Socle Series of Cohomology Groups of Line Bundles on G/B

Short running title: Socle Series of Cohomology Groups

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Dedicated to Professor Xihua Cao on his seventieth birthday

Abstract

The G-module structure of the cohomology groups of line bundles over the flag variety G/B for a semisimple algebraic group G is studied. It is proved that, generically, their socle filtrations satisfy the same sum formula as Andersen's filtrations if Lusztig's conjecture is true. Another purpose of this paper is to calculate the socle series of the H^0's for the p-regular weights close to a chamber wall for the group of type G_2, because the multiplicity is not free in this case.

Introduction Let G be a connected and simply connected semisimple algebraic group over an algebraically closed field of characteristic $p > 0$ and B a Borel subgroup of G. The cohomology group $H^i(\lambda)$ of the line bundle on the flag variety G/B induced from a character λ of B has a G-module structure in a natural way. Weyl modules are precisely the nonvanishing $H^N(\lambda)$, where $N = \dim G/B$. In fact, the Weyl modules are the dual modules of the nonvanishing induced modules $H^0(\lambda)$. In this paper we study the socle series of $H^i(\lambda)$ as G-module.

Concerning the structure of Weyl modules, Jantzen [14] constructed a filtration via a contravariant form. This filtration satisfies a certain sum formula of characters. More recent investigations give more evidence that the Jantzen filtration is the

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radical filtration. Andersen [5] proved that Jantzen's conjecture (the consistency of certain intertwining maps between Weyl modules with the Jantzen filtrations) implies that the Jantzen filtration and the radical filtration coincide. More generally, Andersen [1] constructed a filtration in each $H^{\ell(w)}(w\cdot\lambda)$, which also satisfies a certain sum formula. Generically, Andersen's filtration generalizes the Jantzen filtration to other cohomology groups. So the same question may be asked. Is Andersen's filtration the radical filtration of that cohomology group? In this paper (Theorem 2.5) we prove that Lusztig's conjecture implies that, in the generic situation, the radical filtration of $H^{\ell(w)}(w\cdot\lambda)$ satisfies the same sum formula as Andersen's filtration does. We start from the sum formula (proved by Andersen and Kaneda [6]) for the G_1-radical filtration of $\mathrm{Ind}_B^{G_1B}\lambda$ (G_1 is the Frobenius kernel of G). Then we *lift* this formula to G by using the fact that the radical (or socle) filtration can be gotten by applying the functor $H_1^{\ell(w)} = R^{\ell(w)}\mathrm{Ind}_{G_1B}^{G}$ to the G_1-radical (or socle) filtration of $\mathrm{Ind}_B^{G_1B}(w\cdot\lambda)$ [17]. This gives some evidence for an affirmative answer to the above question in the generic situation. This is not true in some nongeneric situations (see [5] and [18] for examples).

Sullivan [20] and Bai-Wang-Wen [7] show that the G_1-socle filtration of $H^0(\lambda)$ is the induced one from the socle filtration of $\mathrm{Ind}_B^{G_rB}\lambda$ even in the nongeneric situation. But there is no practical way to calculate the induced filtration when a simple factor occurs in different layers in $\mathrm{Ind}_B^{G_rB}\lambda$. However, Sullivan [20] shows that the simple factors and their multiplicities in an induced module from a socle layer of $\mathrm{Ind}_B^{G_rB}\lambda$ depend on the G_rB-structure of that layer. In Section 3, we use Sullivan's result and study the intertwining maps which give some relation on the G_rB-structure of socle layers of different induced modules of G_rB. Some of the G_rB-structure of socle layers is studied. In Section 4, we use the argument developed in Section 3 to calculate the socle series of $H^0(\lambda)$ for the group of type G_2 when the weight λ satisfies the conditions in Lusztig's conjecture. It turns out that the socle filtration of $H^0(\lambda)$, for λ close to a chamber wall (not in the generic position), can be calculated from the generic filtration pattern via the cancellation principle.

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based on the referee's suggestion.

1 Preliminaries

1.1 Basic Notations Let G be a connected and simply connected semisimple algebraic group over an algebraically closed field of characteristic $p > 0$ and T a maximal torus of G. R is the root system of G relative to T and B is a Borel subgroup containing T and corresponding to the negative roots R^-. $S \subseteq R^+$ is the set of simple roots. $X(T)$ is the character group of T and $\langle\ ,\ \rangle$ is a W-invariant inner product on $X(T) \otimes \mathbf{R}$, where $W = N(T)/T$ is the Weyl group which is generated by the simple reflections s_α ($\alpha \in S$) such that $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ ($\lambda \in X(T)$) and $\alpha^\vee$ is the coroot of α. $\ell(w)$ is the length function on W. Let ρ be the half-sum of all positive roots. If $w(\lambda)$ denotes the ordinary action of $w \in W$ on $\lambda \in X(T)$, the new *dot* action of w on λ is defined by $w{\cdot}\lambda = w(\lambda + \rho) - \rho$.

We denote by $X_r(T) = \{\lambda \in X(T) \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle \leq p^r, \forall \alpha \in S\}$ the set of p^r-restricted weights. Any translate of $X_r(T)$ by an element of $p^r X(T)$ is called an r-box. Any weight $\lambda \in X(T)$ has a unique decomposition $\lambda = \lambda^0 + p^r \lambda^1$ with $\lambda^0 \in X_r(T)$ and $\lambda^1 \in X(T)$. With this decomposition, we can define another new action of W on $X(T)$ by $w_r{\cdot}\lambda = \lambda^0 + p^r w{\cdot}\lambda^1$ ($\lambda \in X(T)$, $w \in W$). Instead of $w_1 \cdot \lambda$, we will still denote $w_r \cdot \lambda$ if $r = 1$. Each λ is contained in a unique r-box $X_r(T) + p^r \lambda^1$ with the upper vertex $\lambda^v = p^r \lambda^1 + (p^r - 1)\rho$ and the lower vertex $\lambda_v = p^r \lambda^1 - \rho$. It is easily verified that $(w_r \cdot \lambda)^v = w \cdot (\lambda^v)$. There is a geometric description for the action of $w_r{\cdot}$ in [9]. The affine Weyl group W_p is the group generated by the reflections $s_{\alpha,np}{\cdot}\lambda = s_\alpha{\cdot}\lambda + np\alpha$ with the fixed hyperplane $H_{\alpha,np} = \{x \in X(T) \otimes \mathbf{R} \mid \langle x + \rho, \alpha^\vee \rangle = np\}$ ($\alpha \in R^+$). $(s_\alpha)_*$ (or s_α^*) is the reflection in the nearest lower (or upper) α-hyperplane, i.e., $(s_\alpha)_*{\cdot}\lambda = s_\alpha{\cdot}\lambda + np\alpha$ (or $s_\alpha^*{\cdot}\lambda = s_\alpha{\cdot}\lambda + (n+1)p\lambda$) if $np \leq \langle \lambda + \rho, \alpha^\vee \rangle < (n+1)p$ (or $np < \langle \lambda + \rho, \alpha^\vee \rangle \leq (n+1)p$). Let $C_{p^r} = \{\lambda \in X(T) \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p^r, \forall \alpha \in R^+\}$ be the bottom p^r-alcove. We call $C_0 = C_p$ the fundamental alcove. Any image of C_0 under W_p is called an alcove. $\hat{C}_{p^r} = \{\lambda \in X(T) \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle \leq p^r, \forall \alpha \in R^+\}$ is the upper closure of C_{p^r} and $\bar{C}_{p^r} = \{\lambda \in X(T) \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p^r, \forall \alpha \in R^+\}$ is the closure of C_{p^r}.

1.2 Induction Let G_r be the kernel of the r-th Frobenius map Fr^r in G and

G_rB the pullback of B via Fr^r. For any B-module M, $\mathrm{Ind}_B^{G_rB}M$ is the induced G_rB-module. We denote $Z_r(\lambda) = \mathrm{Ind}_B^{G_rB}\lambda$ (for $\lambda \in X(T)$) by regarding λ as a 1-dimensional B-module. Since G_r is a normal subgroup scheme of G_rB, $\mathrm{Soc}^i_{G_r}Z_r(\lambda)$ is a G_rB-submodule of $Z_r(\lambda)$ for each i and therefore each G_r-socle layer has a G_rB-structure. Specially, $L_r(\lambda) = \mathrm{Soc}_{G_r}Z_r(\lambda)$ is also a simple G_rB-module. In fact all the simple G_rB-modules are obtained in this way. If $\lambda = \lambda^0 + p^r\lambda^1$, then $Z_r(\lambda) \cong Z_r(\lambda^0) \otimes p^r\lambda^1$ and $L_r(\lambda) \cong L_r(\lambda^0) \otimes p^r\lambda^1$. Here $L_r(\lambda^0) = L(\lambda^0)$ is a simple G-module by the Steinberg tensor product theorem. Thus one has the following translation property: $L_r(\lambda + p^r\nu) \cong L_r(\lambda) \otimes p^r\nu$ and $Z_r(\lambda + p^r\nu) \cong Z_r(\lambda) \otimes p^r\nu$ for all $\nu \in X(T)$.

