimes \tilde{u}^+ \to \tilde{U}^+$. On the other hand, the multiplication maps give rise to the isomorphisms

$$\tilde{U}^+ \otimes \tilde{u}^0 \xrightarrow{m} \tilde{U}^+\tilde{u}^0, \tilde{u}^0\tilde{U}^+ \xleftarrow{m} \tilde{u}^0 \otimes \tilde{U}^+.$$

Similarly, we have the isomorphisms: $\tilde{u}^- \otimes \tilde{U}^{\geq 0} \xrightarrow{m} \tilde{u}^-\tilde{U}^{\geq 0} = \tilde{U}^{\geq 0}\tilde{u}^- \xleftarrow{m} \tilde{U}^{\geq 0} \otimes \tilde{u}^-$. Using the commutativity of the algebra $\tilde{U}^0$, we can show that the multiplication map $\psi(G^0) \otimes \tilde{u}^0 \to \tilde{U}^0$ is an isomorphism of $A$-algebras. Putting all the above isomorphisms together one can get the isomorphism in the theorem. ∎

Let $G_1$ be one of the subalgebras $G^0$, $G^+$, $G^-$, $G^{\geq 0}$ and $G^{\leq 0}$ of $G$ and $\tilde{U}_1$ be one the subalgebras $\tilde{U}^0$, $\tilde{U}^+$, $\tilde{U}^-$, $\tilde{U}^{\geq 0}$ and $\tilde{U}^{\leq 0}$ of $\tilde{U}$ corresponding to $G_1$ (i.e., with the same superscript). By considering the restriction of the isomorphism in the theorem we get

**Corollary.** *The multiplication map $m : \tilde{u} \otimes \psi(G_1) \to \tilde{u}\psi(G_1) = \tilde{u}\tilde{U}_1$ is an isomorphism of $A$-modules.* ∎

**5.6.** Let $G_1$ and $\tilde{U}_1$ be as above. Let the PBW type basis be taken according to the tensor product decompositions $G = G^- \otimes G^0 \otimes G^+$ and $\tilde{U} = \tilde{U}^- \otimes \tilde{U}^0 \otimes \tilde{U}^+$ with the same chosen order in the set of positive roots for both $G$ and $\tilde{U}$. It follows from the definition of $\psi$ that $\psi(G_1) \subseteq \tilde{U}_1$ and $\psi$ is compatible with the tensor product decompositions (i.e., $\psi$ has a tensor product decomposition too).

**Lemma.** *The restriction $\psi|_{G_1}$ is a homomorphism of algebras.*

*Proof:* If $G_1 = G^+$ (or $G^-$), this follows from ([14] 8.6). If $G_1 = G^0$ and $\tilde{U}_1 = \tilde{U}^0$, which are commutative, we can use the identity of ([13] 6.2(c))

$$K_{i,t}K_{i,t'} = \sum_{j=0}^{t'} (-1)^{j+1} v^{-tj} \begin{bmatrix} t+j-1 \\ j \end{bmatrix}_{d_i} K_{i,t}K_{i,t'-j} + v^{-tt'} \begin{bmatrix} t+t' \\ t \end{bmatrix}_{d_i} K_{i,t+t'}$$

in $U$ over $\mathbb{Z}[v, v^{-1}]$. In $G$, the above identity becomes

$$\begin{bmatrix} K_i \\ t \end{bmatrix} \begin{bmatrix} K_i \\ t' \end{bmatrix} = \sum_{j=0}^{t'} (-1)^{j+1} \binom{t+j-1}{j} \begin{bmatrix} K_i \\ t \end{bmatrix} \begin{bmatrix} K_i \\ t'-j \end{bmatrix} + \binom{t+t'}{t} \begin{bmatrix} K_i \\ t+t' \end{bmatrix}$$

by taking $v = 1$ and $K_i = 1$ into account. Similarly in $\tilde{U}$ we have

$$\begin{bmatrix} K_i \\ tl \end{bmatrix} \begin{bmatrix} K_i \\ t'l \end{bmatrix} = \sum_{j=0}^{t'l} (-1)^{j+1} \binom{t+j-1}{j} \begin{bmatrix} K_i \\ tl \end{bmatrix} \begin{bmatrix} K_i \\ (t'-j)l \end{bmatrix} + \binom{t+t'}{t} \begin{bmatrix} K_i \\ (t+t')l \end{bmatrix}$$

by identifying $v^l = 1$ and $K_i^l = 1$ and using the fact that $l$ is odd. However, the above identities determine the multiplications in $G^0$ and $\tilde{U}^0$ respectively. Therefore $\psi$ preserves the multiplications in this situation.

