LUSZTIG'S GEOMETRIC APPROACH TO HALL ALGEBRAS

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1. Introduction

The introduction of l-adic cohomology by M. Artin and Grothenieck was motivated by the Weil conjecture concerning the number of rational points of algebraic varieties over finite fields. Since its introduction in the early 60s, l-adic cohomology theory has been used in the representation theory of finite Chevalley groups in a "misterious way". The first application of l-adic cohomology and intersection cohomology theory to reductive groups was by Lusztig in 1975 [L3]. The constuction of intersection cohomology complexes for certain algebraic varieties associated to reductive groups leads the celebrating work was by Deligne and Lusztig [DL] constructing a large class of virtual characters for finite Chevalley groups over algebraically field of characteritic zero. The spirit of the application of the l-adic cohomology groups is the Lefschetz fixed point theorem and Grothendieck trace formula. Such an approach will construct the virtual characters for finite Chevalley groups over all finite fields at once. Later on Lusztig in a serieous of papers realized virtual characters in terms simple persverse sheaves, complexes of representations [L4]. The concept of linear representation theory has ever changed, and the algebraic geometry seams to be play an inevitable role. Although the questions conerned are just simple algebraic question, such as computing the irreducible characters. Another celebrating work is Springer's construction of representations of Weyl groups in term of l-adic cohomology groups of certan subvarieties the nilpotent varieties, realize the Young's construction of irreducible representations of symestric groups in terms of partitions. Just as Springer summarized [Sp] *To a diehard algebraist, it may seam disturbing that in order to solve a concrete problem..., one has invoke arcane theories such as l-adic cohomology and intersection cohomology..., whatever simplications the future might bring, the insights brought by these theories are there to stay.*

In this note Lusztig's geometric construction of the Hall algebras are introduced with an amphasis on the connecting the geometric construction and the combinatorial construction. The advantage of the geometric construction has been summarized by Lusztig in his book.

Lusztig's geometric approach was not a just a coincident. Before Ringel [R1] has constructed the Hall algebraic for Dynkin quivers, lusztig has already constructed the homogeneous part of the quiver with one vertex and one loop in [L4]. In this case, the classes of nilpotent representations of dimension n are exactly the set of nilpotent orbits of the Lie algebra $\mathfrak{gl}_n$ and the filtration numbers are used in [R1] are exactly the number of the rantional points of a fiber in a resolution of the orbit. Thus Lusztig's geometric construction first appears in [L5] seams to be a natural generalization of [L4] to the quiver situation.

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The note is organized as follows. In section two, we will reformulate the definition of Hall algebras of a finite quiver in terms of complex valued functions on the set of isomorphism classes of nilpotent representations of the quiver over a finite field. In section 3 we use Lusztig geometric approach to describe the hall algebras in terms of perverse sheaves. In this section we will recall some of the Grothendieck's results on fixed point theorems and trace formulas. Then the geometric hall algebra is defined in terms of the pur complexes of the l-adic sheaves. We define a surjective homomorphism from the geometric Hall algebra to the Hall algebra definted in section 2. Only these result may be new. All other results represented in this notes are either reinterpretation of Lusztig's construction or Ringel's definition. The connection between Ringel's construction and Lusztig's geometric construction are identified only on the composition algebra level by Green and Lusztig, using the either the generators and relations or "invariant" bilinear forms. However for entire Hall algebra of general quivers, Lusztig's geometrick construction of the Hall algebra still needs to proved to have a generic form. However, for each fixed finite field $\mathbb{F}_q$, Deng and Xiao has come up with a description of the hall algebra (see the their survey in this volume, and the references there for other descriptions of Hall algebras). Hower, the relation and generator sets depends on the finite field $\mathbb{F}_q$. There is still a question of constructing generic subalgebras that contains the composition algebra, this will not be discussed in this note. This note is devoted to understanding Lusztig's philosophy in his approach to quantum groups. The cases when Q is a cyclic quiver is rather understood, by Schiffman [Sch], which gives description of canonical bases of the Hall algebra and the canonical bases of q-Fock spaces. Such a correspondence provides new proof to Lusztig's conjecture on the character formulas of quantum groups at roots of 1 in the cases of the type A. See [VV, 11].

2. Definition of Hall Algebra from Finite Fields

2.1. **Representations of quivers.** Let $Q = (I, H)$ be a finite quiver and k be a field. For each arrow $h \in H$, we denote by $h' \xrightarrow{h} h''$ to indicate the source h' and target h'' of h. A representation of Q is an I-graded vector space $\mathbf{V} = \oplus V_i$ together with an element $x = (x_h)_{h\in H} \in E_{\mathbf{V}} = \prod_{h\in H} \mathrm{Hom}_k(V_{h'}, V_{h''})$. We will denote by $M = (\mathbf{V}, x)$ this representation.

A homomorphism $f : M \to N = (\mathbf{W}, y)$ of Q-representation is an I-graded k-linear map $f = \oplus_{i\in I} f_i : \oplus_{i\in I} V_i \to \oplus_{i\in I} W_i$ such that $y_h f_{h'} = f_{h''} x_h$ for all $h \in H$.

Appearently, $(\mathbf{V}, x)$ and $(\mathbf{V}, y)$ define isomorphic representations if and only if there exists $g = (g_i)_{i\in I} \in G_{\mathbf{V}} = \prod_{i\in I} GL(V_i)$ such that $y_h = g_{h''} x_h g_{h'}^{-1}$ for all $h \in H$. In fact the group $G_{\mathbf{V}}$ acts on the set $\prod_{h\in H} \mathrm{Hom}_k(V_{h'}, V_{h''})$ naturally by conjugation $g \cdot x = (g_{h''} x_h g_{h'}^{-1})_{h\in H}$. $(\mathbf{V}, x)$ and $(\mathbf{V}, y)$ define isomorphic Q-modules if and only if $G_{\mathbf{V}} \cdot x = G_{\mathbf{V}} \cdot y$.

Let kQ be the path algebra. A representation $M = (\mathbf{V}, x)$ is called nilpotent if any directed closed cycle acts on M nilpotently. Let $\mathcal{N}_{\mathbf{V}} \subseteq E_{\mathbf{V}}$ be the set of all x such that $(\mathbf{V}, x)$ is a nilpotent representation of Q.

Given an I-graded vector space $\mathbf{V}$ or a Q-module $M = (\mathbf{V}, x)$, we use $|\mathbf{V}|$ or $|M|$ to denote the dimension vector $|\mathbf{V}| = (\dim(V_i))_{i\in I} \in \mathbb{N}^I$.

2.2. **Over Finite Fields.** Let $k = \mathbb{F}_q$ be a finite field with q elements. For $\nu \in \mathbb{N}^I$, let $\mathcal{H}_\nu(q)$ be the set of all $\mathbb{C}$ valued functions on the set of isomorphism classes of all nilpotent representations M of Q with dimension vector $|M| = \nu$. Then $\mathcal{H}_\nu(q)$ can be identified with

the $\mathbb{C}$-vector space $\mathbb{C}_{G_{\mathbf{V}}}(\mathcal{N}_{\mathbf{V}})$ of all $G_{\mathbf{V}}$-invariant functions on $\mathcal{N}_{\mathbf{V}}$ for a fixed a I-graded k-vector space $\mathbf{V}$ with $|\mathbf{V}| = \nu$.

Let $\mathcal{H}(q) = \oplus_{\nu\in\mathbb{N}^I}\mathcal{H}_\nu(q)$. Define an $\mathbb{N}^I$-graded $\mathbb{C}$-algeba structure by

$$(f_1 * f_2)(E) = \sum_{N\subseteq E} f_1(E/N)f_2(N)$$

for any kQ-module E. The sum is taken over the set of kQ-submodules N of E. Since k is finite, the sum is finite. When $f_1 = \chi_M$ and $f_2 = \chi_N$ be the characteristic function of the isoclasses of N and M respectively, then for each kQ-module E,

$$\chi_M * \chi_N(E) = \#\{L \subseteq E | L \cong N, E/L \cong M\} = g^E_{M,N}$$

which is compatible with the definition given by Ringel [R1].