Let us denote $H^i(M) = R^i\mathrm{Ind}_B^G M$ for each B-module M and $H^i(\lambda) = R^i\mathrm{Ind}_B^G\lambda$ which is the i-th cohomology group of the line bundle induced by the weight λ on the flag variety G/B. Since $G_rB/B \cong G_r/B_r$ is an affine scheme, $\mathrm{Ind}_B^{G_rB}$ is an exact functor from the category of B-modules to the category of G_rB-modules. Applying the transitivity of induction functors and the corresponding Grothendieck spectral sequence one can get

$$R^i\mathrm{Ind}_B^G M \cong R^i\mathrm{Ind}_{G_rB}^G(\mathrm{Ind}_B^{G_rB}M)$$

for each B-module M. We will use the notation $H_r^i(M) = R^i\mathrm{Ind}_{G_rB}^G M$ for each G_rB-module M. Therefore we have $H_r^i(Z_r(\lambda)) = H^i(\lambda)$. It is a well known result of Andersen ([15] II 9.9(3)) that

$$R^i\mathrm{Ind}_{G_rB}^G(V \otimes M^{(r)}) \cong V \otimes (R^i\mathrm{Ind}_B^G M)^{(r)} \tag{1}$$

for a G-module V and a B-module M. Here $E^{(r)}$ is the r-th Frobenius twist for a G-module E, which has the same vector space as E but with the G-action given by the composition $G \xrightarrow{Fr^r} G \to GL(E)$. The Frobenius twist $M^{(r)}$ of the B-module M admits a G_rB-structure. Now the following theorem follows from Bott's theorem ([15] II 5.5).

Theorem (Andersen [4]) *Let $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda^1 \in W{\cdot}\bar{C}_0$. Then*

$$H_r^i(L_r(\lambda)) = \begin{cases} L(w_r{\cdot}\lambda) & \text{if } w{\cdot}\lambda^1 \in X_+(T) \text{ for some } w \in W \text{ and } i = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

Here $L(w_r{\cdot}\lambda) = L(\lambda^0) \otimes L(w{\cdot}\lambda^1)^{(r)}$ is the simple G-module with highest weight $w_r{\cdot}\lambda$.

1.3 Characters Let E be a T-module (throughout this paper, all modules are finite dimensional). The character of E is defined to be $\mathrm{ch}E = \sum_{\lambda\in X(T)} \dim E_\lambda\, e(\lambda)$, an element in the group ring $\mathbf{Z}[X(T)]$ (with basis $\{e(\lambda) \mid \lambda \in X(T)\}$). If E is a B-module, $\chi(E) = \sum(-1)^i \mathrm{ch}H^i(E)$ is the Euler character of E. Especially, we denote $\chi(\lambda) = \sum(-1)^i \mathrm{ch}H^i(\lambda)$. So $\chi(w{\cdot}\lambda) = (-1)^{\ell(w)}\chi(\lambda)$ can be calculated via Weyl's character formula ([15], II 5.10). If M is a G-module and $L(\lambda)$ is a simple G-module, $[M : L(\lambda)]$ is the multiplicity of $L(\lambda)$ in M. Since all the characters of simple G-modules form a $\mathbf{Z}$-basis of $\mathbf{Z}[X(T)]^W$ (the W-invariants with the natural W-action on the group ring $\mathbf{Z}[X(T)]$), we denote by $[\chi : L(\lambda)]$ the coefficient of $\mathrm{ch}L(\lambda)$ in a virtual character χ, an element of $\mathbf{Z}[X(T)]^W$.

Let $\lambda \in X(T)$ and $\alpha \in R$. We define

$$\begin{array}{ll} C_\alpha(\lambda) & = \sum_{0<kp<\langle\lambda+\rho,\alpha^\vee\rangle} \chi(s_\alpha{\cdot}\lambda + kp\alpha) \quad \text{if } \langle\lambda+\rho,\alpha^\vee\rangle > 0, \\ C_\alpha(\lambda) & = \chi(\lambda) - C_{-\alpha}(\lambda) \qquad\qquad\qquad\qquad \text{if } \langle\lambda+\rho,\alpha^\vee\rangle < 0. \end{array}$$

It can be verified that $C_{w(\alpha)}(w{\cdot}\lambda) = (-1)^{\ell(w)}C_\alpha(\lambda)$ for all $w \in W$.

2 Sum formula for socle filtrations

In this section we discuss the sum formula for the socle series of $H^{\ell(w)}(w{\cdot}\lambda)$ for a generic p-regular weight λ. Throughout this section we fix $r = 1$.

2.1 For $\alpha \in R^+$ and $\lambda \in X(T)$, we set $\lambda_j^\alpha = (s_\alpha^*)^j{\cdot}\lambda$ throughout this section. Now let α be a simple root until 2.4, and P be the minimal parabolic subgroup of G containing B and the root subgroup U_α. Let P_1 be the Frobenius kernel in P. The P_1B-module $\mathrm{Ind}_B^{P_1B}(\lambda) = Z_1^\alpha(\lambda)$ has two composition factors: $L_1^\alpha(\lambda)$ as the socle and $L_1^\alpha((s_\alpha)_*{\cdot}\lambda)$ as the head (cf. [8]). Therefore there is an induced map $\phi :\ Z_1(\lambda_1^\alpha) \to Z_1(\lambda)$ via induction functor $\mathrm{Ind}_{P_1B}^{G_1B}$.

Lemma $\sum_{j>0}(-1)^{j+1} chZ_1(\lambda_j^\alpha) = chInd_{P_1B}^{G_1B} L_1^\alpha(\lambda) = ch(Im(\phi))$.

Proof: It follows from the structure of $Z_1^\alpha(\lambda_j^\alpha)$ that there is an exact sequence of P_1B-modules

$$\to Z_1^\alpha(\lambda_j^\alpha) \to Z_1^\alpha(\lambda_{j-1}^\alpha) \to \cdots \to Z_1^\alpha(\lambda_1^\alpha) \xrightarrow{\phi} L_1^\alpha(\lambda) \to 0.$$

Applying the exact functor $\mathrm{Ind}_{P_1B}^{G_1B}$ to the above exact sequence, we get the exact sequence of G_1B-modules

$$\to Z_1(\lambda_j^\alpha) \to Z_1(\lambda_{j-1}^\alpha) \to \cdots \to Z_1(\lambda_1^\alpha) \to \mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda) \to 0.$$

The last surjective map is the induced map ϕ to its image, which is a submodule of $Z_1(\lambda)$. Since each simple module as composition factor occurs in finitely many of $Z_1(\lambda_j^\alpha)$, the sum formula in the lemma follows from this exact sequence. ∎

2.2 Let us denote by $Gr(G)$ and $Gr(G_rB)$ the Grothendieck groups of G and G_rB respectively. For each G_rB-module M, we define

$$\chi_r(M) = \sum_i (-1)^i \mathrm{ch} H_r^i(M).$$

It follows that χ_r defines a group homomorphism $\chi_r : \ Gr(G_rB) \to Gr(G)$. Here we regard the character of a G-module as an element of $Gr(G)$. In particular $\chi_r(Z_r(\mu)) = \chi(\mu)$.