Recall from ([13] 2.3 (g9), (g10)), we have the following identities in $U$ over $\mathbb{Z}[v, v^{-1}]$

$$\begin{aligned}\left[\begin{smallmatrix} K_i;-c \\ t \end{smallmatrix}\right] &= \sum_{j=0}^{t}(-1)^j v^{c(t-j)}\left[\begin{smallmatrix} c+j-1 \\ j \end{smallmatrix}\right]_{d_i} K^j \left[\begin{smallmatrix} K_i \\ t-j \end{smallmatrix}\right], \\ \left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] &= \sum_{j=0}^{t} v^{c(t-j)}\left[\begin{smallmatrix} c \\ j \end{smallmatrix}\right]_{d_i} K^{-j} \left[\begin{smallmatrix} K_i \\ t-j \end{smallmatrix}\right]\end{aligned}$$

for $t, c \geq 0$ ($c > 0$ in the first identity). Similarly we can get the corresponding formulas in $G$ and $\tilde{U}$ as follows: in $G$ we have

$$\begin{aligned}\left[\begin{smallmatrix} K_i;-c \\ t \end{smallmatrix}\right] &= \sum_{j=0}^{t}(-1)^j \binom{c+j-1}{j} \left[\begin{smallmatrix} K_i \\ t-j \end{smallmatrix}\right], \\ \left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right] &= \sum_{j=0}^{t}\binom{c}{j} \left[\begin{smallmatrix} K_i \\ t-j \end{smallmatrix}\right]\end{aligned}$$

and in $\tilde{U}$ we have

$$\begin{aligned}\left[\begin{smallmatrix} K_i;-cl \\ tl \end{smallmatrix}\right] &= \sum_{j=0}^{t}(-1)^j \binom{c+j-1}{j} \left[\begin{smallmatrix} K_i \\ (t-j)l \end{smallmatrix}\right], \\ \left[\begin{smallmatrix} K_i;cl \\ tl \end{smallmatrix}\right] &= \sum_{j=0}^{t}\binom{c}{j} \left[\begin{smallmatrix} K_i \\ (t-j)l \end{smallmatrix}\right].\end{aligned}$$

This identities shows that $\psi(\left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix}\right]) = \left[\begin{smallmatrix} K_i;lc \\ lt \end{smallmatrix}\right]$ for all $c \in \mathbf{Z}$.

Let $G_1 = G^{\geq 0}$ (or $G^{\leq 0}$). Since $\psi$ is multiplicative restricted to $G^0$ and $G^+$ (or $G^-$), using $E_j^{(c)}\left[\begin{smallmatrix} K_i \\ t \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_i;-ca_{ij} \\ t \end{smallmatrix}\right] E_j^{(c)}$ one can see that $\psi_{G_1}$ is multiplicative. ∎

**5.7.** Let $X$ be the weight lattice of the root system $R$ of $\underline{g}$ and $S = \{\alpha_1, \cdots, \alpha\}$ the set of simple roots in $R$. Each $\lambda \in X$ defines an algebra homomorphism (still denoted by $\lambda$) $\lambda : \tilde{U}^0 \to A$ such that $\lambda(K_i) = v^{d_i\langle\lambda, \alpha_i^\vee\rangle}$ and $\lambda(\left[\begin{smallmatrix} K_i \\ l \end{smallmatrix}\right]) = \left[\begin{smallmatrix} \langle\lambda,\alpha_i^\vee\rangle \\ l \end{smallmatrix}\right]_{d_i}$. Let $I(\lambda)$ be the kernel of $\lambda$. Then $I(\lambda)$ is generated by

$$\left\{K_i - v^{d_i\langle\lambda,\alpha_i^\vee\rangle}, \left[\begin{smallmatrix} K_i \\ l \end{smallmatrix}\right] - \left[\begin{smallmatrix} \langle\lambda,\alpha_i^\vee\rangle \\ l \end{smallmatrix}\right]_{d_i} \mid i = 1, \cdots, n\right\}.$$

If $M$ is a $\tilde{U}^0$-module and $\lambda \in X$, set $M_\lambda = \{m \in M \mid hm = \lambda(h)m \text{ for all } h \in \tilde{U}^0\}$. Then $\sum_{\lambda\in X} M_\lambda$ is a $\tilde{U}^0$-submodule of $M$. $M$ is said to have a weight space decomposition if $M = \sum_{\lambda\in X} M_\lambda$. If $M$ is $A$-flat, then $\sum_{\lambda\in X} M_\lambda = \oplus_{\lambda\in X} M_\lambda$. If $A$ is a field and $\dim_A M$ is finite, the character of $M$ is defined as $\mathrm{ch}M = \sum_\lambda \dim_A M_\lambda e(\lambda)$, which is an element of the group ring $\mathbb{Z}[X]$ with basis $\{e(\lambda) \mid \lambda \in X\}$.

For a finite set $\{\lambda_1, \cdots, \lambda_s\}$ of weights, set $I(\lambda_1, \cdots, \lambda_s) = I(\lambda_1) \cap I(\lambda_2) \cap \cdots \cap I(\lambda_s)$. A $\tilde{U}^0$-module $M$ which is finitely generated over $A$ has a weight space decomposition if and only if $I(\lambda_1, \ldots, \lambda_s) \subseteq \mathrm{Ann}_{\tilde{U}^0}(M)$ for some finite set $\{\lambda_1, \ldots, \lambda_s\}$. Let $\mathcal{I}$ be the admissible set of ideals of $\tilde{U}$ corresponding to the category of integrable $\tilde{U}$-modules. It is proved in [1] (1.29) that $I \in \mathcal{I}$ if and only if it contains some $I(\lambda_1, \cdots, \lambda_s)$ and $E_i^{(n)}$, $F_i^{(m)}$ for all $n > n_I$, $m > m_I$ and $i = 1, \cdots, n$ with some fixed $n_I, m_I$.