2.3. Given two kQ-representations N and M of dimension vectors τ and μ respectively. Let $\mathbf{V}$ be a fixed I-graded k-vector space with $|\mathbf{V}| = \tau+\mu$. Let $\Phi_{\tau,\mu}$ be the set of all I-graded subspaces $\mathbf{L}$ of $\mathbf{V}$ such that $|\mathbf{L}| = \mu$. Given each $x \in \mathcal{N}_{\mathbf{V}}$, we have a kQ-representation $E = (\mathbf{V}, x)$. Define

$$\tilde{\Phi}_{\tau,\mu} = \{(\mathbf{L}, x) \in \Phi_{\tau,\mu} \times \mathcal{N}_{\mathbf{V}} \mid x\mathbf{L} \subseteq \mathbf{L}\}.$$

Then the group $G_{\mathbf{V}}$ acts on $\Phi_{\tau,\mu}$ and on $\Phi_{\tau,\mu} \times \mathcal{N}_{\mathbf{V}}$ by the diagonal action. The subset $\tilde{\Phi}_{\tau,\mu}$ is $G_{\mathbf{V}}$-invariant. For two fixed I-graded k-vector spaces $\mathbf{W}$ and $\mathbf{T}$ such that $|\mathbf{W}| = \mu$ and $|\mathbf{T}| = \tau$, there are maps

$$\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}} \xleftarrow{\kappa} \tilde{\Phi}_{\tau,\mu} \xrightarrow{\pi} \mathcal{N}_{\mathbf{V}} \tag{2.3.1}$$

defined by $\kappa(\mathbf{L}, x) = (x', x'')$ where $x'' = x_{\mathbf{L}}$ and x' is the induced quotient of $x : \mathbf{V}/\mathbf{L} \to \mathbf{V}/\mathbf{L}$ and with fixed isomorphism $\mathbf{L} \cong \mathbf{W}$ and $\mathbf{V}/\mathbf{L} \cong \mathbf{T}$. π is just the projection to the second factor. Note that map κ is not well-defined, it depends on the choice of the isomorphisms $\mathbf{L} \cong \mathbf{W}$ and $\mathbf{V}/\mathbf{L} \cong \mathbf{T}$. However the induced maps

$$\mathcal{N}_{\mathbf{T}}/G_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}/G_{\mathbf{W}} \xleftarrow{\kappa} \tilde{\Phi}_{\tau,\mu} \xrightarrow{\pi} \mathcal{N}_{\mathbf{V}}/G_{\mathbf{V}} \tag{2.3.2}$$

are well defined and independent of the choice of the isomorphisms. Therefore there are well defined maps

$$\mathbb{C}_{G_{\mathbf{T}}}(\mathcal{N}_{\mathbf{T}}) \otimes \mathbb{C}_{G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{W}}) \xrightarrow{\kappa^*} \mathbb{C}_{G_{\mathbf{V}}}(\tilde{\Phi}_{\tau,\mu}) \xrightarrow{\pi_!} \mathbb{C}_{G_{\mathbf{V}}}(\mathcal{N}_{\mathbf{V}}). \tag{2.3.3}$$

Here that maps $\pi_!$ and κ^* are defined as follows. Let $\phi : X \to Y$ be map between two finite sets and $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ be the spaces of $\mathbb{C}$-valued functions on X and Y respectively. Then ϕ induces the map $\phi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ by $\phi^*(f)(x) = f(\phi(x))$ for all $x \in X, f \in \mathbb{C}(Y)$ and the map $\phi_! : \mathbb{C}(X) \to \mathbb{C}(Y)$ by

$$\phi_!(h)(y) = \sum_{x\in\phi^{-1}(y)} h(x)$$

for all $y \in Y$ and $h \in \mathbb{C}(X)$. Thus the multiplication in defined in 2.2 can be written as

$$f_1 * f_2 = \pi_!(\kappa^*(f_1 \otimes f_2)).$$

2.4. **Lusztig's Setting.** Fix $\nu, \tau, \mu \in \mathbb{N}^I$ such that $\nu = \tau + \mu$. Let $\mathbf{V}$ be an I-graded k-vector space of dimension vector ν and $\mathbf{W} \subseteq \mathbf{V}$ an I-graded k-subspace of dimension μ. Set $\mathbf{T} = \mathbf{V}/\mathbf{W}$ which is also I-graded k-space and has dimension τ. Set

$$\mathbf{F} = \{x \in \mathcal{N}_{\mathbf{V}} \mid x\mathbf{W} \subseteq \mathbf{W}\}.$$

Each $x \in \mathbf{F}$ defines $(x_1, x_2) \in \mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}$ by taking quotient and restriction. Let $P = \mathrm{Stab}_{G_{\mathbf{V}}}(\mathbf{F})$, then there is a natural homomorphism of groups $P \to G_{\mathbf{T}} \times G_{\mathbf{W}}$, which is onto with kernel U. If G is a group and X is a G-set, for any subgroup $H \subseteq G$, we define $G \times^H X = (G \times X)/\langle (gh, x) = (g, hx)\rangle$.

$$\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}} \xleftarrow{\kappa} \mathbf{F} \xrightarrow{\iota} \mathcal{N}_{\mathbf{V}}, \tag{2.4.1}$$

$$\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}} \xleftarrow{p_1} G_{\mathbf{V}} \times^U \mathbf{F} \xrightarrow{p_2} G_{\mathbf{V}} \times^P \mathbf{F} \xrightarrow{p_3} \mathcal{N}_{\mathbf{V}}. \tag{2.4.2}$$

Here the maps are defined as follows: $p_1(g, x) = x$, $p_2(g, x) = (g, x)$ and $p_3(g, x) = g \cdot x$ for all $x \in \mathbf{F}$ and $g \in G_{\mathbf{V}}$.

For each function $h \in \mathbb{C}_{G_{\mathbf{T}} \times G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}})$, $p_1^*(h)$ is in $\mathbb{C}_{G_{\mathbf{V}}}(G_{\mathbf{V}} \times^U \mathbf{F})$. In fact, $p_1^*(h)$ takes a constant value on each fiber $p_2^{-1}(g, x) = gP \times^U x$, i.e., one has

$$p_1^*(\mathbb{C}_{G_{\mathbf{T}} \times G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}) \subseteq p_2^*(\mathbb{C}_{G_{\mathbf{V}}}(G_{\mathbf{V}} \times^U \mathbf{F}). \tag{2.4.3}$$

Note that p_2 is surjective implies that p_2^* is an embedding. Therefore there is a unique map

$$(p_2)_\flat : p_1^*(\mathbb{C}_{G_{\mathbf{T}} \times G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}) \to \mathbb{C}_{G_{\mathbf{V}}}(G_{\mathbf{V}} \times^P \mathbf{F}).$$

Lusztig defined the induction $\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} : \mathbb{C}_{G_{\mathbf{T}} \times G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}) \to \mathbb{C}_{G_{\mathbf{V}}}(\mathcal{N}_{\mathbf{V}})$ by

$$\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}(h) = (p_3)_!(p_2)_\flat p_1^* h \tag{2.4.4}$$

Note that $G_{\mathbf{V}}$ is a reductive group and P is a parabolic subgroup with unipotent radical U and Levi factor $G_{\mathbf{T}} \times G_{\mathbf{W}}$. The induction is a natural generalization of the parabolic induction of a character h of $G_{\mathbf{T}} \times G_{\mathbf{W}}$ to a character of $G_{\mathbf{V}}$.

Let Φ be the the set of all I-graded subspaces $\mathbf{L}$ of $\mathbf{V}$ with $|\mathbf{L}| = |\mathbf{W}|$. Then Φ is identified with $G_{\mathbf{V}}/P$ by $gP \mapsto g\mathbf{W}$. Therefore the map $(g, x) \mapsto (g\mathbf{W}, g \cdot x)$ identifies $G_{\mathbf{V}} \times^P \mathbf{F}$ with the subset $\tilde{\Phi} = \{(\mathbf{L}, y) \in \Phi \times \mathcal{N}_{\mathbf{V}} \mid x\mathbf{L} \subseteq \mathbf{L}\}$. We still use p_3 to be the map $\tilde{\Phi} \to \mathcal{N}_{\mathbf{V}}$ defined by p_3 under the identification.

Given any $y \in \mathcal{N}_{\mathbf{V}}$, $E = (\mathbf{V}, y)$ is a kQ-module and $p_3^{-1}(y)$ is the set of all kQ-submodules $L = (\mathbf{L}, y|_{\mathbf{L}})$ of E with $|L| = \mu$.