Lemma *If $\alpha \in S$ and $0 < \langle \lambda + \rho, \alpha^\vee \rangle < p$, then $\chi_1 \left(Ind_{P_1B}^{G_1B} L_1^\alpha(\lambda) \right) = \chi(\lambda)$.*

Proof: In this case, λ is a restricted weight with respect to the root system $\{\alpha, -\alpha\}$. Therefore the simple P_1B-module $L_1^\alpha(\lambda)$ is a P-module which is isomorphic to $\mathrm{Ind}_B^P \lambda$. Note that P/P_1B and P/B are complete varieties. We can use (1) and the generalized tensor identity to show

$$R^i\mathrm{Ind}_{P_1B}^P L_1^\alpha(\lambda) \cong R^i\mathrm{Ind}_B^P L_1^\alpha(\lambda) \cong R^i\mathrm{Ind}_B^P \lambda \cong \begin{cases} L_1^\alpha(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can use the argument in 1.2 and a collapsing spectral sequence, arising from $\mathrm{Ind}_B^G = \mathrm{Ind}_P^G \circ \mathrm{Ind}_B^P$, to get

$$R^i\mathrm{Ind}_{G_1B}^G(\mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda)) \cong R^i\mathrm{Ind}_{P_1B}^G L_1^\alpha(\lambda) \cong R^i\mathrm{Ind}_P^G(\mathrm{Ind}_{P_1B}^P L_1^\alpha(\lambda))$$

$$\cong R^i\mathrm{Ind}_P^G(\mathrm{Ind}_B^P L_1^\alpha(\lambda)) \cong R^i\mathrm{Ind}_P^G(\mathrm{Ind}_B^P \lambda) \cong \mathrm{Ind}_B^G \lambda.$$

Now the lemma follows from the definition of χ_1. ∎

2.3 Lemma *If $\alpha \in S$, then $\chi_1(Ind_{P_1B}^{G_1B} L_1^\alpha(\lambda)) = C_{-\alpha}(\lambda)$.*

Proof: First we assume $\langle \lambda+\rho, \alpha^\vee \rangle < 0$. There is an $n \geq 0$ such that $0 < \langle \lambda_n^\alpha+\rho, \alpha^\vee \rangle < p$. By applying the exact functor $\mathrm{Ind}_{P_1B}^{G_1B}$ to an exact sequence of P_1B-modules we can get the exact sequence of G_1B-modules

$$0 \to \mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda_n^\alpha) \to Z_1(\lambda_n^\alpha) \to \cdots \to Z_1(\lambda_1^\alpha) \to \mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda) \to 0.$$

Now we apply the homomorphism χ_1 to it and use Lemma 2.2 to get

$$\begin{aligned} \chi_1(\mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda)) &= \sum_{j=1}^{n-1} (-1)^{j+1} \chi(\lambda_j^\alpha) + (-1)^{n+1} \left(\chi(\lambda_n^\alpha) - \chi_1(\mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda_n^\alpha)) \right) \\ &= \sum_{j=1}^{n-1} (-1)^{j+1} \chi(\lambda_j^\alpha). \end{aligned}$$

One can calculate that n satisfies $-np < \langle \lambda+\rho, \alpha^\vee \rangle < (-n+1)p$. Thus

$$\begin{aligned} \sum_{j=1}^{n-1} (-1)^{j+1} \chi(\lambda_j^\alpha) &= -\left(\sum_{0<2j<n} \chi(\lambda_{2j}^\alpha) + \sum_{0\leq 2j<n-1} \chi(s_\alpha \cdot \lambda_{2j+1}^\alpha) \right) \\ &= -\left(\sum_{0<2j<n} \chi(\lambda + jp\alpha) + \sum_{0\leq 2j<n-1} \chi(\lambda + (n-1-j)p\alpha) \right) \\ &= -\sum_{0<j<n} \chi(\lambda + jp\alpha) = -C_\alpha(s_\alpha \cdot \lambda) = C_{-\alpha}(\lambda). \end{aligned}$$

Here $\lambda_{2j+1}^\alpha = s_\alpha \cdot \lambda + (-n+1+j)p\alpha$. This proves the first case.

Next we assume $0 \leq np < \langle \lambda+\rho, \alpha^\vee \rangle < (n+1)p$ for some n. There is a weight μ such that $0 < \langle \mu+\rho, \alpha^\vee \rangle < p$ and $\lambda = \mu_n^\alpha$. Using a similar argument to that in the first case we can get the exact sequence of G_1B-modules

$$0 \to \mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda) \to Z_1(\mu_n^\alpha) \to \cdots \to Z_1(\mu_1^\alpha) \to \mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\mu) \to 0.$$

Similarly we apply the homomorphism χ_1 to it and use Lemma 2.2 to get

$$\begin{aligned} \chi_1\left(\mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda)\right) &= \sum_{j=1}^{n} (-1)^{n-j} \chi(\mu_j^\alpha) + (-1)^n \chi_1(\mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\mu)) \\ &= (-1)^n \sum_{j=0}^{n} (-1)^j \chi(\mu_j^\alpha). \end{aligned}$$

Note $s_\alpha{\cdot}\mu_j^\alpha = (s_\alpha{\cdot}\lambda)_{n-j}^\alpha$ for all $j = 0, \cdots, n$. It follows from the result we proved in the first case that

$$\begin{aligned}
\chi_1\left(\mathrm{Ind}_{P_1B}^{G_1B} L_1^\alpha(\lambda)\right) &= \sum_{j=0}^{n}(-1)^{n-j+1}\chi((s_\alpha{\cdot}\lambda)_{n-j}^\alpha) \\
&= -\chi(s_\alpha{\cdot}\lambda) + C_{-\alpha}(s_\alpha{\cdot}\lambda) \\
&= \chi(\lambda) - C_\alpha(\lambda) = C_{-\alpha}(\lambda). \blacksquare
\end{aligned}$$

2.4 Each simple G_1T-module extends to a simple G_1B-module and every simple G_1B-module is still simple if it is restricted to G_1T. Thus we can identify the Grothendieck groups of G_1T and G_1B in a natural way. Let W act on $Gr(G_1T)$ via $w{\cdot}L_1(\lambda) = L_1(w_r{\cdot}\lambda)$ and on $Gr(G)$ via the sign character, i.e., $w{\cdot}L(\lambda_0) = (-1)^{\ell(w)}L(\lambda_0)$. It turns out that the homomorphism $\chi_1 :\ Gr(G_1T) \to Gr(G)$ of Abelian groups is consistent with the W-actions. Indeed, by using (1), we have

$$\begin{aligned}
\chi_1(w{\cdot}L_1(\lambda)) &= \sum_i (-1)^i \mathrm{ch} H_1^i(L_1(\lambda^0 + pw{\cdot}\lambda^1)) \\
&= \sum_i (-1)^i \mathrm{ch}(L(\lambda^0) \otimes H^i(w{\cdot}\lambda^1)^{(1)}) \\
&= \mathrm{ch}L(\lambda^0)\left((-1)^{\ell(w)}\chi(\lambda^1)^{(1)}\right) = (-1)^{\ell(w)}\mathrm{ch}L(\lambda^0)\chi(\lambda^1)^{(1)} \\
&= (-1)^{\ell(w)}\chi_1(L_1(\lambda)).
\end{aligned}$$

On the other hand, we know (cf. Lin [17] or Jantzen [15] II Ch 9)

$$[Z_1(w{\cdot}\lambda)] = w_r{\cdot}[Z_1(\lambda)].$$

Here $[Z_1(\lambda)] = \{(\mu, n) \in X(T) \times \mathbf{N} \mid [Z_1(\lambda) : L_1(\mu)] = n\}$ is the decomposition pattern of $Z_1(\lambda)$ in terms of simple modules and their multiplicities. W acts on the patterns simply by acting on the highest weights of the simple modules via $w_r{\cdot}$.

Proposition *Let $\alpha \in R^+$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \not\equiv 0 \pmod p$. Then*

$$\chi_1\left(\sum_{j>0}(-1)^{j+1} Z_1(\lambda_j^\alpha)\right) = C_{-\alpha}(\lambda).$$

Proof: Let $w \in W$ such that $\beta = w(\alpha)$ is simple. Set $\mu = w{\cdot}\lambda$. Then one can easily check $\mu_j^\beta = w{\cdot}(\lambda_j^\alpha)$. So the w-action yields

$$\begin{aligned} w{\cdot}\left(\sum_{j>0}(-1)^{j+1}Z_1(\lambda_j^\alpha)\right) &= \sum_{j>0}(-1)^{j+1}Z_1(w{\cdot}\lambda_j^\alpha) \\ &= \sum_{j>0}(-1)^{j+1}Z_1(\mu_j^\beta). \end{aligned}$$

Applying the map χ_1 on the both sides and using Lemma 2.3 and Lemma 2.1, we get

$$(-1)^{\ell(w)}\chi_1(\sum_{j>0}(-1)^{j+1}Z_1(\lambda_j^\alpha)) = C_{-\beta}(\mu) = (-1)^{\ell(w)}C_{-\alpha}(\lambda). \blacksquare$$

2.5 Assuming Lusztig's conjecture, Andersen and Kaneda [6] proved that, for a p-regular weight λ (requiring $p \geq h$, the Coxeter number of the root system), $Z_1(\lambda)$ is rigid (i.e., the socle and radical filtrations coincide) as G_1-module and its G_1-radical filtration can be calculated from the generic inverse Kazhdan-Lusztig polynomials [19]. The Loewy length of $Z_1(\lambda)$ as G_1-module is $N+1 = \#R^+ + 1$. They further established a sum formula for the G_1-radical filtration:

$$\sum_{j>0}\mathrm{ch}(\mathrm{Rad}^j_{G_1}Z_1(\lambda)) = \sum_{\alpha\in R^+}\sum_{i>0}(-1)^{i+1}\mathrm{ch}Z_1(\lambda_i^\alpha).$$

Using the rigidity, the sum formula can be reformulated in terms of the socle filtration in $G_r(G_1T)$ or $G_r(G_1B)$ as

$$\sum_{j\leq N}\mathrm{Soc}^j_{G_1}Z_1(\lambda) = \sum_{\alpha\in R^+}\sum_{i>0}(-1)^{i+1}Z_1(\lambda_i^\alpha).$$

If λ is *generic* (e.g., $5(h-1) < \langle\lambda^1+\rho, \alpha^\vee\rangle \leq p - 5(h-1)$ for $\alpha \in R^+$), it has been proved in [17] that $\mathrm{Soc}^j_G H^{\ell(w)}(w{\cdot}\lambda) = H_1^{\ell(w)}(\mathrm{Soc}^j_{G_1}Z_1(w{\cdot}\lambda))$ for all $w \in W$.