Consider the map $\pi : \tilde{U} \to G$. Then $\pi(\begin{bmatrix} K_i \\ ml \end{bmatrix}) = \begin{bmatrix} K_i \\ m \end{bmatrix} = \binom{H_i}{m}$ with $H_i = E_iF_i - F_iE_i$ in $G$. Similarly, for each integral weight $\lambda$ we define the ideal $I''(\lambda)$ of $G^0$ as the kernel of the algebra homomorphism $\lambda : G^0 \to A$ with $\lambda(\begin{bmatrix} K_i \\ t \end{bmatrix}) = \binom{\langle \lambda, \alpha^\vee \rangle}{t}$ and $I''(\lambda_1, \cdots, \lambda_s) = I''(\lambda_1) \cap \cdots \cap I''(\lambda_s)$. Let $\mathcal{I}''$ be the flat admissible set of ideals in $G$ corresponding to the category of rational $G$-modules. Then following [7] (II 1.20), one has the similar results that $I'' \in \mathcal{I}''$ if and only if $I''$ contains $I''(\lambda_1, \cdots, \lambda_s)$ and all $E_i^{(n)}$ and $F_i^{(m)}$ with $n > n_{I''}$ and $m > m_{I''}$ for some integral weights $\lambda_1, \cdots, \lambda_s$ and some integers $n_{I''}$ and $m_{I''}$. Let $X_l = \{\mu \in X \mid 0 \leq \langle \mu, \alpha_i^\vee \rangle < l, i = 1, \cdots, n\}$. Each $\lambda \in X$ admits a decomposition $\lambda = l\lambda^1 + \lambda^0$ with $\lambda^0 \in X_l$ and $\lambda^1 \in X$.

**Lemma.** *The sets $\mathcal{I}''$ and $\pi(\mathcal{I})$ of ideals in $G$ are equivalent.*

*Proof:* If $\lambda$ is an integral weight, the ideal $I(\lambda)$ contains the ideal of $\tilde{U}^0$ generated by $\left\{ \begin{bmatrix} K_i \\ l \end{bmatrix} - \langle \lambda^1, \alpha_i^\vee \rangle \right\}$. Here $\begin{bmatrix} \langle \lambda, \alpha_i^\vee \rangle \\ l \end{bmatrix}_{d_i} = \langle \lambda^1, \alpha_i^\vee \rangle$. Thus we have $\psi(I''(\lambda_1)) \subseteq I(\lambda)$ since $\psi|_{G^0}$ is a homomorphism of algebras. Hence $\psi(I''(\lambda_1^1, \cdots, \lambda_s^1)) \subseteq I(\lambda_1, \cdots, \lambda_s)$ and

$$I''(\lambda_1, \cdots, \lambda_s) \subseteq \pi\psi(I''(\lambda_1^1, \cdots, \lambda_s^1)) \subseteq \pi(I(\lambda_1, \cdots, \lambda_s)).$$

Now suppose $I \in \mathcal{I}$. There exist $\lambda_1, \cdots, \lambda_s$ and $n_I$, $m_I$ such that $I(\lambda_1, \cdots, \lambda_s) \subseteq I$ and $E_i^{(n)}, F_i^{(m)} \in I$ for all $n > n_I$, $m > m_I$ and $i$. If $I''$ is the ideal of $G$ generated by $I''(\lambda_1^1, \cdots, \lambda_s^1)$ and $\{E_i^{(n)}, F_i^{(m)} \mid nl > n_I, ml > m_I\}$, then $I'' \subseteq \pi(I)$. Conversely, if $I'' \in \mathcal{I}''$ contains $I''(\lambda_1, \cdots, \lambda_s)$ and $\{E_i^{(n)}, F_i^{(m)} \mid n > n_{I''}, m > m_{I''}\}$, we set $I$ equal to the ideal of $\tilde{U}$ generated by $I(l\lambda_1, \cdots l\lambda_s)$ and $\{E_i^{(n)}, F_i^{(m)} \mid n > ln_{I''}, m > lm_{I''}\}$. Then $I \in \mathcal{I}$ and $\pi(I) \subseteq I''$. ∎

**5.8.** Let $G_1$ and $\tilde{U}_1$ be as in 5.6. The admissible sets $\mathcal{I}_1''$ and $\mathcal{I}_1$ of ideals in $G_1$ and $\tilde{u}\tilde{U}_1$ inherited from $\mathcal{I}''$ and $\mathcal{I}$ respectively can be described as follows:

(i) If $G_1 = G^+$ (or $G^-$) and $\tilde{U}_1 = \tilde{U}^+$ (or $\tilde{U}^-$), then $\mathcal{I}_1''$ and $\mathcal{I}_1$ are the sets of all $A$-cofinite ideals.