Lemma 2.5. *$h = f_1 \times f_2 \in \mathbb{C}_{G_{\mathbf{T}} \times G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}) = \mathbb{C}_{G_{\mathbf{T}}}(\mathcal{N}_{\mathbf{T}}) \otimes_{\mathbb{C}} \mathbb{C}_{G_{\mathbf{W}}}(\mathcal{N}_{\mathbf{W}})$, then $\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}(f_1 \times f_2) = f_1 * f_2$.*

Proof. One can simply take $f_1 = \chi_M$ and $f_2 = \chi_N$ for $M = (\mathbf{T}, x')$ and $N = (\mathbf{W}, x'')$ to verify that

$$\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}(f_1 \times f_2)(y) = \#\{(\mathbf{L}, y) \mid g \cdot y \in \mathbf{F}, \kappa(g \cdot y) \in G_{\mathbf{T}} \cdot x' \times G_{\mathbf{W}} \cdot x'') \text{ for some } g \in G_{\mathbf{V}}\}.$$

The right hand side is exactly $g_{M,N}^E$ as in the definition of Ringel [R1] $\square$

2.6. **Twirsted Multiplication.** $\alpha : \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{C}^*$ be a cocycle (of the abelian group $\mathbb{Z}I$. Define the twisted multiplication $*_\alpha : \mathcal{H}_\tau(q) \otimes \mathcal{H}_\mu(q) \to \mathcal{H}_{\tau+\mu}(q)$ by

$$f_1 *_\alpha f_2 = \alpha(\tau, \mu) f_1 * f_2,$$

for $f_1 \in \mathcal{H}_\tau(q)$ and $f_2 \in \mathcal{H}_\mu(q)$. Then with the new multiplication $*_\alpha$, $\mathcal{H}(q)$ remains to be an assocative algebra. And two cocycles defines an isomorphic algebra is and only if the two cocycles differ by a coboundary.

3. Lusztig's Geometric Construction

3.1. In this section we will denote $k = \bar{\mathbb{F}}_q$. If $V = k^n$ is a k-vector space, we denote by $Fr : V \to V$ be the map $(x_i) \mapsto (x_i^q)$, which is called the Frobenius morphism. The F-fixed point set V^{Fr} is the natural $\mathbb{F}_q$ subspace $\mathbb{F}_q^n$. Such an $\mathbb{F}_q$ subspace is called an $\mathbb{F}_q$-structure of V, which uniquely determines the Frobenius morphism. So when we say a k-vector space has an $\mathbb{F}_q$ structure is equivalent to saying that V is equipped with an Frobenius morphism.

Regarding V as an algebraic variety (with the fixed coordinate $V = k^n$. The Frobenius morphism $Fr : V \to V$ is defined by homomorphism of k-algebra $Fr^\# : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ such that $Fr^\#(x_i) = x_i^q$. Identifying $k[x_1, \ldots, x_n] = k \times_{\mathbb{F}_q} \mathbb{F}_q[x_1, \ldots x_n]$, $Fr^\#(f) = f^q$ for all $f \in \mathbb{F}_q[x_1, \ldots x_n]$. In general an algebraic variety X over k is said to have an $\mathbb{F}_q$-structure, if there is an $\mathbb{F}_q$-variety X_0 such that $X = X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(k)$. The natural homomorphism $Fr_0 : X_0 \to X_0$ of $\mathbb{F}_q$-schemes (by raising the q-th power of coordinate algebras of affine open sets) extends, by change of basis to a morphism $Fr : X \to X$ of k-schemes, which we call the Frobenius morphism. Let X^{Fr} be the closed points of X that is fixed by Fr. Then X^{Fr} is the set of $\mathbb{F}_q$ rational points. Note that the Frobenius map Fr induces a homeomorphism $X(k) \to X(k)$ of topological spaces.

Let l be a prime number that is different from the characteristic of k. An l-dic sheaf is a projective system $\mathcal{F}$ of constructable etale sheaves $\mathcal{F}_n$ of $\mathbb{Z}/l^n\mathbb{Z}$-modules such that $\mathcal{F}_n \otimes_{\mathbb{Z}/l^n\mathbb{Z}} \mathbb{Z}/l^{n-1}\mathbb{Z} \cong \mathcal{F}_{n-1}$. The l-adic cohomology $H^i(X, \mathcal{F})$ of an l-adic sheaf $\mathcal{F}$ is defined

$$H^i(X, \mathcal{F}) = (\varprojlim H^i(X, \mathcal{F}_n) \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l.$$

Where $H^i(X, \mathcal{F}_n)$ the etale cohomology of the etale sheaf $\mathcal{F}_n$, which is a $\mathbb{Z}/l^n\mathbb{Z}$-module. The readers are refered to [C, Appendix] for a gentle and brief induction to l-adic sheaves and l-adic cohomology.

If $\mathcal{F}_0 = (\mathcal{F}_{0,n})$ is an l-adic sheaf on X_0, with the natural projection $\pi : X \to X_0$, then $\mathcal{F} = (\pi^*(\mathcal{F}_{0,n})$ is an l-adic sheaf on X.

Lemma 3.2. *If $\mathcal{F}_0$ is an etale sheaf over X_0, then there is an isomorphism $\Psi : Fr^*\mathcal{F} \to \mathcal{F}$ of etale sheaves over X.*

Proof. Let $q : X \to X$ be the morphism of schemes locally defined by the raising to the q-th power of elements in the structure sheaf $\mathcal{O}_X$ and $\tau_X : X \to X$ be the automorphism defined by the Galois group element $\tau : k \to k$ defined by $\tau(a) = a^q$. Note that both morphisms are not over k. However $Fr = \tau_X^{-1} \circ q$ is defined over k. Both q and τ define homeomorphisms of the underline topological spaces. Hence Fr is a homeomorphism of the underline topological spaces of the scheme X. Since

$$Fr^*(\mathcal{F})(V \to X) = \varinjlim_U \mathcal{F}(U \to X)$$

where the direct limit is taken over all etale morphism $U \to X$ such that diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \phi\big\downarrow & & \big\downarrow\rho \\ X & \xrightarrow{Fr} & X \end{array}$$

commutes However, $\mathcal{F}(U \to X) = \lim\limits_{\longrightarrow U_0} \mathcal{F}_0(U \xrightarrow{\rho_0} X_0)$ with the direct limit taken over all etale morphisms $U \xrightarrow{\rho_0} X_0$ over $\mathbb{F}_q$ such that the diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \rho\big\downarrow & & \big\downarrow\rho_0 \\ X & \xrightarrow{\pi} & X_0 \end{array}$$

commutes. Note that $Fr_0^*(\mathcal{F}_0) = \mathcal{F}_0$ since Fr_0 is identity over the underline topological space. Hence we have $Fr^*\pi^*(\mathcal{F}_0) \cong \pi^* Fr_0^* \mathcal{F}_0$. □

Note that when $x \in X_0(\mathbb{F}_q)$, one has $Fr^*(\mathcal{F})_x = \mathcal{F}_x$. For a given complex $\mathcal{G}^{\cdot}$ of sheaves such that stalks of at each point is finite dimensional and a chain map $\phi^{\cdot} : \mathcal{G}^{\cdot} \to \mathcal{G}^{\cdot}$, then we have one defines

$$\mathrm{Tr}(\phi_a : \mathcal{G}_a^{\cdot} \to \mathcal{G}_a^{\cdot}) = \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{Tr}(\phi_a^i : \mathcal{G}_a^i \to \mathcal{G}_a^i) = \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{Tr}(\phi_a^i : H^i(\mathcal{G}_a^{\cdot}) \to H^i(\mathcal{G}_a^{\cdot})).$$

Theorem 3.3. *(Lefschetz fixed point formula for Frobenius Morphisms) Let $f_0 : X_0 \to Y_0$ be a morphism of algebraic schemes over a finite field $\mathbb{F}_q$ and left $\mathcal{G}_0^{\cdot}$ be a complex of l-adic sheaves on X_0, and let $F : X \to Y$ be the base change and $\mathcal{G}^{\cdot}$ be the complex of l-adic sheaves over X. For any positive integer d, let Fr_X^d and Fr_Y^d be the iteration of Fr d-times respectively and let $\Psi_{\mathcal{G}^{\cdot}}^d : (Fr_X^d)^*(\mathcal{G}^{\cdot} \to \mathcal{G}^{\cdot}$ be the isomorphism of sheaves. $\mathcal{K}^{\cdot} = Rf_!(\mathcal{G}^{\cdot})$ is quasi-isomorphic to a complex of l-adic sheaves over Y and*

$$\sum_{a\in X^{Fr_X^d}} \mathrm{Tr}(Fr_{\mathcal{G}^{\cdot},a}^d : \mathcal{G}_a^{\cdot} \to \mathcal{G}_a^{\cdot}) = \sum_{b\in X^{Fr_Y^d}} \mathrm{Tr}(Fr_{f_!\mathcal{G}^{\cdot},b}^d : Rf_!\mathcal{G}^{\cdot})_b \to (Rf_!\mathcal{G}^{\cdot})_b)$$