Theorem *Let $\lambda \in X_+(T)$ be generic and p-regular. If Lusztig's conjecture holds, then*

$$\sum_{j\leq N} ch(Soc^j_G H^{\ell(w)}(w{\cdot}\lambda)) = \sum_{\alpha\in w^{-1}(R^-)} C_\alpha(\lambda)$$

for all $w \in W$.

Proof: Note that $\mathrm{ch}H_1^{\ell(w)}(\mathrm{Soc}_{G_1}^j Z_1(w{\cdot}\lambda)) = (-1)^{\ell(w)}\chi_1(\mathrm{Soc}_{G_1}^j Z_1(w{\cdot}\lambda))$. By using Proposition 2.4 and Andersen-Kaneda's sum formula we mentioned above, we can get

$$\begin{aligned}\sum_{j\le N}\mathrm{ch}(\mathrm{Soc}_G^j H^{\ell(w)}(w{\cdot}\lambda)) &= (-1)^{\ell(w)}\chi_1(\sum_{j\le N}\mathrm{Soc}_{G_1}^j Z_1(w{\cdot}\lambda))\\ &= (-1)^{\ell(w)}\sum_{\alpha\in R^+}\chi_1(\sum_{i>0}(-1)^{i+1}Z_1((w{\cdot}\lambda)_i^\alpha))\\ &= (-1)^{\ell(w)}\sum_{\alpha\in R^+}C_{-\alpha}(w{\cdot}\lambda) = \sum_{\alpha\in w^{-1}(R^-)}C_\alpha(\lambda). \blacksquare\end{aligned}$$

Remark Since all $Z_1(\lambda)$ are rigid for p-regular weight λ, the $H^{\ell(w)}(w{\cdot}\lambda)$ are also rigid in the generic situation [17]. So the sum formula can also be written in terms of the radical filtration as

$$\sum_{j>0}\mathrm{ch}(\mathrm{Rad}_{G_1}^j H^{\ell(w)}(w{\cdot}\lambda)) = \sum_{\alpha\in w^{-1}(R^-)}C_\alpha(\lambda).$$

3 The socle series of induced modules

3.1 In this section, we fix an integer $r>0$ and $\lambda\in X_+(T)$. Bai-Wang-Wen [7] and, independently, Sullivan [20] have proved a formula relating the socle series of $Z_r(\lambda)$ and the socle series of $H^0(\lambda)$ for a dominant weight λ under certain circumstances. The theorem stated here can be found in [7] and [20].

Theorem *Let $\lambda\in X_+(T)$. Suppose that every dominant weight μ with $[Z_r(\lambda) : L_r(\mu)]\neq 0$ satisfies $\mu^1\in\bar{C}_0$. Then $Soc_G^i H^0(\lambda) = H_r^0(Soc_{G_r}^i Z_r(\lambda))$ for all $i>0$.* $\blacksquare$

Remark Suppose we know the socle series of $Z_r(\lambda)$ (e.g., the socle series of $Z_1(\lambda)$ can be calculated from the generic inverse Kazhdan-Lusztig polynomial if Lusztig's conjecture is proved). If we can show that $H_r^j(\mathrm{Soc}_{G_r}^i Z_r(\lambda)) = 0$ for all $j>0$ and $i>0$, then $\mathrm{ch}(\mathrm{Soc}_G^i H^0(\lambda)) = \chi_r(\mathrm{Soc}_{G_r}^i Z_r(\lambda))$. In fact this works in the generic situation as we have discussed in 2.5 since all the composition factors of $Z_r(\lambda)$ have highest weights in the dominant chamber. If the group is of either type A_2 or B_2 and λ^v is in C_{p^2}, the socle series of $H^0(\lambda)$ can be easily calculated from the generic one by using the inequality in Theorem 3.3 of [7] since all the composition factors of $Z_r(\lambda)$ have multiplicity one. The above mentioned vanishing of H_r^k on socle filtration can also be established by

using Proposition 3.2 of Sullivan [20] in these two cases with $r = 1$. Sullivan [20] has studied the relation between the G_rB-structure of a G_r-semisimple G_rB-module and the composition factors of the induced G-module from it . In this section we develop some arguments, which will be used in Section 4 to calculate the socle series of $H^0(\lambda)$ with $\lambda^v \in C_{p^2}$ for the group of type G_2.

3.2 From now on, we fix a dominant weight λ and assume that each composition factor $L_r(\mu)$ of $Z_r(\lambda)$ satisfies $\mu^1 \in W \cdot \bar{C}_0$. Then the conditions of Theorem 3.1 are satisfied. For notational simplicity we denote $Z^i = \mathrm{Soc}^i_{G_r} Z_r(\lambda)$ and $Z = Z_r(\lambda)$ and fix a dominant weight μ till the end of this section.

Lemma *Let M be a G_rB-module such that the 1-parts of the highest weights of all composition factors of M are in $W \cdot \bar{C}_0$. If $[H^\ell_r(M) : L(\mu)] \neq 0$, then there exists $w \in W$ with $\ell(w) = \ell$ such that $[M : L_r(w_r \cdot \mu)] \neq 0$.*

Proof: Consider a composition series $M = M^0 \supset M^1 \supset \cdots \supset M^t = 0$. If $[M : L_r(w_r \cdot \mu)] = 0$ for all $w \in W$ with $\ell(w) = \ell$, then we have $[H^\ell_r(M^i/M^{i+1}) : L(\mu)] = 0$ for $i = 0, \cdots, t-1$ by Theorem 1.2. Therefore, $[H^\ell_r(M^i) : L(\mu)] \leq [H^\ell_r(M^{i+1}) : L(\mu)]$ following the exact sequence $H^\ell_r(M^{i+1}) \to H^\ell_r(M^i) \to H^\ell_r(M^i/M^{i+1})$. Iterating the above inequality we have $[H^\ell_r(M) : L(\mu)] \leq [H^\ell_r(M^i) : L(\mu)]$ for all i and therefore $[H^\ell_r(M) : L(\mu)] = 0$. ∎

Remark Recall from [9] that the action of $w_r\cdot$ can be described geometrically as follows: $w_r\cdot$ translates the p^r-box with upper vertex μ^v to the p^r-box with upper vertex $w_r \cdot \mu^v$. Look at the decomposition pattern of M. If none of the upper vertices of the p^r-boxes containing a highest weight of composition factors of M is in a w-chamber (the one containing $w \cdot X_+(T)$) with $\ell(w) = \ell$, then $H^\ell_r(M) = 0$ by this lemma.

Following the exact sequence $H^\ell_r(Z^j) \to H^\ell_r(Z^j/Z^{j-1}) \to H^{\ell+1}_r(Z^{j-1})$, one can see that if $[H^{\ell+1}_r(Z^{j-1}) : L(\mu)] = [H^\ell_r(Z^j) : L(\mu)] = 0$, then $[H^\ell_r(Z^j/Z^{j-1}) : L(\mu)] = 0$. The following proposition extends Proposition 3.2 of Sullivan in [20].

Proposition *Let j be the minimal number such that $[Z : L_r(w_r\cdot\mu)] = [Z^j : L_r(w_r\cdot\mu)]$ for all $w \in W$ and $\ell = \max\{\ell(w) \mid [Z : L_r(w_r\cdot\mu)] \neq 0\}$. Then we have the following:*

(i) $[H_r^k(Z^j) : L(\mu)] = 0$ *for all* $k > 0$ *and* $[H_r^\ell(Z^j/Z^{j-1}) : L(\mu)] = 0$.

(ii) If $[Z^{j-1} : L_r(w_r \cdot \mu)] = 0$ *for all* $w \in W$ *with* $\ell(w) \geq \ell$, *then* $[H_r^k(Z^j/Z^{j-1}) : L(\mu)] = 0$ *for all* $k \geq \ell - 1$ *(if* $\ell \geq 2$*).*

(iii) If all composition factors of the form $L_r(w_r \cdot \mu)$ *(*$w \in W$*) occur in only one socle layer of* Z*, then* $[H_r^k(Z^i) : L(\mu)] = 0$ *for all* $i > 0$ *and* $k > 0$.