(ii) If $G_1 = G^0$ and $\tilde{U}_1 = \tilde{U}^0$, then $\mathcal{I}_1''$ consists of all $I''(\lambda_1, \cdots, \lambda_s)$ and $\mathcal{I}_1$ consists of all $I(\lambda_1, \cdots, \lambda_s)$.

(iii) If $G_1 = G^{\geq 0}$ (or $G^{\leq 0}$) and $\tilde{U}_1 = \tilde{U}^{\geq 0}$ (or $\tilde{U}^{\leq 0}$), then $\mathcal{I}_1''$ consists of all ideals $I''$ of $G_1$, containing some $I''(\lambda_1, \cdots, \lambda_s)$ and $\{E_i^n, \mid n > n_{I''}\}$ (or $\{F_i^n, \mid n > n_{I''}\}$). The set $\mathcal{I}_1$ for $\tilde{U}_1$ can be described similarly.

A similar argument as in 5.7 can show that $\pi_1(\mathcal{I}_1)$ and $\mathcal{I}_1''$ are equivalent. Here $\pi_1 : G_1 \to \tilde{U}_1$ is the restriction map of $\pi$.

**Lemma.** *If $I \in \mathcal{I}_1$, then there exists $I'' \in \mathcal{I}_1''$ such that $\psi(I'') \subseteq I$.*

*Proof:* Let $G_1 = G^+$ (or $G^-$) and $\tilde{U}_1 = \tilde{U}^+$ (or $\tilde{U}^-$). If $I \in \mathcal{I}_1$ contains $\{E_i^{(n)} \mid n > n_I\}$ (or $\{F_i^{(n)} \mid n > n_I\}$ ), consider the ideal $I''$ generated by the set $\{E_i^{(n)} \mid ln > n_I\}$ (or $\{F_i^{(n)} \mid ln > n_I\}$ ). It is clear that the image of the generating set for $I''$ under $\psi$ is contained in $I$. Then $I \supseteq \psi(I'')$ since $\psi$ is a homomorphism of algebras from $G_1$ to $\tilde{u}\tilde{U}_1$. For $G_1 = G^0$, this has been proved in 5.6 and (ii). For $G_1 = G^{\geq 0}$ (or $G^{\leq 0}$), the lemma follows from the description of the sets $\mathcal{I}_1''$ and $\mathcal{I}_1$ in (iii) and a similar argument using the fact that $\psi_{G_1}$ is a homomorphism of algebras. ∎

**5.9. Theorem.** *If $\tilde{U}_1$ is one of the subalgebras $\tilde{U}^0$, $\tilde{U}^{\geq 0}$ and $\tilde{U}^{\leq 0}$ of $\tilde{U}$, then the induction functor $\mathrm{Ind}_{\tilde{u}}^{\tilde{u}\tilde{U}_1}$ is exact.*

*Proof:* It follows from Corollary 5.5 that the map $\mathrm{Hom}_{\tilde{u}}(\tilde{u}\tilde{U}_1, M) \to \mathrm{Hom}_A(G_1, M)$, given by $f \mapsto \tilde{f}$ with $\tilde{f}(g) = f(\psi(g))$, is a natural isomorphism as $A$-modules for each $\tilde{u}$-module $M$. Denote by $C_1$ and $C_1''$ the functors for $\tilde{u}\tilde{U}_1$ and $G_1$ respectively. We now show that $f \in C_1\mathrm{Hom}_{\tilde{u}}(\tilde{u}\tilde{U}_1, M)$ if and only if $\tilde{f} \in C_1''\mathrm{Hom}_A(G_1, M)$. In fact, by Lemma 5.8, $\tilde{f} \in C_1''\mathrm{Hom}_A(G_1, M)$ if $f \in C_1\mathrm{Hom}_{\tilde{u}}(\tilde{u}\tilde{U}_1, M)$. Conversely, if $\tilde{f}(I'') = 0$ for some $I'' \in \mathcal{I}_1''$, $\tilde{u}\psi(I'')$ is a cofinite left ideal of $\tilde{u}\tilde{U}_1$ and the quotient $\tilde{u}\tilde{U}_1$-module $\tilde{u}\tilde{U}_1/(\tilde{u}\psi(I''))$ is in $\mathcal{C}_1$. Since

$$(\tilde{u}\psi(I'')f)(\tilde{u}\tilde{U}_1) = f(\tilde{u}\tilde{U}_1\tilde{u}\psi(I'')) = f(\tilde{u}\psi(I'')) = \tilde{u}f(\psi(I'') = 0,$$

we have $f \in C_1\mathrm{Hom}_{\tilde{u}}(\tilde{u}\tilde{U}_1, M)$. Note that $C_1''\mathrm{Hom}_A(G_1, M) = H_{\mathcal{I}_1''}(M)$. This shows that the two functors $\mathrm{Ind}_{\tilde{u}}^{\tilde{u}\tilde{U}_1}$ and $H_{\mathcal{I}_1''}$ are isomorphic. Thus the theorem follows from the exactness of the latter one (cf. 1.5). ∎

# 6 Structure of higher derived functors

**6.1.** Throughout this section, we assume that $A$ is a field of characteristic 0 in addition to the assumption in 5.3. We will consider the induction for the following pairs of Hopf subalgebras of the quantum algebra $\tilde{U}$: $(\tilde{u}^{\leq 0}, \tilde{u})$, $(\tilde{u}^{\leq 0}\tilde{U}^0, \tilde{u}\tilde{U}^0)$ and $(\tilde{U}^{\leq 0}, \tilde{u}\tilde{U}^{\leq 0})$. For each Hopf subalgebra mentioned here, we consider the admissible category inherited from the admissible category of $\tilde{U}$-modules.