In particular, if $\mathcal{G}^{\cdot} = \mathcal{G}$ is a constructable l-adic sheaf concentrated at degree 0, and $Y = \mathrm{Spec}(k)$ then we have

$$\sum_{a\in X^{Fr_X^d}} \mathrm{Tr}(Fr_{\mathcal{G}^{\cdot},a}^d : \mathcal{G}_a^{\cdot} \to \mathcal{G}_a^{\cdot}) = \sum_{i=0}^{2\dim X} (-1)^i \,\mathrm{Tr}(Fr^d : H_c^i(X,\mathcal{G}) \to H_c^i(X,\mathcal{G})).$$

The special case when $\mathcal{G} = \bar{\mathbb{Q}}_l$ is a constant l-adic sheaf, one gets the Lefschetz fixed point formula

$$\#X_0(\mathbb{F}_{q^d}) = \sum_{i=0}^{2\dim X} (-1)^i \dim H_c^i(X, \bar{\mathbb{Q}}_l).$$

We quote a result of Deligne which will be used later no frequently.

Theorem 3.4. *If $f : X \to S$ be a separable and of finite type of schemes, then for each sheaf $\mathcal{F}$ and each geometric point $s \in S$, the stalk of the qth cohomology sheaf $R^q f_! \mathcal{F}_s$ is $H^q_c(X_s, \mathcal{F})$, where X_s is the fiber of s in X under f.*

In general for an l-adic sheaf $\mathcal{F}$ over X, a Weil sheaf of $\mathcal{F}$ is a pair $(\mathcal{F}, \phi_{\mathcal{F}})$ with $\phi_{\mathcal{F}} : Fr^*_X(\mathcal{F}) \to \mathcal{F}$ be an isomorphism. (See [KW, I.1]). In particular, for each $x \in X^{Fr}$ we have $Fr^*_X(\mathcal{F})_x = \mathcal{F}_x$. Let $\chi_{\mathcal{F},\phi} : X^{Fr} \to \bar{\mathbb{Q}}_l$ be the function defined by

$$\chi_{\mathcal{F},\phi}(x) = \mathrm{Tr}(\phi_x : \mathcal{F}_x \to \mathcal{F}_x) \tag{3.4.1}$$

Then $\chi_{\mathcal{F},\phi}$ is a function on $X(\mathbb{F}_q)$. More generally, if $\mathcal{F}^{\cdot}$ is bounded complexes of l-adic sheaves on X with an quasi-isomorphism $\phi^{\cdot} : \mathcal{F}^{\cdot} \to \mathcal{F}^{\cdot}$, then Then one generalize the above definition $\chi_{\mathcal{F}^{\cdot},\phi^{\cdot}}$ naturally. Note that $\mathcal{F}^{\cdot}$ and $\phi^{\cdot}$ can be taken in the derived category $\mathcal{D}^b(X, \bar{\mathbb{Q}}_l)$ of the category of all l-adic sheaves.

3.5. [BBD, 5.1.3] Let $\mathcal{P}_q(X)$ be the category objects in $D^b(X)$ of l-adic sheaves, that has is an Weil sheave structure with respect to the Frobenius morphism Fr determined by q. Then each object $\mathcal{F}$ in $\mathcal{P}_q(X)$ determines a function on $X^{Fr} \to \bar{\mathbb{Q}}_l$ by $\chi_{\mathcal{F}}(x) = \mathrm{Tr}(\phi_{\mathcal{F}} : \mathcal{F}_x \to \mathcal{F})$. Let $K_0(\mathcal{P}_q(X))$ be the Grothendieck group of $\mathcal{P}_q(X)$. Clearly we have a homomorphism of abelian group $K_0(\mathcal{P}_q(X)) \to \bar{\mathbb{Q}}_l(X^{Fr})$.

Definition 3.6. [BBD, 5.1.5] An l-adic sheaf $\mathcal{F}_0$ on X_0 is called *pure* of weight w if for each $n \in \mathbb{N}$ and each point $x \in X_0(\mathbb{F}_{q^n})$, each eigenvalue λ of $Fr^n_{\mathcal{F}} : \mathcal{F}_x \to \mathcal{F}_x$ has the property $|\lambda| = (q^n)^{w/2}$. Here $|\lambda|$ is the complex absolute value under a fix embedding of the field of algebraic numbers in $\bar{\mathbb{Q}}_l$ into $\mathbb{C}$.

$\mathcal{F}_0$ is called *mixed* is $\mathcal{F}_0$ has a finite filtration such that each subquotient is pure. A complex $\mathcal{F}_0^{\cdot}$ is called mixed if each cohomology sheaf $\mathcal{H}^i(\mathcal{F}_0^{\cdot})$ is mixed.

A complex $\mathcal{F}_0^{\cdot}$ in $D^b_c(X_0\bar{\mathbb{Q}}_l)$ is called pure if eigenvalues of $Fr^n_{\mathcal{H}^i(\mathcal{F}^{\cdot})}$ are of the form λ such that $|\lambda| = (q^n)^{(w+i)/2}$. When $f : X_0 \to Y_0$ is proper, then $f_* = f_!$ which sends pure complex of weight w to pure complex of weight w [BBD, 5.1.13]. By the semisimplicity theorem [BBD, 5.3.8], all pure complexes in $D^b_c(X_0, \bar{\mathbb{Q}}_l)$ are semisimple. For any pure complex $\mathcal{F}^{\cdot}$ in $D^b_c(X_0, \bar{\mathbb{Q}}_l)$, then the function $\chi_{\mathcal{F}^{\cdot},n} : X_0(\mathbb{F}_{q^n}) \to \bar{\mathbb{Q}}_l$ defined by

$$\chi_{\mathcal{F}^{\cdot}}(x) = \mathrm{Tr}(Fr^n_{\mathcal{F}^{\cdot}} : \mathcal{F}^{\cdot}_x \to \mathcal{F}^{\cdot}_x)$$

3.7. **Over algebraically closed fields.** One repeats the construction of 2.4, over the algebraically closed field k for each $\mathbf{V}$ with an fixed $\mathbb{F}_q$-structure (i.e., by just fixing a basis of $\mathbf{V}$). Then $\mathbf{W}$ and $\mathbf{T}$ can be taken to have $\mathbb{F}_q$-structures compatible to the $\mathbb{F}_q$-structure of $\mathbf{V}$. Then $\mathcal{N}_{\mathbf{V}}$, $\mathcal{N}_{\mathbf{T}}$, and $\mathcal{N}_{\mathbf{W}}$ are algebraic varieties with an $\mathbb{F}_q$-structure. The subset $\mathbf{F}$ is a closed subset of $\mathcal{N}_{\mathbf{V}}$ which is also has a compatible $\mathbb{F}_q$-structure. The maps κ and ι are all compatible with the Frobenius morphisms.

With the construction, the algebraic groups $G_{\mathbf{V}}$, $G_{\mathbf{T}}$, $G_{\mathbf{W}}$, P, and U all have compatible $\mathbb{F}_q$-structures. So the algebraic varieties $G_{\mathbf{V}} \times_U \mathbf{F}$ and $G_{\mathbf{V}} \times_P \mathbf{F}$ have also $\mathbb{F}_q$-structures and the morphisms p_1, p_2, and p_3 all commute with the Frobenius morphisms. Thus in the following, morphisms are all defined over $\mathbb{F}_q$

$$\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}} \xleftarrow{\kappa} \mathbf{F} \xrightarrow{\iota} \mathcal{N}_{\mathbf{V}}, \tag{3.7.1}$$

$$\mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}} \xleftarrow{p_1} G_{\mathbf{V}} \times^U \mathbf{F} \xrightarrow{p_2} G_{\mathbf{V}} \times^P \mathbf{F} \xrightarrow{p_3} \mathcal{N}_{\mathbf{V}}. \tag{3.7.2}$$

Applying the Fr-fixed point sets, one recovers the construction in 2.4. All most all the constructions in 2.4 works over infinite field except the construction of $(p_3)_!$, cannot be defined as sum of the values over the fibers.