Proof: Note that $H_r^k(Z) = 0$ for all $k > 0$ by Kempf's vanishing theorem. It follows from the induced long exact sequence from $0 \to Z^j \to Z \to Z/Z^j \to 0$ and the above lemma that $[H_r^k(Z^j) : L(\mu)] = 0$ for all $k > 0$. The second part of (i) follows from the above exact sequence and the definition of ℓ. (iii) is a special case of (ii). The assumption in (ii) implies $[H_r^\ell(Z^i/Z^{i-1}) : L(\mu)] = 0$ for all $i \neq j$. Now the proposition follows from the following exact sequence:

$$H_r^{\ell-1}(Z^j) \to H_r^{\ell-1}(Z^j/Z^{j-1}) \to H_r^\ell(Z^{j-1}) \to H_r^\ell(Z^j) \to H_r^\ell(Z^j/Z^{j-1}) \to H_r^{\ell+1}(Z^{j-1}). \blacksquare$$

3.3 In this and next subsection we follow the notations of Sullivan [20]. Let $[\mu]$ be the block of μ for B. It follows from Andersen [2](Corollary 2.4) that $w_r \cdot \mu \subseteq [\mu]$. For a B-module N, $N_{[\mu]} = \oplus_{\nu \in [\mu]} N_\nu$ is a B-submodule of N and $N = \oplus N_{[\mu]}$. For a G_r-semisimple G_rB-module M Sullivan [20] proved that

$$M = \bigoplus_{\mu^0 \in X_r(T)} L(\mu^0) \otimes N(\mu^0)^{(r)}. \tag{2}$$

Here $N(\mu^0) = \mathrm{Hom}_{G_r}(L(\mu^0), M)^{(-r)}$ is a B-module. If $\mu = \mu^0 + p^r\mu^1 \in X_+(T)$, we define $M_{[\mu]} = L(\mu^0) \otimes (N(\mu^0)_{[\mu^1]})^{(r)}$. Let $M^0_{[\mu]}$ be the maximal submodule of $M_{[\mu]}$ not containing any composition factors of the form $L_r(w_r \cdot \mu)$ with $w \in W$. Then $M^0_{[\mu]} = L(\mu^0) \otimes (N(\mu^0)^0_{[\mu^1]})^{(r)}$, where $N(\mu^0)^0_{[\mu^1]}$ is the maximal submodule of $N(\mu^0)_{[\mu^1]}$ not containing any weights in $w \cdot \mu^1$. Denote $M^1_{[\mu]} = M_{[\mu]}/M^0_{[\mu]}$. If we denote $N(\mu^0)^1_{[\mu^1]} = N(\mu^0)_{[\mu^1]}/N(\mu^0)^0_{[\mu^1]}$, then $M^1_{[\mu]} = L(\mu^0) \otimes (N(\mu^0)^1_{[\mu^1]})^{(r)}$.

Lemma *(Sullivan)* $[H_r^0(M) : L(\mu)] = [Soc_{G_rB} M^1_{[\mu]} : L_r(\mu)]$. $\blacksquare$

Let $\phi : M \to E$ be a G_rB-homomorphism between G_r-semisimple G_rB-modules M and E. Then for each $\mu \in X_+(T)$ there is an induced map $\phi_\mu : M_{[\mu]} \to E_{[\mu]}$. In fact ϕ induces a B-map:

$$\mathrm{Hom}_{G_r}(L(\mu^0), M) \to \mathrm{Hom}_{G_r}(L(\mu^0), E)$$

which is consistent with the Frobenius twists.

Since $\phi_\mu(M^0_{[\mu]})$ contains no composition factors of the from $L_r(w_r{\cdot}\mu)$, we have $\phi_\mu(M^0_{[\mu]}) \subseteq E^0_{[\mu]}$. Therefore ϕ induces a map $\phi^1_\mu :\ M^1_{[\mu]} \to E^1_{[\mu]}$.

Proposition *If ϕ kills no composition factors of the form $L_r(w_r{\cdot}\mu)$ with $w \in W$, then $[H^0_r(M) : L(\mu)] \leq [H^0_r(E) : L(\mu)]$. Moreover, if $[E/Im\phi : L_r(w_r{\cdot}\mu)] = 0$ for all $w \in W$, then $[H^k_r(M) : L(\mu)] = [H^k_r(E) : L(\mu)]$ for all $k \geq 0$.*

Proof: To show the first statement, we show that the induced map ϕ^1_μ is injective. In fact ϕ_μ kills no simple factor of the form $L_r(w_r{\cdot}\mu)$ with $w \in W$. Therefore, ϕ^1_μ kills no simple factor in $\mathrm{Soc}_{G_rB}M^1_{[\mu]}$ which contains only the simple factors of such form. Now the first statement follows from the injectivity of ϕ^1_μ and the lemma of Sullivan.

Note that $[H^k_r(\mathrm{Ker}(\phi)) : L(\mu)] = [H^k_r(E/\mathrm{Im}(\phi)) : L(\mu)] = 0$ for all $k \geq 0$ by the second assumption. Considering the two induced long exact sequences from $0 \to \mathrm{Ker}\phi \to M \to \mathrm{Im}\phi \to 0$ and $0 \to \mathrm{Im}\phi \to E \to E/\mathrm{Im}\phi \to 0$, we can get

$$[H^k_r(M) : L(\mu)] = [H^k_r(\mathrm{Im}(\phi)) : L(\mu)] = [H^k_r(E) : L(\mu)]$$

for all $k \geq 0$. ∎

3.4 Recall from Andersen [2] (Corollary 2.4) that $\mathrm{Ext}^1_B(\mu, \nu) \neq 0$ if and only if $\mu = \nu + p^j\alpha$ for some $j \geq 0$ and $\alpha \in S$.

Lemma *Let $\alpha \in S$ and N be a B-module. If N has a quotient $\bar{N}$ such that $Soc_B\bar{N} \supseteq N_\nu \oplus N_{\nu+i\alpha}$ for a weight ν and an integer i with $p > i \geq 0$, then for each n with $i \leq n < p$, there is a quotient N' of N such that $Soc_BN' = N_\nu \oplus N_{\nu+n\alpha}$.*

Proof: We use induction on n. If $n = i$, then we can choose a maximal submodule of $\bar{N}$ not containing weights ν and $\nu+i\alpha$ (since N is finite dimensional). The quotient N' of $\bar{N}$ by this submodule contains only the weights ν and $\nu+i\alpha$ in its socle. Therefore it has the required socle. Now suppose N'_n is the quotient found for $n \geq i$ and $n+1 < p$. Let $\phi :\ N \to N'_n$ be the quotient map. $\mathrm{Ker}(\phi)$ has a quotient consisting of all weight spaces $\mathrm{Ker}(\phi)_\eta$ with $\eta \geq \nu + n\alpha$. This implies that N has a quotient $\bar{N}_n$ with the following properties:

(i) $\dim(\bar{N}_n)_\eta = \dim N_\eta$ for all $\eta \geq \nu + n\alpha$ or $\eta = \nu$;

(ii) $\mathrm{Soc}_B \bar{N}_n \supseteq N_\nu \oplus N_{\nu+n\alpha}$;

(iii) $\eta \geq \nu$ for each weight η of $\bar{N}_n$.

Next, we show that the only weight η of $\bar{N}_n$ with $\mathrm{Ext}^1_B(\nu + (n+1)\alpha, \eta) \neq 0$ is $\nu + n\alpha$. In fact, $\nu + (n+1)\alpha = \eta + p^j\beta$ for some $j \geq 0$ and $\beta \in s$ as we mentioned before this lemma. On the other hand, $0 \leq \eta - \nu = (n+1)\alpha - p^j\beta$. This forces $j = 0$ and $\beta = \alpha$ since $n + 1 < p$.

Now let us consider the quotient $\bar{N}_n/N_{\nu+n\alpha}$, which is also a quotient of N and $\mathrm{Soc}_B(\bar{N}_n/N_{\nu+n\alpha}) \supseteq N_\nu \oplus N_{\nu+(n+1)\alpha}$. We can follow the argument used in the proof for $n = i$ to get the required quotient with the expected B-socle for $n + 1$. ∎

Let M be a G_r-semisimple G_rB-module and $\mu \in X_+(T)$. The following properties are easily verified:

(i) $M_{[\mu]} \otimes p^r\xi = (M \otimes p^r\xi)_{[\mu+p^r\xi]}$ for all $\xi \in X(T)$;

(ii) $M_{[\mu]} = M_{[\mu+p^r\xi]}$ for all $\xi \in \mathbf{Z}R$ (the root lattice);

(iii) $M_{[\mu]} \otimes p^r\xi = L(\mu^0) \otimes ((N(\mu^0) \otimes \xi)_{[\mu^1+\xi]})^{(r)}$ for all $\xi \in X(T)$ (cf. (2).