**Lemma.** *Let $(\tilde{U}_1, \tilde{U}_2)$ be one of the pairs of subalgebras of $\tilde{U}$ mentioned above. Then*

(i) $\tilde{U}_2 = \tilde{U}_1 \otimes \tilde{u}^+$ *as $A$-modules.*

(ii) *With the given admissible categories $\mathcal{C}_i$ and the functors $C_i$, $C_1(M) = C_2(M)$ for all $\tilde{U}_2$-modules $M$.*

*Proof:* (i) follows from the tensor product decomposition $\tilde{u} = \tilde{u}^- \otimes \tilde{u}^0 \otimes \tilde{u}^+$. To show (ii), one only need to show $C_1(M) \subseteq C_2(M)$ for each $\tilde{U}_2$-module $(M)$. For each $m \in C_1(M)$, let $N = \tilde{U}_1 m$ be the $\tilde{U}_1$-submodule generated by $m$. Then the $\tilde{U}_2$-submodule generated by $m$ is $\tilde{U}_2 m = \tilde{u}^+ N$, which is $A$-finite. In the case $(\tilde{U}_1, \tilde{U}_2) = (\tilde{u}^{\leq 0}, \tilde{u})$ we are done since the categories consist of all locally $A$-finite modules. For the remaining two situations, we have $\tilde{U}^0 \subseteq \tilde{U}_1 \subseteq \tilde{U}_2$. An $A$-finite $\tilde{U}_i$-module is in $\mathcal{C}_i$ if and only if it has a weight space decomposition. In our case $C_1(M) \subseteq \oplus_\mu M_\mu$. Here $M_\mu$ is the weight space of $\mu$ in $M$. Following [11] (5.2), $\oplus_\mu M_\mu$ is a $\tilde{U}_2$-submodule of $M$. Therefore we have $\tilde{U}_2 m \subseteq \oplus_\mu M_\mu$ and $\tilde{U}_2 m$ is in $\mathcal{C}_2$. ∎

**Corollary. (APW [2] 1.3)** *The induction functor $\mathrm{Ind}_{\tilde{U}_1}^{\tilde{U}_2}$ is exact.*

*Proof:* It follows from (i) of the lemma that $\tilde{U}_2$ is a free $\tilde{U}_1$-module. Since we have assumed that $A$ is a field, the corollary follows from 3.8. ∎

**6.2.** For each integral weight $\lambda$, there is a 1-dimensional $\tilde{U}^{\leq 0}$-module $A_\lambda$, on which $\tilde{U}^0$ acts through the character $\lambda$ and $\tilde{U}^-$ acts trivially. Thus $A_\lambda$ is a $\tilde{U}_1$-module through restriction. It follows from (i) in Lemma 6.1 that $\mathrm{Hom}_{\tilde{U}_1}(\tilde{U}_2, A_\lambda) \cong \mathrm{Hom}_A(\tilde{u}^+, A)$, which is finite-dimensional. In case $(\tilde{U}_1, \tilde{U}_2)$ is of the last two situations, we can choose the dual basis of $\tilde{u}^+$ corresponding to a PBW type basis of $\tilde{u}^+$ and use the commutation relations

$$E_j^{(n)} K_i = v^{-d_i a_{ij} n} K_i E_j^{(n)} \text{ and } E_j^{(n)} \begin{bmatrix} K_i \\ l \end{bmatrix} = \begin{bmatrix} K_i; -na_{ij} \\ l \end{bmatrix} E_j^{(n)}$$

to give a weight space decomposition to $\mathrm{Hom}_{\tilde{U}_1}(\tilde{U}_2, A_\lambda)$ as a $\tilde{U}^0$-module via the restriction. Recall that the $\tilde{U}_2$-structure is given by $(yf)(x) = f(xy)$ for each $y \in \tilde{U}_2$, $f \in \mathrm{Hom}_{\tilde{U}_1}(\tilde{U}_2, A_\lambda)$. This shows $\mathrm{Ind}_{\tilde{U}_1}^{\tilde{U}_2} A_\lambda = \mathrm{Hom}_{\tilde{U}_1}(\tilde{U}_2, A_\lambda)$ , which has character

$$e(\lambda)\mathrm{ch}(\mathrm{Ind}_{\tilde{U}_1}^{\tilde{U}_2} A) = e(\lambda) \prod_{\alpha \in R^+} \frac{1 - e(-l\alpha)}{1 - e(-\alpha)}.$$

Furthermore one can show that $\mathrm{Ind}_{\tilde{u}^{\leq 0}}^{\tilde{u}}(A_\lambda) \cong \mathrm{Ind}_{\tilde{u}^{\leq 0}\tilde{U}^0}^{\tilde{u}\tilde{U}^0}(A_\lambda) \cong \mathrm{Ind}_{\tilde{U}^{\leq 0}}^{\tilde{u}\tilde{U}^{\leq 0}}(A_\lambda)$ as $\tilde{u}$-modules and $\tilde{u}\tilde{U}^0$-modules respectively.