3.8. **Induction.** Note that the map P_3 is proper with $G_{bbfv} \times^P \mathbf{F}$ being smooth. Thus by [BBD, 5.4.2], the derived functor $(p_3)_! = (p_3)_* : D_c^b(G_{\mathbf{V}} \times^P \mathbf{F}, \bar{\mathbb{Q}}_l) \to D_c^b(\mathcal{N}_{\mathbf{V}}, \bar{\mathbb{Q}}_l)$ sends pure complexes of weight w to pure complexes of weight w.

Let $\mathcal{F}$ (resp. $\mathcal{G}$) be $G_{\mathbf{T}}$-equivariant (resp. $G_{\mathbf{W}}$-equivariant) pure complexes of weight w_1 (resp. weight w_2) in $D_c^b(\mathcal{N}_{\mathbf{T}}, \bar{\mathbb{Q}}_l)$ (resp. in $\mathcal{N}_{\mathbf{W}}, \bar{\mathbb{Q}}_l$). Then $\mathcal{F} \otimes \mathcal{G}$ is a $G_{\mathbf{T}} \times G_{\mathbf{W}}$-equivariant pure complex of weight $w_1 + w_2$. Then $p_1^*(\mathcal{F} \otimes \mathcal{G})$ is also pure of weight $w_1 + w_2$. Since $p_1^*(\mathcal{F} \otimes \mathcal{G})$ is $G_{\mathbf{V}} \times P$-equivariant there is an $G_{\mathbf{V}}$-equivariant complex $\mathcal{E}$ in $D_c^b(G_{\mathbf{V}} \times^P F, \bar{\mathbb{Q}}_l)$ such that $p^*(\mathcal{E}) = p_1^*(\mathcal{F} \otimes \mathcal{G})$. Since p_2 is surjective, $\mathcal{E}$ also of weight $w_1 + w_2$. Therefore the complex

$$\mathcal{K} = (p_3)_!(\mathcal{E})$$

is also a $G_{\mathbf{V}}$-equivariant pure complex in $D_c^b(\mathcal{N}_{\mathbf{V}}, \bar{\mathbb{Q}}_l)$ of weight $w_1 + w_2$. We denote the complex $(p_3)_!(\mathcal{E})$ by $\widetilde{\mathrm{Ind}}(\mathcal{F}, \mathcal{G})$.

Note that if $\mathcal{F}$ is a pure complex of weight w, then the complex $\mathcal{F}[-w]$ is of pure of weight 0.

Let $\mathcal{C}_q(\mathcal{N}_{\mathbf{V}})$ be the category of all complexes in $D_c^b(\mathcal{N}, \bar{\mathbb{Q}}_l)$ which is pure of weight $w \in \mathbb{Z}$. Let $\mathcal{K}_0(\mathcal{C}_q(\mathcal{N}_{\mathbf{V}}))$ be the Grothendieck group, which is a $\mathbb{Z}[v, v^{-1}]$-module with $v[\mathcal{F}] = \mathcal{F}[1]$. Let

$$\mathcal{K}(q) = \oplus_{\mathbf{V}} \mathcal{K}_0(\mathcal{C}_q(\mathcal{N}_{\mathbf{V}})).$$

Theorem 3.9. *(a) $\mathcal{K}_q$ is an $\mathbb{Z}[v, v^{-1}]$-algebra with multiplication $\mathcal{F} * \mathcal{G} = \widetilde{\mathrm{Ind}}(\mathcal{F}, \mathcal{G})$.*

(b) The map $\chi : \mathcal{K}_q \to \mathcal{H}(q)$ defined by $\chi([\mathcal{F}]) = \chi_{\mathcal{F}}$ and $v \mapsto q^{1/2}$.

Proof. (a) Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be pure complexes in $D_c^b(\mathcal{N}_{\mathbf{V}_1}, \bar{\mathbb{Q}}_l)$, $D_c^b(\mathcal{N}_{\mathbf{V}_2}, \bar{\mathbb{Q}}_l)$, and $D_c^b(\mathcal{N}_{\mathbf{V}_3}, \bar{\mathbb{Q}}_l)$ respectively. We consider $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \mathbf{V}_3$ with the flag partial flag $W^i = \mathbf{V}_i \oplus \cdot \oplus \mathbf{V}_3$. Let $\mathbf{F}' = \{x \in \mathcal{N}_{\mathbf{V}} \mid x\mathbf{W}^i \subseteq \mathbf{W}^i\}$ and $P' = \mathrm{Stab}_{G_{\mathbf{V}}}(\mathbf{F}')$. Let U' be the unipotant radical, then we have the following

$$\mathcal{N}_{\mathbf{V}_1} \times \mathcal{N}_{\mathbf{V}_2} \times \mathcal{N}_{\mathbf{V}_3} \xleftarrow{p_1'} G_{\mathbf{V}} \times^{U'} \mathbf{F}' \xrightarrow{p_2'} G_{\mathbf{V}} \times^P \mathbf{F}' \xrightarrow{p_3'} \mathcal{N}_{\mathbf{V}}. \tag{3.9.1}$$

Then $(\mathcal{F}_1 * \mathcal{F}_2) * \mathcal{F}_3 = (p_3')_! \mathcal{E}' = \mathcal{F}_1 * (\mathcal{F}_2 * \mathcal{F}_3)$ where $(p_2')^*(\mathcal{E}) = (p_1')^*(\mathcal{F}_1 \times \mathcal{F}_2 \otimes \mathcal{F}_3)$.

(b). It is clear that χ is additive. Given pure complexes $\mathcal{F}$, and $\mathcal{G}$, and $x \in \mathcal{N}_{\mathbf{V}}(\mathbb{F}_q)$, we consider the stalk at x of the i-th cohomology sheaf of $\mathcal{F} * \mathcal{G}$

$$(\mathcal{H}^i(\mathcal{F} * \mathcal{G}))_x = (H^i(p_3)_!(\mathcal{E}))_x = H_c^i(X_x, \mathcal{E})$$

by Deligne's Theorem 3.4. Here $X_x = p_3^{-1}(x) \subseteq G_{\mathbf{V}} \times^P F$ is a close subvariety and has a compatible $\mathbb{F}_q$ structure. The Frobenius morphism on X_x is compatible with the Frobenius morphism at x under the map p_3. Since $\mathcal{E}$ is defined over $\mathbb{F}_q$ we have

$$chi_{\mathcal{F}*\mathcal{G}}(x) = \mathrm{Tr}(Fr_{\mathcal{F}*\mathcal{G},x}) = \sum_{i \in \mathbb{Z}} (-1)^i \,\mathrm{Tr}(Fr : H_c^i(X_s, \mathcal{E})) = \sum_{a \in X_x(\mathbb{F}_q)} \mathrm{Tr}(Fr_a : \mathcal{E}_a).$$

Here the last equality follows from [FK, II 3.15]. However, for any $a \in G_{\mathbf{V}} \times^P \mathbf{F}$, take any $y \in G_{\mathbf{V}} \times^U \mathbf{F}$ such that $p_2(y) = a$, take $(x_1, x_2) = p_1(y) \in \mathcal{N}_{\mathbf{T}} \times \mathcal{N}_{\mathbf{W}}$. By the definition, we have $\mathcal{F}_{x_1} \otimes \mathcal{G}_{x_2} = \mathcal{E}_a$. If $a \in X_x(\mathbb{F}_q)$, we can take y also fixed by Fr and thus x_1 and

x_2 both are fixed by Fr. Here x_1 and x_2 independent of choice of y upto an action by an element of the $G_{\mathbf{T}}$ and $G_{\mathbf{W}}$. Since $\mathcal{F}$ and $\mathcal{G}$ are $G_{\mathbf{T}}$ and $G_{\mathbf{W}}$-equivariant, then trace

$$\mathrm{Tr}(Fr_{x_1,x_2} : \mathcal{F}_{x_1} \otimes \mathcal{G}_{x_2}) = \chi_{\mathcal{F}}(x_1)\chi_{\mathcal{G}}(x_2)$$

Thus we have for each $x \in \mathcal{N}_{\mathbf{V}}(\mathbb{F}_q)$,

$$\chi_{\mathcal{F}*\mathcal{G}}(x) = \sum_{a \in X_x(\mathbb{F}_q)} \chi_{\mathcal{F}}(x_1)\chi_{\mathcal{G}}(x_2) = \chi_{\mathcal{F}} * \chi_{\mathcal{G}}(x).$$