Let $\alpha \in S$ and $\nu = (s_\alpha)_r{\cdot}\mu$. Choose $\xi \in X(T)$ such that $\langle \xi, \alpha^\vee \rangle = -1$. Then $(s_\alpha)_r{\cdot}(\mu + p^r(\xi + \alpha)) = \nu + p^r\xi$.

Proposition *Suppose ν and μ are the only highest weights of composition factors of M in $W_r{\cdot}\mu$. If $[H^0_r(M) : L(\mu)] = [M : L_r(\mu)]$, then*

$$[H^0_r(M \otimes p^r\xi) : L(\mu + p^r(\xi + \alpha))] = [M : L_r(\mu + p^r\alpha)]$$

for any $\xi \in X(T)$ with $\langle \xi, \alpha^\vee \rangle = -1$ such that $\nu + p^r\xi$ and $\mu + p^r(\alpha + \xi)$ are the only highest weights of composition factors of $M \otimes p^r\xi$ in $W_r{\cdot}(\mu + p^r(\alpha + \xi))$.

Proof: It follows from Lemma 3.3 that the assumption in the proposition implies $\mathrm{Soc}_{G_rB} M^1_{[\mu]} = L_r(\nu)^s \oplus L_r(\mu)^t$, where $s = [M : L_r(\nu)]$ and $t = [M : L_r(\mu)]$. By the definition of $M^1_{[\mu]}$ we have $\mathrm{Soc}_B N(\mu^0)^1_{[\mu^1]} = N(\mu^0)_{\nu^1} \oplus N(\mu^0)_{\mu^1}$. By the lemma above, there is a quotient N' of $N(\mu^0)_{[\mu^1]}$ such that $\mathrm{Soc}_B N' = N(\mu^0)_{\nu^1} \oplus N(\mu^0)_{\mu^1+\alpha}$.

Since $N(\mu^0)_{[\mu^1]} \otimes \xi = (N(\mu^0) \otimes \xi)_{[\mu^1+\xi]} = (N(\mu^0) \otimes \xi)_{[\mu^1+\xi+\alpha]}$, $N' \otimes \xi$ is a quotient of $(N(\mu^0) \otimes \xi)_{[\mu^1+\xi+\alpha]}$ such that $\mathrm{Soc}_B(N' \otimes \xi) = (N(\mu^0) \otimes \xi)_{\nu^1+\xi} \oplus (N(\mu^0) \otimes \xi)_{\mu^1+\xi+\alpha}$. Therefore $N' \otimes \xi = (N(\mu^0) \otimes \xi)^1_{[\mu^1+\xi=\alpha]}$ and $(M \otimes p^r\xi)^1_{[\mu+p^r(\xi+\alpha)]} = L(\mu^0) \otimes (N' \otimes \xi)^{(r)}$ for any ξ chosen as in the proposition. Thus by using Lemma 3.3 once again, the proposition follows from the following calculation:

$$[H^0_r(M \otimes p^r\xi) : L(\mu + p^r(\xi + \alpha)] = \dim(\mathrm{Soc}_B(N' \otimes \xi))_{\mu^1+\xi+\alpha}$$

$$= \dim(N(\mu^0) \otimes \xi)_{\mu^1+\xi+\alpha} = \dim N(\mu^0)_{\mu^1+\alpha} = [M : L_r(\mu + p^r\alpha)]. \blacksquare$$

3.5 Let $\mu \in X_+(T)$. We still use the notations in 3.2 and assume in this subsection $[H^k_r(Z^i/Z^{i-1}) : L(\mu)] = 0$ for all $i > 0$ and $k > 1$ (this happens when the weight λ close to only one chamber wall in the bottom p^2-alcove for $r = 1$). We study $[H^1_r(Z^i) : L(\mu)]$ in this subsection. First, by induction one can show that $[H^k_r(Z^i) : L(\mu)] = 0$ for all $i > 0$ and $k > 1$.

Proposition *Let $i > 0$. If $[H^1_r(Z^i) : L(\mu)] = 0$, then*

(i) $[H^1_r(Z^i/Z^{i-1}) : L(\mu)] = 0$ *and* $[H^0_r(Z^i/Z^{i-1}) : L(\mu)] = [\chi_r(Z^i/Z^{i-1}) : L(\mu)]$;

(ii) $[H^1_r(Z^{i-1}) : L(\mu)] \leq [\chi_r(Z^i/Z^{i-1}) : L(\mu)]$.

Proof: This can be easily checked from the induced exact sequence in 3.2. $\blacksquare$

Remark By using this proposition inductively (from a higher socle layer to a lower one), one can get the following algorithm: If i is the maximal integer such that $[\chi_r(Z^i/Z^{i-1}) : L(\mu)] \neq 0$, then $[H^1_r(Z^j) : L(\mu)] = 0$ for all $j \geq i$. This algorithm will be used in the calculation in next section for the group of type G_2.

4 Socle series of H^0 for the group of type G_2

4.1 Jantzen has calculated the composition factors of $H^0(\lambda)$ in his unpublished paper for the group of type G_2 before Lusztig formulated his conjecture. However, Lusztig has checked his conjecture in this case based on Jantzen's calculation. So Lusztig's conjecture has been verified in this situation. We define the socle series pattern of $Z_1(\lambda)$ by attaching numbers to alcoves such that the number of times a

number j appears in an alcove is the multiplicity of the simple factor with highest weight in this alcove in the j-th socle layer Z^j/Z^{j-1}. Now we can use the results of Andersen and Kaneda [6] to get the socle series pattern of $Z_1(\lambda)$ through the generic inverse Kazhdan-Lusztig polynomials. So we will start from these twelve types of socle series patterns based on the twelve alcove types. The calculation in this section is for p-regular weights λ (the weights inside an alcove). We denote the twelve alcove types by the numbers $1, 2, \cdots, 12$ following the strong linkage order with 1 to be the bottom alcove. However the alcove $s_{2\beta+3\alpha,p}\cdot C_0$ is of type 6 (we denote by α and β the two simple roots with α short). In this section we fix a dominant weight λ as in 3.2 and use the notations there, but $r = 1$. Our goal is to show $H_1^k(Z^i) = 0$ for all $k > 0$ and $i > 0$.

Take a socle series pattern of $Z_1(\lambda)$ and place the chamber walls properly such that the weight λ is in the dominant chamber. Look at the upper vertices of the boxes containing a highest weight of composition factors of $Z_1(\lambda)$. One finds that none of those vertices is in a w-chamber with $\ell(w) > 2$. Therefore, by Lemma 3.2 (rather the remark) $H_1^k(M) = 0$ for all $k > 2$ and any subquotient M of $Z_1(\lambda)$. We will show $H_1^k(Z^i) = 0$ for $k = 2$ in 4.2 and then for $k = 1$ in 4.3.

4.2 For $k = 2$, we only need to show that $H_1^2(Z^i/Z^{i-1}) = 0$ for all i by showing that all the possible composition factors as described in Lemma 3.2 are not in it. Table 1 lists all λ's with the potential composition factors of highest weight μ and the associated w as in Lemma 3.2 for some i, i.e., $[Z_1(\lambda) : L_1(w_r\cdot\mu)] \neq 0$ and $\ell(w) = 2$. In this table we denote $\lambda = (n, m; i)$ if $\lambda^1 = n\omega_\alpha + m\omega_\beta$ and λ^0 is in the alcove of type i. Here ω_α and ω_β are the fundamental weights.