By [17], all simple $\tilde{u}\tilde{U}^0$-modules are parameterized by $X$. For each $\lambda \in X$ we denote by $\hat{L}_1(\lambda)$ the simple $\tilde{u}\tilde{U}^0$-module of highest weight $\lambda$. The set of simple $\tilde{u}$-modules are parameterized by $X_l$. For each $\lambda \in X_l$ the simple $\tilde{U}$-module $L(\lambda)$ (we will use $L(\mu)$

to denote the simple $\tilde{U}$-module of the dominant highest weight $\mu \in X$) is still simple when restricted to $\tilde{u}$ ([11] (7.1)). In general, each $\lambda \in X$ admits a decomposition $\lambda = \lambda^0 + l\lambda^1$ with $\lambda^0 \in X_l$ and $\lambda^1 \in X$. Then one has $\hat{L}_1(\lambda) = l\lambda^1 \otimes L(\lambda^0)$, which has a $\tilde{u}\tilde{U}^{\leq 0}$-module structure and is $\tilde{u}\tilde{U}^{\leq 0}$-simple also. All simple $\tilde{u}\tilde{U}^{\leq 0}$-modules are obtained in this way. By the Frobenius reciprocity,

$$\mathrm{Hom}_{\tilde{U}^2}(\hat{L}_1(\mu), \mathrm{Ind}_{\tilde{U}_1}^{\tilde{U}_2}(A_\lambda)) = \mathrm{Hom}_{\tilde{U}_1}(\hat{L}_1(\mu), A_\lambda)$$

and a standard highest weight theory argument, one can show that $\mathrm{Ind}_{\tilde{U}_1}^{\tilde{U}_2}(A_\lambda)$ has a unique simple submodule $\hat{L}_1(\lambda)$.

**6.3.** The simple $\tilde{U}$-module $L(\lambda)$ is still simple when restricted to $\tilde{u}$ if $\lambda \in X_l$. Specially, $L((l-1)\rho)$ is simple as $\tilde{u}$-module, which is a quotient module of $Z((l-1)\rho) = \tilde{u} \otimes_{\tilde{u}^{\leq 0}} A_{(l-1)\rho}$ [17]. Now it follows from APW [1] that $L((l-1)\rho) = St = \mathrm{Ind}_{\tilde{U}^{\leq 0}}^{\tilde{U}}((l-1)\rho)$, which has the character of "Steinberg module" and is injective as $\tilde{U}$-module [1](9.10). Comparing the dimensions we have $L((l-1)\rho) = Z((l-1)\rho)$, which is simple as $\tilde{u}$-module. Following [17] (2.16) we consider a $Z$-filtration of the $\tilde{u}$-injective hull of $L((l-1)\rho)$ and conclude that $Z((l-1)\rho)$ is injective as $\tilde{u}$-module. Thus we have shown that the injective $\tilde{U}$-module $St = \mathrm{Ind}_{\tilde{U}^{\leq 0}}^{\tilde{U}}((l-1)\rho)$ restricted to $\tilde{u}$ is the injective $\tilde{u}$-module $Z((l-1)\rho)$.

**Proposition. (APW [2] 4.8)** *The induction functor* $\mathrm{Ind}_{\tilde{u}}^{\tilde{U}}$ *is exact.*

*Proof:* This follows from 3.7. ∎

If we set $(H, D, G) = (\tilde{U}, \tilde{u}, G)$ in 4.9, all the conditions we assumed there are satisfied. (The conditions in 4.5 were verified in 5.7.) Denote by $Q_1(\lambda)$ the injective hull of the $\tilde{u}$-module $L_1(\lambda)$ for $\lambda \in X_l$ and $Q(\lambda)$ the injective hull of the $\tilde{U}$-module $L(\lambda)$ for $\lambda \in X_+$.

**Corollary. (APW [2] 4.6)** *If $\lambda \in X_l$, then the injective $\tilde{u}$-module $Q_1(\lambda)$ is the restriction of $Q(\lambda)$ to $\tilde{u}$.* ∎

**6.4. Theorem.** *If $M$ is a $G^{\leq 0}$-module, then there is an isomorphism of $\tilde{U}$-modules* $R^n\mathrm{Ind}_{\tilde{u}\tilde{U}^{\leq 0}}^{\tilde{U}}(\pi_1^*(M)) \cong \pi^*(R^n\mathrm{Ind}_{G^{\leq 0}}^{G}(M))$.