This proves that χ is a homomorphism rings. □

Theorem 3.10. *The homomorphism χ is surjective.*

Proof. Let $f \in \mathcal{H}_\nu(q)$ be a function. We only need to consider that the characteristic function χ_E with $E = (\mathbf{V}, x)$ with $x \in \mathcal{N}_{\mathbf{V}}(\mathbb{F}_q)$. Let $\mathcal{O} = G_{\mathbf{V}} \cdot x \subseteq \mathcal{N}_{\mathbf{V}}$ is a smooth subvariety. Let $f : \mathcal{O} \to \mathcal{N}_{\mathbf{V}}$ be the inclusion, which is defined over $\mathbb{F}_q$. Note that f is a quasi-finite with smooth source. Then $\mathcal{F} = f_!(\bar{\mathbb{Q}}_l[)$. We will show that $\mathcal{F}$ is pure and thus, by the trace formula, we have $\chi_{\mathcal{F}} = \chi_E$. □

3.11. **Geometric Characterisation Hall functions.** Given $\mathbf{V}$ over k with a fixed basis such that the $\mathbb{F}_q$-structure is the $\mathbb{F}_q$-subspace generated by the basis. If $E = (\mathbf{V}, x)$ is a nilpotent representation of Q over k. We say that E is defined over $\mathbb{F}_q$ if there is a representation M_0 of Q over $\mathbb{F}_q$ such that $M \cong M_0 \otimes_{\mathbb{F}_q} k$. Now $E = (\mathbf{V}, x)$ is defined over $\mathbb{F}_q$ if and only if the orbit $G_{\mathbf{V}}x$ contains an $\mathbb{F}_q$-rational point (i.e., fixed by the Frobenius Fr). Thus by replacing x with x_0 and we can assume that $Fr(x) = x$. Thus the orbit $\mathcal{O}_E = G_{\mathbf{V}}x \subseteq \mathcal{N}_{\mathbf{V}}$ is Fr-stable. Thus the isomorphism classes of kQ-representations that are defined over $\mathbb{F}_q$ are in one-to-one correspondence to $G_{\mathbf{V}}$-orbits that contains an $\mathbb{F}_q$-rational point.

Fix an I-graded subspace $\mathbf{W} \subseteq \mathbf{V}$ and set $\mathbf{W} = \mathbf{V}/\mathbf{W}$ with compatible $\mathbb{F}_q$-structures. For any nilpotent kQ-representations M, N, and E defined over $\mathbb{F}_q$ such that $|E| = |\mathbf{V}|$, $|M| = |\mathbf{T}|$, and $|N| = |\mathbf{W}|$, there exist $x' \in \mathcal{N}_{\mathbf{T}}$, $x'' \in \mathcal{N}_{\mathbf{W}}$ and $x \in \mathcal{N}_{\mathbf{V}}$ all fixed by Fr defining representations M, N, and E of kQ respectively. Thus the subset $p_2(p_1^{-1}(\mathcal{O}_M \times \mathcal{O}_N))$ is a constructable subvariety of $\tilde{\Phi}_{\tau,\nu}$, which is also defined over $\mathbb{F}_q$. The fiber $p_3^{-1}(x)$ is a closed subvariety of $\tilde{\Phi}_{\tau.\nu}$.

Let $X^E_{M,N} = p_2(p_1^{-1}(\mathcal{O}_M \times \mathcal{O}_N)) \cap p_3^{-1}(x)$. Then $X^E_{M,N}$ is subvariety of $\tilde{\Phi}_{\tau,\nu}$ with compatible $\mathbb{F}_q$-structure (i.e., $X^E_{M,N}$ is Fr-stable). Thus the reformulation gives

Lemma 3.12. $g^E_{M,N} = \#(X^E_{M,N}(\mathbb{F}_q))$.

Let us fix $\mathbb{F}_qQ$-representations M_0, N_0, E_0, with $|M| + |N| = |E|$. For any $e \in \mathbb{N}$ let $M_r = M_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$ be the $\mathbb{F}_{q^r}Q$-representation. Define $g^E_{M,N}(q^r) = g^{E_r}_{M_r,N_r}$. Thus $g^E_{M,N}$ is a function of q^r. If it is a polynomial of q^r, this polynomial is called a Hall polynomial. In general we only call it a Hall function. One can form the ζ-function

$$\zeta(t) = \zeta^E_{M,N}(t) = \exp(\sum_{r=1}^{\infty} g^E_{M,N}(q^r)t^r/r).$$

Let $H^i_c(X, \bar{\mathbb{Q}}_l)$ be the l-adic cohomology of the variety $X = X^E_{M,N}$ and $Fr^r : X \to X$ the r-th composition of the Frobenius morphism, then there is induced $\bar{\mathbb{Q}}_l$-linear map

$Fr^r : H^i(X, \bar{\mathbb{Q}}_l) \to H^i(X, \bar{\mathbb{Q}}_l)$ (are the vector spces finite dimensional?) Lefschetz fix point formula implies

$$g^E_{M,N}(q^r) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(Fr^r : H^i(X, \bar{\mathbb{Q}}_l) \to H^i(X, \bar{\mathbb{Q}}_l)). \tag{3.12.1}$$

Thus tools involved in proving Weil conjecture could be used.

3.13. **From sheaves to functions.** Let $\mathbf{V}$ be an I-graded k-vector space with a fixed $\mathbb{F}_q$-structure and let $\underline{i} = (i_1, i_2, \ldots, i_n)$ be a sequnece of elements in I such that $\sum_{j=1}^n i_j = |\mathbf{V}| \in \mathbb{N}I$. Let E_{i_j} be the simple nilpotent representation of Q at i_j and its defined over any fixed finite field $\mathbb{F}_q$. Over a finite field

$$E_{i_1} * E_{i_2} * \cdots E_{i_n} = \sum_E g^E_{E_{i_1},\ldots,E_{i_n}}(q) E$$

Here the summation is taken over all isomorphism classes of Q-modules which is defined over $\mathbb{F}_q$ and

$$g^E_{E_{i_1},\ldots,E_{i_n}}(q) = \#\{(E = E^1 \supseteq E^1 \supseteq \cdots \supseteq E^{n+1} = E \,|; E^j/E^{j+1} \cong E_{i_j}\}$$

is the filtration number. Here the filtrations are defined over $\mathbb{F}_q$.

To interpret the filtration numbers, we set

$$\Phi_{\underline{i}} = \{\mathfrak{f} = (\mathbf{V} = \mathbf{L}^1 \supseteq \mathbf{L}^1 \supseteq \cdots \supseteq \mathbf{L}^{n+1} = 0) \mid |\mathbf{L}^j/\mathbf{L}^{j+1}| = i_j.$$

Note that $\Phi_{\underline{i}}$ is a projective variety with an $\mathbb{F}_q$-structure and

$$\tilde{\Phi}_{\underline{i}} = \{(\mathfrak{f}, x) \in \Phi_{\underline{i}} \times \mathcal{N}_{\mathbf{V}} | x\mathbf{L}^j \subseteq \mathbf{L}^j\}$$

is a smooth algebraic variety and defined over $\mathbb{F}_q$. The projection $\pi : \tilde{\Phi}_{\underline{i}} \to \mathcal{N}_{\mathbf{V}}$ is proper morphism of algebraic variety. Once again the filtration number $g^E_{E_{i_1},\ldots,E_{i_n}}(q)$ is the number of $\mathbb{F}_q$-rational points in the fiber $\pi^{-1}(x)$ (for $x \in \mathcal{N}(\mathbb{F}_q)$).

Now let $\bar{\mathbb{Q}}_l$-be the constant l-adic sheaf on $\tilde{\Phi}_{\underline{i}}$, then by Deligne's theorem the element $E_{i_1} * E_{i_2} * \cdots E_{i_n}$ can be viewed as

$$\sum_{\mathcal{O} \subseteq \mathcal{N}_{\mathbf{V}}} (\sum_s (-1)^s \operatorname{Tr}(Fr : R^s \pi_!(\bar{\mathbb{Q}}_l))) \chi_{\mathcal{O}}.$$

Here that sum is taken over $G_{\mathbf{V}}$-orbits $\mathcal{O}$ in $\mathcal{N}_{\mathbf{V}}$ which is has an $\mathbb{F}_q$-rational point. Here we use $\mathcal{L}_{\mathcal{O}}$ to denote the representation of Q defined by the orbit for a moment. Later on, one will replace $\chi_{\mathcal{O}}$ by simple perverse sheaf $\mathcal{L}_{\mathcal{O}}$ which is supported on $\mathcal{O}$ and is $G_{\mathbf{V}}$-equivarient perverse sheaf on $\mathcal{N}_{\mathbf{V}}$.