By inspection of the socle series patterns, we find that for each $\mu \in X_+(T) \cap W_p \cdot \lambda$ there are at most two socle layers of $Z_1(\lambda)$ containing the composition factors of the form $L_1(y_r \cdot \mu)$ with $y \in W$ (cf. Remark 3.2). For each dominant weight μ, let the j-th (j_0-th) layer be the highest (lowest) layer containing a composition factor of the form $L_1(y_r \cdot \mu)$ with $y \in W$. So j is the same one as defined in Proposition 3.2 if j exists. Thus we only need to show $[H_1^2(Z^i/Z^{i-1}) : L(\mu)] = 0$ for $i = j,\ j_0$. We first use Proposition 3.2. If $j = j_0$, we are done by Proposition 3.2(iii). So we only consider those μ such that $j < j_0$ in in rest of this subsection and 4.3. In this case, $\ell = 2$. Then

λ	w	μ
(1,0;1)	$s_\beta s_\alpha$	(0,0;1)
(1,0;2)	$s_\beta s_\alpha$	(0,0;3)
(2,0;1)	$s_\beta s_\alpha$	(0,0;1), (0,0;2), (0,0;3)
(1,0;2)	$s_\beta s_\alpha$	(0,0;1), (0,0;2)
(2,0;2)	$s_\beta s_\alpha$	(0,0;1)
(1,0;3)	$s_\beta s_\alpha$	(0,0;1), (0,0;2)
(0,1;3)	$s_\alpha s_\beta$	(0,0;1)
(0,0;4)	$s_\alpha s_\beta$	(0,0;1)
(1,0;7)	$s_\beta s_\alpha$	(0,0;1), (0,0;2)
(1,0;8)	$s_\beta s_\alpha$	(0,0;1)
(0,0;9)	$s_\beta s_\alpha$	(0,0;1)

Table 1: Possible highest weights μ of composition factors in $H_1^2(Z^i/Z^{i-1})$

$[H_1^2(Z^j/Z^{j-1}) : L(\mu)] = 0$ by Proposition 3.2(i). Therefore we only need to consider those cases such that some $L_1(w_r \cdot \mu)$ (with w as in Table 1) is in the j_0-th layer. This happens only in the first two rows in Table 1. In the case of the first row, one can see from Figure 1 by drawing the chamber walls correctly (λ should be in the alcove with 1) that there are two numbers 3 and 5 in the alcove containing the weight μ (one of the marked alcoves), but the alcove containing the weight $w_r \cdot \mu$ is the one with a circled 3, which indicates that the composition factor $L_1(w_r \cdot \mu)$ is in the third layer (the j_0-th layer).

In the case of the first two rows, $j_0 = j - 2$ ($j = 5$ in the first row and $j = 6$ in the second row). Recall that the composition factors of the form $L_1(y_r \cdot \mu)$ ($y \in W$) appear only in the j-th and j_0-th layer and $[H^2(Z^j/Z^{j-1}); L(\mu)] = 0$. It follows from the long exact sequence arising from the exact sequence $0 \to Z^{j-2} \to Z^{j-1} \to Z^{j-1}/Z^{j-2} \to 0$ that $[H_1^2(Z^{j-2}) : L(\mu)] = [H_1^2(Z^{j-1}) : L(\mu)]$ since Z^{j-1}/Z^{j-2} has no composition factors of the form $L_1(y_r \cdot \mu)$ ($y \in W$). However, $[H_1^k(Z^{j-3}) : L(\mu)] = 0$ for all $k \geq 0$, therefore, $[H_1^2(Z^{j-2}) : L(\mu)] = [H_1^2(Z^{j-2}/Z^{j-3}) : L(\mu)]$. Now we only need to show $[H_1^1(Z^j/Z^{j-1}) : L(\mu)] = 0$ since in the exact sequence $H_1^1(Z^j/Z^{j-1}) \to H_1^2(Z^{j-1}) \to$

$H_1^2(Z^j)$ we have $[H_1^2(Z^j) : L(\mu)] = 0$ by Proposition 3.2(i).

To show $[H_1^1(Z^j/Z^{j-1}) : L(\mu)] = 0$ in the cases of the first two rows in the table, we consider the map $\phi : \ Z_1(\lambda) \to Z_1((s_\alpha)_*\cdot\lambda)$. The character of Im$\phi$ can be calculated by using Lemma 2.1. By comparing the multiplicity of each composition factor in the image of ϕ and its positions in the socle series of both $Z_1(\lambda)$ and $Z_1((s_\alpha)_*\cdot\lambda)$, one can see that the induced map $M = \mathrm{Soc}_{G_1}^j Z_1(\lambda)/\mathrm{Soc}_{G_1}^{j-1} Z_1(\lambda) \to \mathrm{Soc}_{G_1}^{j-1} Z_1((s_\alpha)_*\cdot\lambda)/\mathrm{Soc}_{G_1}^{j-2} Z_1((s_\alpha)_*\cdot\lambda) = E$ has no simple factor of the form $L_1(y_r\cdot\mu)$ in its kernel and cokernel. Therefore, Proposition 3.3 implies that $[H_1^k(M) : L(\mu)] = [H_1^k(E) : L(\mu)]$. For the weight $(s_\alpha)_* \cdot \lambda$ and μ, one can check that $\ell = 1$ (cf. Proposition 3.2) and E is the highest layer of $Z_1((s_\alpha)_* \cdot \lambda)$ containing any composition factor of the form $L_1(y_r \cdot \mu)$, therefore $[H_1^k(E) : L(\mu)] = 0$ by Proposition 3.2(i) for $k \geq 1$.

4.3 We call a simple module $L(\mu)$ well placed in $H^0(\lambda)$ if its positions in the socle series of $H^0(\lambda)$ are determined through the cancellation principle, equivalently, $[H_1^k(\mathrm{Soc}_{G_1}^i Z_1(\lambda)) : L(\mu)] = 0$ for all i and $k > 0$. As we have shown the vanishing of H_1^2 on the socle filtration of $Z_1(\lambda)$ in 4.2, we only need to show the vanishing of H_1^1 by showing $[H_1^1(Z^i/Z^{i-1}) : L(\mu)] = 0$ for all the possible candidates $L(\mu)$ as described in Lemma 3.2. Now we fix a such μ. Let j and j_0 be as in 4.2 so that the composition factors of the form $L_1(y_r \cdot \mu)$ $(y \in W)$ appear only in the j-th and j_0-th layer. If $j = j_0$ we are done by Proposition 3.2(iii). We only need to show $[H_1^1(Z^{j_0}/Z^{j_0-1}) : L(\mu)] = 0$ since $[H_1^1(Z^j/Z^{j-1}) : L(\mu)] = 0$ follows from the exact sequence $H_1^1(Z^j) \to H_1^1(Z^j/Z^{j-1}) \to H_1^2(Z^{j-1})$ by 4.2 and Proposition 3.2(i). We will show $[H_1^1(Z^{j_0}/Z^{j_0-1}) : L(\mu)] = 0$ by considering three cases and assuming $j_0 < j$.

Case 1 If $[\chi_1(Z^j/Z^{j-1}) : L(\mu)] = 0$, then we can use Proposition 3.5 recursively (rather the remark) to get $[H_1^1(Z^i) : L(\mu)] = 0$ for $i \geq j_0$. However, $[H_1^1(Z^i) : L(\mu)] = 0$ for $i < j_0$ follows from Lemma 3.2.

Case 2 In this case we assume $[\chi_1(Z^j/Z^{j-1}) : L(\mu)] \neq 0$ and $\langle \lambda + \rho, \gamma^\vee \rangle > p$ for all simple roots $\gamma \in S$. Denote $M = Z^{j_0}/Z^{j_0-1}$. By inspection of the socle series of $Z_1(\lambda)$ under the conditions assumed in this case, one finds that $[M : L_1((s_\gamma)_r\cdot\mu)] \leq 1$ for any simple root $\gamma \in S$. If $[H_1^1(M) : L(\mu)] \neq 0$, then $[M : L_1((s_\gamma)_r\cdot\mu)] = 1$ for a simple root $\gamma \in S$. It can be checked under the assumptions in this case that $[M : L_1((s_\eta)_r \cdot \mu)] = 0$ for the other simple root η (i.e., there is no cancellation in the other chamber wall for

μ at the j_0-th layer). Thus we have $[H_1^0(M) : L(\mu)] = [M : L_1(\mu)]$ from the following:

$$\begin{aligned}[H_1^0(M) : L(\mu)] &= [\chi_1(M) : L(\mu)] + [H_1^1(M) : L(\mu)] \\ &= [M : L_1(\mu)] - [M : L_1((s_\gamma)_r \cdot \mu)] + [H_1^1(M) : L(\mu)].\end{aligned}$$

Choose $\xi = -\omega_\gamma + m\omega_\eta$ with appropriate m such that ξ satisfies the conditions in Proposition 3.4. Thus we have

$$[H_1^0(M \otimes p\xi) : L(\mu + p(\gamma + \xi))] = [M : L_1(\mu + p\gamma)].$$

It is easily seen that $M \otimes p\xi = \mathrm{Soc}_{G_1}^{j_0} Z_1(\lambda + p\xi)/\mathrm{Soc}_{G_1}^{j_0-1} Z_1(\lambda + p\xi)$ and $\langle \lambda + p\xi + \rho, \gamma^\vee \rangle = \langle \lambda + \rho, \gamma^\vee \rangle - p$. If $L(\mu + p(\gamma + \xi))$ is well placed in $H^0(\lambda + p\xi)$, we can get

$$\begin{aligned}&[H_1^0(M \otimes p\xi) : L(\mu + p(\gamma + \xi))] = [\chi_1(M \otimes p\xi) : L(\mu + p(\gamma + \xi))] \\ &= [M \otimes p\xi : L_1(\mu + p(\gamma + \xi))] - [M \otimes p\xi : L_1((s_\gamma)_r \cdot (\mu + p(\gamma + \xi)))].\end{aligned}$$