*Proof:* By 5.9, $\mathrm{Ind}_{\tilde{u}}^{\tilde{u}\tilde{U}^{\leq 0}}$ is exact and by the above proposition $\mathrm{Ind}_{\tilde{u}}^{\tilde{U}}$ is exact. Now the theorem follows from 5.7 and 4.7. ∎

**Corollary.** *If $\lambda = \lambda^0 + l\lambda^1$ with $\lambda^0 \in X_l$, and $\hat{L}_1(\lambda) = l\lambda^1 \otimes L(\lambda^0)$ is the simple $\tilde{u}\tilde{U}^{\leq 0}$-module, then*

$$R^n\mathrm{Ind}_{\tilde{u}\tilde{U}^{\leq 0}}^{\tilde{U}}\hat{L}_1(\lambda) = \begin{cases} L(\lambda^0 + lw\cdot\lambda^1) & \text{if } w\cdot\lambda^1 \in X_+ \text{ and } n = l(w) \text{ for some } w \in W, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof:* Here we use the generalized tensor identity and the above formula keeping in mind that $R^n\mathrm{Ind}_{G^{\leq 0}}^{G}(\lambda^1) = L(w\cdot\lambda^1)$ if $w\cdot\lambda^1 \in X_+$ and $n = l(w)$ for some $w \in W$ and 0 otherwise since the characteristic of the field is 0. Then we use Lusztig 's tensor product theorem. ∎

**6.5.** It follows from APW [1] (1.13) (or rather its proof) that the characters of all finite dimensional $\tilde{U}$-module are $W$-invariant in the group ring $\mathbb{Z}[X]$. One can also use the argument of Lusztig that the character of each irreducible module is a $\mathbb{Z}$-linear combination of the characters of the universal highest weight modules, whose characters are given by Weyl's character formula [10]. For notational simplicity, we denote $\hat{u} = \tilde{u}\tilde{U}^0$. So $\hat{L}_1(\lambda)$ is the simple $\hat{u}$-module with highest weight $\lambda$. For a module $M$ of finite length, and a simple module $L$ in an appropriate category, we denote by $[M : L]$ the multiplicity of $L$ in $M$.

**Proposition.** *Let $M$ be a finite dimensional $\tilde{U}$-module. Then*

(i) $\mathrm{Soc}^i_{\tilde{u}}M = \mathrm{Soc}^i_{\tilde{U}}M = \mathrm{Soc}^i_{\hat{u}}M$ *for* $i \geq 1$.

(ii) *For any $w \in W$, and $\lambda \in X$,*

$$\left[\mathrm{Soc}^i_{\tilde{u}}M/\mathrm{Soc}^{i-1}_{\tilde{u}}M : \hat{L}_1(\lambda^0 + l\lambda^1)\right] = \left[\mathrm{Soc}^i_{\tilde{u}}M/\mathrm{Soc}^{i-1}_{\tilde{u}}M : \hat{L}_1(\lambda^0 + lw\lambda^1)\right].$$

(iii) *For any finite dimensional $\hat{u}$-module $N$,* $\mathrm{Soc}^i_{\tilde{u}}N = \mathrm{Soc}^i_{\hat{u}}N$.

*Proof:* Note that $\hat{u}/\tilde{u}$ is isomorphic to the Hopf algebra $G^0$ and all rational $G^0$-modules are semisimple. Thus (i) and (iii) follows from Lemma 4.9. To show (ii), let us recall that

$$\mathrm{Soc}_{\tilde{u}}M = \sum_{\lambda^0\in X_l} \mathrm{Hom}_{\tilde{u}}(L_1(\lambda^0), M)\otimes L_1(\lambda^0).$$

Here $\mathrm{Hom}_{\tilde{u}}(L_1(\lambda^0), M) = \pi^*(E)$ as $\tilde{U}$-module for some rational $G$-module $E$. If $\mathrm{ch}E = \sum_{\mu\in X} a_\mu e(\mu)$, then $\mathrm{ch}(\pi^*(E)) = \sum_{\mu\in X} a_\mu e(l\mu)$. Since $\mathrm{ch}E$ is $W$ invariant, the proposition follows from

$$[\mathrm{Soc}_{\tilde{u}}M : \hat{L}_1(\lambda^0 + lw(\lambda^1))] = \dim \mathrm{Hom}_{\tilde{u}}(L_1(\lambda^0), M)_{lw(\lambda^1)}.$$ ∎

**6.6.** For each $\lambda \in X$, we denote $\hat{Z}'_1(\lambda) = \mathrm{Ind}^{\tilde{u}\tilde{U}^0}_{\tilde{u}\leq 0\tilde{U}^0}(A_\lambda)$, which is isomorphic to $\mathrm{Ind}^{\tilde{u}\tilde{U}^{\leq 0}}_{\tilde{U}^{\leq 0}}(A_\lambda)$ as $\tilde{u}\tilde{U}^0$-modules. We will denote the latter by $\hat{Z}'_1(\lambda)$ as well. By comparing the dimensions, we have $\mathrm{ch}\hat{Z}'_1((l-1)\rho) = \mathrm{ch}L((l-1)\rho)$ which is $W$-invariant. On the other hand, one can verify that $\mathrm{ch}\hat{Z}'_1(\lambda) = e(\lambda-(l-1)\rho)\mathrm{ch}\hat{Z}'_1((l-1)\rho)$. Following a similar calculation as in Jantzen's book [7] (II, 9.13) we have, for $w \in W$,

(i) $[\hat{Z}'_1(\lambda) : \hat{L}_1(\mu^0 + l\mu^1)] = [\hat{Z}'_1(w\cdot\lambda) : \hat{L}_1(\mu^0 + l(w\cdot\mu^1))]$.