4. Composition Algebras

4.1. In this section we outline Lusztig's geometric construction of compisition algebra. The composition algebra is the subalgebra of $\mathcal{H}(q)$ generated by simple nilpotent representations of Q. In this section k is the algebraic closure of $\mathbb{F}_q$.

Fix $\nu \in \mathbb{N}I$ and I-graded vector space Let $\underline{\nu} = (\nu^1, \ldots, \nu^r)$ be a sequence of elements of I with $\sum_{j=1}^r \nu^r = \nu = |\mathbf{V}|$. Fix an I-graded filtration $\mathfrak{f}_0 = (\mathbf{V} = \mathbf{L}^1 \supseteq \mathbf{L}^2 \supseteq \cdots \supseteq \mathbf{L}^{r+1} = \{0\}$. By extending the basis of $\mathbf{L}^j$ to get a k-basis of $\mathbf{V}$. We will use $\mathbf{V}^{Fr}$ to denote the $\mathbb{F}_q$ span

of the basis, which defines the $\mathbb{F}_q$-structure on V with the Frobenius map Fr defined by raising the coefficients under this basis to qth power. Then $\mathbf{L}_0$ is invarient under Fr. Set

$$\Phi_{\underline{\nu}} = \{\mathfrak{f} = (\mathbf{V} = \mathbf{L}^1 \supseteq \cdots \supseteq \mathbf{L}^{r+1} = \{0\} \mid |\mathbf{L}^j/\mathbf{L}^{j+1}| = \nu^j\}. \tag{4.1.1}$$

An element $\mathfrak{f}$ in $\Phi_{\underline{\nu}}$ is called a flag of type $\underline{\nu}$. Note that the algebraic group $G_{\mathbf{V}}$ acts on $\Phi_{\underline{\nu}}$ transitively with the stablizer P at the special flag $\mathfrak{f}_0$. Then P is a parabolic subgroup $G_{\mathbf{V}}$ and $\Phi_{\underline{\nu}} = G_{\mathbf{V}}/P$ is a complete algebraic variety. Note that the $\mathbb{F}_q$-structure fixed above naturally induces an $\mathbb{F}_q$-structure with Frobenius morphism also denoted by Fr and P is also defined over $\mathbb{F}_q$. Thus the algebraic variety $\Phi_{\underline{\nu}}$ has an $\mathbb{F}_q$-structure and the action of $G_{\mathbf{V}}$ is compatible with the Frobenius morphisms.

An element $x = (x_h)_{h\in H} \in \mathcal{N}_{\mathbf{V}}$ is said to stablize $\mathfrak{f}$ if $x_h\mathbf{L}^j_{h'} \subseteq \mathbf{L}^j_{h''}$ for all j and $i \in I$. We simply write $x\mathfrak{f} \subseteq \mathfrak{f}$. Define

$$\tilde{\Phi}_{\underline{\nu}} = \{(\mathfrak{f}, x) \in \Phi_{\underline{\nu}} \times \mathcal{N}_{\mathbf{V}} \mid x\mathfrak{f} \subseteq \mathfrak{f}.\} \tag{4.1.2}$$

$\tilde{\Phi}$ is a closed smooth subvariety of $\Phi_{\underline{\nu}} \times \mathcal{N}_{\mathbf{V}}$. Therefore the projection map $\pi_{\underline{\nu}} : \tilde{\Phi}_{\underline{\nu}} \to \mathcal{N}_{\mathbf{V}}$ to the second component is a proper morphism of algebraic varieties.

Let U be the unipotent radical of P. If $\nu^j = \sum_{i\in I} c^j_i i$, then the dimension of U and the dimension of $\Phi_{\underline{\nu}}$ are equal and is

$$\sum_{i\in I, j<j'} c^j_i c^{j'}_i. \tag{4.1.3}$$

The first projection $\tilde{\Phi}_{\underline{\nu}} \to \Phi_{\underline{\nu}}$ is a vector bundle of rank

$$\sum_{h\in H, j\leq j'} c^j_{h'} c^{j'}_{h''}. \tag{4.1.4}$$

Thus $\tilde{\Phi}_{\underline{\nu}}$ is smmoth and has dimension of

$$f(\underline{\nu}) = \sum_{i\in I, j<j'} c^j_i c^{j'}_i + \sum_{h\in H, j\leq j'} c^j_{h'} c^{j'}_{h''}. \tag{4.1.5}$$

Note that $G_{\mathbf{V}}$ acts on $\tilde{\Phi}_{\underline{\nu}}$ diagonally and the both projections to the first and second components are $G_{\mathbf{V}}$-equivariant morphisms.

Let $\mathbf{1}$ be the constant l-adic sheaf on $\tilde{\Phi}_{\underline{\nu}}$, which can be naturally regarded as an complex in $D^b(\tilde{\Phi}_{\underline{\nu}}$ concentrated at degree zero. Its shift $\mathbf{1}[f(\underline{\nu})]$ concentrated at degree $-f(\underline{\nu}0$ is simple perverse sheaf in $D^b(\tilde{\Phi}_{\underline{\nu}})$ (see [BBD, Thm 4.3.1]) for classification of simple perverse sheaves).

Theorem 4.2. *If $f : X \to Y$ is a proper morphism of algebraic varieties, then for any simple perverse sheaf $\mathcal{F}$ in $D^b(X)$ then $f_!(\mathcal{F})$ is semisimple complex.*

Let $L_{\underline{\nu}} = (\pi_{\underline{\nu}})_!(\mathbf{1})$. Let $\mathcal{P}_\nu$ be the isoclass of all simple perverse sheaves L such that $L[i]$ is direct summand of $L_{\underline{\nu}}$ for some sequence $\underline{\nu}$ such that $\sum_j \nu^j = \nu$. Let $\mathcal{Q}_\nu$ be the semisimple subcategory of $D^b_c(\mathcal{N}_{\mathbf{V}}, \bar{\mathbb{Q}}_l)$ of finite direct sums of $L[i]$ with $L \in \mathcal{P}_\nu$ and $i \in \mathbb{Z}$. Let $\tilde{\mathcal{K}} = \sum_\nu \mathcal{K}_0(\mathcal{Q}_\nu)$. Then $\tilde{\mathcal{K}}$ ans associative algebra with multiplication defined by $L_1[i] * L_2[j] = \widetilde{\mathrm{Ind}}(L_1[i], L_2[j])$. Note that $L_1[i] * L_2[j]$ is in $\mathcal{Q}_\nu$ if $L_1 \in \mathcal{P}_{\nu_1}$ and $L_2 \in \mathcal{P}_{\nu_2}$ with $\nu = \nu_1 + \nu_2$ since the functors $p_1^*, (p_3)_!$ are additive functors preseving the direct sums.

Theorem 4.3 (Lusztig)**.** *(a). Let $L \in \mathcal{P}_\nu$. If $L[i]$ is a direct summand of $L_{\underline{\nu}}$ for some $\underline{\nu} = (\nu^1, \ldots, \nu^r)$ discrete, i.e., for each $h \in H$, one of the coefficients of h' and h'' has to be zero, then $L[i]$ is pure of weight i* [L2, 5.3].

(b). Suppose $L_1 \in \mathcal{P}_\tau$ and $L_2 \in \mathcal{P}_\mu$ satisfy (a). For any $L \in \mathcal{P}_\nu$ ($\nu = \tau + \mu$), regarding $\mathrm{Hom}(L_1[-i], L_1 * L_2)$ as an l-adic sheaf on $\mathrm{Spec}(\mathbb{F}_q)$, then the Frobenius morphism on $\mathrm{Hom}(L_1[-i], L_2 * L_3)$ has only one eigen value which is $(\sqrt{q})^{-i-m}$ where $m = \sum_{h\in H} \tau_{h'}\mu_{h''} + \sum_{i\in I} \tau_i\mu_i$ and $L_1 \in \mathcal{P}_\tau$ and $L_3 \in \mathcal{P}_\mu$.