(ii) If $[\hat{Z}_1'(\lambda) : \hat{L}_1(\mu^0 + l\mu^1)] \neq 0$, then $l\mu^1 + \mu^0 \uparrow \lambda \uparrow l(\mu^1 + 2\rho) + w_0(\mu^0)$.

Here $\uparrow$ stands for the strong linkage order defined in $X$ with respect to the integer $l$. The strong linkage principle in the "infinitesimal" level can be derived from 4.6, 6.4 and the strong linkage principle at the "global" level in [1]. One can also find it in [2].

**6.7.** Recall the transitivity of induction functors. It follows from the exactness of $\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{u}\tilde{U}\leq 0}$ that

$$R^i\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(\lambda) = R^i\mathrm{Ind}_{\tilde{u}\tilde{U}\leq 0}^{\tilde{U}}(\hat{Z}_1'(\lambda)).$$

Here we consider $\hat{Z}_1'(\lambda)$ as $\tilde{u}\tilde{U}^{\leq 0}$-module. Since $\tilde{u}$ is a normal Hopf subalgebra and every simple $\tilde{u}$-module is the restriction of a simple $\tilde{u}\tilde{U}^{\leq 0}$-module, for each $i > 0$, $\mathrm{Soc}_{\tilde{u}}^i \hat{Z}_1'(\lambda)$ is a $\tilde{u}\tilde{U}^{\leq 0}$-submodule of $\hat{Z}_1'(\lambda)$ by 4.8(i) and yields a $\tilde{U}$-module $R^j\mathrm{Ind}_{\tilde{u}\tilde{U}\leq 0}^{\tilde{U}}(\mathrm{Soc}_{\tilde{u}}^i \hat{Z}_1'(\lambda))$.

**Theorem.** *Let $\lambda = \lambda^0 + l\lambda^1 \in X_+$ such that $5(h-1) < \langle \lambda^1 + \rho, \alpha^\vee \rangle$ for all positive roots $\alpha$. Then, for all $w \in W$,*

(i) *$R^i\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda) \neq 0$ if and only if $i = l(w)$. Furthermore $\mathrm{ch}R^{l(w)}\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda) = \mathrm{ch}\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(\lambda)$, which is given by Weyl's character formula.*

(ii) *For each $j$,*

$$\mathrm{Soc}_{\tilde{U}}^j R^{l(w)}\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda) = R^{l(w)}\mathrm{Ind}_{\tilde{u}\tilde{U}\leq 0}^{\tilde{U}}(\mathrm{Soc}_{\tilde{u}}^j \hat{Z}_1'(w\cdot\lambda)),$$

$$\mathrm{Rad}_{\tilde{U}}^j R^{l(w)}\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda) = R^{l(w)}\mathrm{Ind}_{\tilde{u}\tilde{U}\leq 0}^{\tilde{U}}(\mathrm{Rad}_{\tilde{u}}^j \hat{Z}_1'(w\cdot\lambda)).$$

(iii) *for $\mu = \mu^0 + l\mu^1 \in X_+$ and $i \geq 0$,*

$$[\mathrm{Soc}_{\tilde{U}}^j R^{l(w)}\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda) : L(\mu)] = [\mathrm{Soc}_{\tilde{U}}^j \mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda^0 + l\lambda^1) : L(\mu^0 + l(w\cdot(\mu^1 - \lambda^1) + \lambda^1))].$$

$$[\mathrm{Rad}_{\tilde{U}}^j R^{l(w)}\mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda) : L(\mu)] = [\mathrm{Rad}_{\tilde{U}}^j \mathrm{Ind}_{\tilde{U}\leq 0}^{\tilde{U}}(w\cdot\lambda^0 + l\lambda^1) : L(\mu^0 + l(w\cdot(\mu^1 - \lambda^1) + \lambda^1))].$$

*Proof:* Following a similar argument as in [7] (II, 11.14), one can show that the assumption in the theorem implies that all $\tilde{u}\tilde{U}^{\leq 0}$-composition factors of $\tilde{Z}'(w\cdot\lambda)$ have the form $\tilde{L}(\mu^0 + lw\cdot(\mu^1))$ with $\mu^1 \in X_+ - \rho$. (i) follows immediately from Corollary 6.4. To prove (ii), one can copy the argument in [9] (Section 2). But one has to be a little bit careful about the proof of 2.4 in that paper since it is not clear if we can have an analogue argument in the quantum group case. However the result of that Lemma is the consequence of 6.6. The argument for the radical series follows from the Serre duality which is proved by APW in [1]. (iii) is a consequence of (ii). ∎

*Remark.* There is a similar argument as in [9] Section 5 to discuss the vanishing properties of these higher derived functors at a weight. These vanishing properties

are much nicer than in the classical case. Finally there is a connection between the structure of the induced modules here and the structure of Weyl modules in generic situation (their highest weights may not be in the bottom $p^2$-alcove) by taking $l = p^r$. This will be discussed else where.

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