Clearly, set $\mathbf{B}$ of all simply perverse sheaves satisfying (a) span a $\mathbb{Z}[v, v^{-1}]$-submodule of $\mathcal{K}_q$, which is closed under multiplication. Further more the action of v is given by $\sqrt{a}$. This is the composition algebra with $\mathbf{B}$ be $\mathbb{Z}[v, v^{-1}]$-basis.

Theorem 4.4 (Lusztig)**.** *Under the map χ, $\mathbb{Z}[v, v^{-1}]\mathbf{B}$ is mapped onto the subalgebra of $\mathcal{H}(q)$ generated by all simple representations of Q.*

5. Appendix

5.1. **Schemes.** Let A a commutative ring, $\mathrm{Spec}(A)$ is the set of all prime ideals of A. $\mathrm{Spec}(A)$ is equipped with the Zariski topology with closed sets being $V(I) = \{p \in \mathrm{Spec}(A) \mid p \supseteq I\}$. For any $f \in A$, let $D_f = \{p \in \mathrm{Spec}(A) \mid f \in P\}$. Then the set D_f form a basis for the topology. $D_f = \mathrm{Spec}(A)$ if and only of f is nilpotent. The assigment $D_f \mapsto A_f$, where A_f is the localization of A with respect to the multiplicative set $\{f^n \mid n \in \mathbb{N}\}$ extends to a sheaf of rings over $\mathrm{Spec}(A)$, which is denoted by $\mathcal{O}$ and called the structure sheaf of the scheme $(\mathrm{Spec}(A), \mathcal{O})$. The pair $(\mathrm{Spec}(A), \mathcal{O})$ is called the affine scheme defined by A. In general a scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally of the type $(\mathrm{Spec}(A), \mathcal{O})$.

Let $(Y, \mathcal{O}_Y)$ be another scheme. A morphism $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(\phi, \phi^\#)$ with $\phi : X \to Y$ is a continous map and an morphism of sheaves of rings $\phi^\# : \mathcal{O}_Y \to \phi_*(\mathcal{O}_X)$. If $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, and $(\phi, \phi^\#)$ is morphism of schemes, then for the morphism $\phi^\#(Y) : B = \mathcal{O}_Y(Y) \to \phi_*(\mathcal{O}_X)(Y) = \mathcal{O}_X(\phi^{-1}(Y)) = \mathcal{O}_X(X) = A$ is a homomorphism of rings and the map $\phi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is given by $\phi(p) = (\phi^\#)(p)$ for all $p \in \mathrm{Spec}(A)$. In particular A is a B-algebra.

Fix a scheme S, a scheme X with basis S is a pair (X, ψ) with $\psi : X \to S$ being a morphism of schemes. Let Sch_S be the cateogry of all schemes with base S, which has objects $X \xrightarrow{\psi} S$ and a morphism $\alpha : (X \xrightarrow{\psi} (Y \xrightarrow{\phi} S)$ is a morphism $\alpha : X \to Y$ of schemes such that $\psi = \phi \circ \alpha$. Such α is called a morphism over S. Let $\mathrm{Hom}_{\mathrm{Sch}_S}(X, Y)$ be the set of all S-morphisms. For example, if $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, and $Y = \mathrm{Spec}(B)$ are affine schemes, then A and B are S-algebras, and $\alpha^\# : B \to A$ is a homophisms of S-algebras.

For two schemes X and Y based on S, the fiber product $X \times_S Y$ is a scheme based on S together with a pair of morphisms $P_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ over S which define the product in the category Sch_S. In particular, $X \times_S Y$ can be viewed as the schemes with basis X or Y under the morphism p_1 or p_2.

5.2. **Etale Topology.** A morphism of schemes $\phi : U \to X$ is called etale if

(a) ϕ is finitely generated, i.e., each point of $x \in X$ has an affine open subset V such that $phi^{-1}(V)$ is a union of affine open subsets U_α of U such that $\mathcal{O}(U_\alpha)$ is a finitely generated $\mathcal{O}_X(V)$-algebra and

(b) The local homomorphism $\phi_x : \mathcal{O}_{X,\phi(x)} \to \mathcal{O}_{U,x}$ for all $x \in U$ are local-etale, i.e., $\mathcal{O}_{U,x}$ is a localization of a finitely generated $\mathcal{O}_{X,\phi(x)}$-algebra and $\mathcal{O}_{X,\phi(x)}/m_{\phi(x)} \subseteq \mathcal{O}_{U,x}/m_x$ is a finite separable extension. Let $Et(X)$ be the category of all etale morphisms to X. Morphisms are morphisms over X (and thus also etale.)

5.3. **Etale Sheaves.** An etale sheaf contravariant functor $\mathcal{F} : Et(X) \to Ab$ satisfying the property

a $\mathcal{F}(U\dot{\cup}V) = \mathcal{F}(U) \times \mathcal{F}(V)$;

b If $U \to V$ is a surjective morphism in $Et(X)$ the sequence

$$\mathcal{F}(V) \to \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_V U)$$

is exact.

Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ be a sheaf on Y then the sheafication of the presheaf $f^{-1}(\mathcal{F})(V) = \lim\limits_{\longrightarrow U \supset F(V)} \mathcal{F}(U)$ defines a sheaf on Y denoted by $f^{-1}(\mathcal{F})$.

Given any ordinary sheaf $\mathcal{G}$ on X, define $\mathcal{G}_{et}(U \xrightarrow{\phi} X) = \phi^{-1}(\mathcal{G})(U)$ is an etale sheave. In particular, the constant sheave A on a scheme X is of the form $\phi^{-1}(A)$ where $\phi : X \to \{pt\}$ is a morphism schemes.

For a fixed positive integer n such that n is invertible, let $(\mathbb{Z}/n\mathbb{Z})_U$ be the constant sheaf of rings over any toplogical space U. Let $\mathbb{Z}/n\mathbb{Z}$ be the etale sheave over X such that $\mathbb{Z}/n\mathbb{Z}(U \xrightarrow{\phi} X) = (\mathbb{Z}/n\mathbb{Z})(U)$. Then the etale sheaf $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the sheave of $\mu_n(U \xrightarrow{\phi} X) = \{f \in \mathcal{O}_U | f^n = 1\}$ as a sheaf of abelian groups.

A constructable etale sheaf $\mathcal{F}$ of abelian groups is torsion etale sheaf (i.e., the stalk $\mathcal{F}_x$ and $\mathcal{F}(U)$ are torsion groups for each geometrick point x and etale morhpism $U \to X$ etale) with finite assending chain condition on subsheaves. In particular the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ on X is constructable.

Let $(f, f^{\#}) : X \to Y$ be a morphism of schemes $f_!$ is the left adjoint to f^* $(f^*(\mathcal{G})_x = \mathcal{G}_{f(x)}$ Where $f_!(\mathcal{F})$ is the sheafiication of the presheaf $(V \to Y) = \oplus_{\phi \in \mathrm{Hom}_Y(V,X)} \phi^*(\mathcal{F})(V)$.

5.4. **Grothendieck Trace Formula.** More general than the l-adic cohomology over schemes are l-adic sheaves. There is good brief quick summary of definitions of l-sheaves in the Appendix of Carter's book for those readers who wants to have get acquented with the l-adic sheaves.

Let X_0 be a scheme over the field $\mathbb{F}_q$ and $X = X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}\, k$ is a scheme over k. Let $\mathcal{G}_0$ be an etale sheaf of abelian groups on X_0 and $\mathcal{G}$ be the inverse image of $\mathcal{G}_0$ over X (there is a natural projection $X \to X_0$ of schemes.) Let $Fr_X = Fr_0 \times 1 : X \to X$ be the Frobenius morphism of k-schemes. Then the inverse image $Fr_0^*(\mathcal{G}_0)$ is a sheave of abelian groups and there is a (geometric) Frobenius isomorphism of etale sheaves $Fr_{\mathcal{G}_0} : Fr_0^*(\mathcal{G}_0) \to \mathcal{G}_0$. In fact $Fr_{\mathcal{G}_0}$ defines an isomorphism of functors $Fr_)^*$ to Id from the category of etale sheaves on X_0 to itsself. For any field extension $\mathbb{F}_q \to F$, with $X_F = X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(F)$, and $Fr_{\mathcal{G}_F} = Fr_{\mathcal{G}_0} \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(F)$.

Note that the set of geometric points over k of X_F is the set $X_F(k) = \mathrm{Hom}_{Sch/F}(\mathrm{Spec}(k), X_F)$ for any F-scheme X_F, ($F \subseteq k$ is a field extension.)

Theorem 5.5. *Grothendieck trace formula*

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