Here $(s_\gamma)_r \cdot (\mu + p(\gamma + \xi)) = (s_\gamma)_r \cdot \mu + p\xi$. Thus

$$[M \otimes p\xi : L_1((s_\gamma)_r \cdot (\mu + p(\gamma + \xi)))] = [M \otimes p\xi : L_1((s_\gamma)_r \cdot \mu + p\xi)] = [M : L_1((s_\gamma)_r \cdot \mu)],$$

$$[M \otimes p\xi : L_1(\mu + p(\gamma + \xi))] = [M : L_1(\mu + p\gamma)].$$

Combining the above calculations, we get $[M : L_1(\mu + p\gamma)] = [M : L_1(\mu + p\gamma)] - [M : L_1((s_\gamma)_r \cdot \mu)]$. Therefore $[M : L_1((s_\gamma)_r \cdot \mu)] = 0$ and this contradicts the assumption $[H_1^1(M) : L(\mu)] \neq 0$. This shows that the well-placement of $L(\mu + p(\gamma + \xi))$ in $H^0(\lambda + p\xi)$ implies the well-placement of $L(\mu)$ in $H^0(\lambda)$. Thus it will be enough to show $[H_1^1(M) : L(\mu)] = 0$ when $0 < \langle \lambda + \rho, \gamma^\vee \rangle < p$ for a simple root γ (this will be done in Case 3).

Remark To check the well-placement of $L(\mu + p(\gamma + \xi))$ in $H^0(\lambda + p\xi)$, one can use the same pattern by redrawing the γ-wall one strip closer to λ. Then the alcove containing the weight $\mu + p(\gamma + \xi)$ is the translate of the alcove containing the weight μ by $p(\gamma + \xi)$. It should be pointed out that the two alcoves containing λ and $\lambda + p\xi$ respectively are of the same type. This fact will be used in the Case 3.

In Figure 1 we show how this works. Here λ is of type 1 in the box with lower vertex $p(\omega_\alpha + n\omega_\beta) - \rho$. Applying the argument in Case 1, one can see that all $L(\mu)$ are well

placed in $H^0(\lambda)$ except for μ in one of the three alcoves in the dominant chamber with dark sides. $L_1((s_\alpha)_r\cdot\mu)$ has multiplicity one in the j_0-th layer M ($j_0 = 3$ or 4). But when we redraw the α-wall as in the figure we can see the cancellation goes smoothly along the new wall according to the argument in Case 1. So the cancellation has to go smoothly along the original wall by the above argument.

Case 3 In this case we suppose that $0 < \langle \lambda + \rho, \gamma^\vee \rangle < p$ for a simple root γ and that μ satisfies $[\chi_1(\mathrm{Soc}_{G_1}^{j_0} Z_1(\lambda)/\mathrm{Soc}_{G_1}^{j_0-1} Z_1(\lambda)) : L(\mu)] \neq 0$. If λ is in one of the alcoves of types 1,2,3,4,5,6,9, there is no weight μ such that the conditions in this case are satisfied. So combining the arguments in Case 1 and Case 2 and the remark in Case 2, we are done if λ is in one of these aloves. In the remaining of this case we consider only those λ such that there is a weight μ satisfying the conditions in this case. There are eight cases at alcove types 7, 8, 10, 11, 12 along the α-wall and two cases at alcove types 11, 12 along the β-wall. Choose a simple root ζ and consider the map $\phi : \ Z_1(s_\zeta^*\cdot\lambda) \to Z_1(\lambda)$. $\mathrm{ch}(\mathrm{Im}(\phi))$ can be calculated by Lemma 2.1. The positions of $L_1((s_\gamma)_r\cdot\mu)$ and $L_1(\mu)$ in the socle series of both $Z_1(s_\zeta^*\cdot\lambda)$ and $Z_1(\lambda)$ and their multiplicities in $\mathrm{Im}(\phi)$ are given in Table 2, where $\gamma = \alpha$ in the first 8 rows and $\gamma = \beta$ in the last two rows. By comparing the positions of these two simple factors in both $Z_1(s_\zeta^*\cdot\lambda)$ and $Z_1(\lambda)$ and their multiplicities in the image, one can see that the induced map

$$\phi_{j_0} : \ \mathrm{Soc}_{G_1}^{j_0+1} Z_1(s_\zeta^*\cdot\lambda)/\mathrm{Soc}_{G_1}^{j_0} Z_1(s_\zeta^*\cdot\lambda) \to \mathrm{Soc}_{G_1}^{j_0} Z_1(\lambda)/\mathrm{Soc}_{G_1}^{j_0-1} Z_1(\lambda)$$

has no factors of the form $L_1(w_r\cdot\mu)$ in its kernel and cokernel. So Proposition 3.3 shows that

$$H_1^k(\mathrm{Soc}_{G_1}^{j_0+1} Z_1(s_\zeta^*\cdot\lambda)/\mathrm{Soc}_{G_1}^{j_0} Z_1(s_\zeta^*\cdot\lambda)) : L(\mu)] = [H_1^k(\mathrm{Soc}_{G_1}^{j_0} Z_1(\lambda)/\mathrm{Soc}_{G_1}^{j_0-1} Z_1(\lambda)) : L(\mu)]$$

for all $k \geq 0$. Thus the well-placement of $L(\mu)$ in $H^0(s_\zeta^*\cdot\lambda)$ implies the well-placement of $L(\mu)$ in $H^0(\lambda)$. In all of the listed cases, the well-placement of $L(\mu)$ in $H^0(s_\zeta^*\cdot\lambda)$ can be checked through the arguments in Case 1 and 2. One should be careful here since the argument of Case 2 depends on the one in Case 3. However, $s_\eta^* \cdot \lambda$ is in one of the alcoves of type 1–6, for which the argument in Case 2 does not need Case 3 since they are not in the list of Table 2 and the reduction in Case 2 does not change the alcove type.

λ	μ	ζ	pos. in $Z_1(s_\zeta^*\cdot\lambda)$		mult. in $\mathrm{Im}(\phi)$		pos. in $Z_1(\lambda)$	
			μ	$(s_\gamma)_r\cdot\mu$	μ	$(s_\gamma)_r\cdot\mu$	μ	$(s_\gamma)_r\cdot\mu$
(0,n;7)	(0,n;1)	α	3,5	3	2	1	2,4	2
(0,n;7)	(1,n-1;4)	α	4,4,6	4	3	1	3,3,5	3
(0,n;8)	(0,n;1)	α	4,6	4	2	1	3,5	3
(0,n;10)	(1,n-1;1)	α	4,4,4,6	4	4	1	3,3,3,5	3
(0,n;11)	(1,n-1;1)	β	3,5	3,5	2	2	2,4,4	2,4
(0,n;12)	(0,n;4)	α	3,5,5	3,5	3	2	2,4,4	2,4
(0,n;12)	(1,n-1;1)	β	3	3	1	1	2,4	2
(0,n;12)	(1,n-1;2)	β	4,6	4,6	2	2	3,5,5	3,5
(n,0;11)	(n,0;1)	β	3,3,3,5	3	4	1	2,2,2,4	2
(n,0;12)	(n,0;1)	β	4,4,4,6	4	4	1	3,3,3,5	3

Table 2: $L_1(w_r \cdot \mu)$ is not in the kernel and cokernel of ϕ_{j_0}

Finally, let us discuss the situation when λ is close to both chamber wall. Note that the arguments in Case 1 and Case 2 work as long as the conditions are satisfied in each of the two cases. In Case 3, ζ has been deliberately chosen such that $s_\zeta^*\cdot\lambda$ stays in the dominant chamber. But one has to be careful since the kernel of ϕ_{j_0} may have composition factors $L_1(w_r \cdot \mu)$ with $w \neq 1, s_\gamma$. This happens when $n = 0$ in the following pairs of $\{\lambda, \mu\}$: $\{(0, n; 7), (0, n; 1)\}$, $\{(0, n; 8), (0, n; 1)\}$, $\{(n, 0; 11), (n, 0; 1)\}$, $\{(n, 0; 12), (n, 0; 1)\}$. Fortunately, these cases have been included in Case 1 since $[\chi_1(Z^j/Z^{j-1}) : L(\mu)] = 0$.

4.4 Let us conclude the calculation in this section for the group of type G_2. The socle series of $Z_1(\lambda)$ is calculated from the generic inverse Kazhdan-Lusztig polynomials as shown in [6]. In the nongeneric case, the inverse polynomial is constructed by Andersen [3] and then reformulated by Kaneda [16] in terms of the generic inverse polynomials. It has been checked in the G_2 case that the filtration pattern determined from the inverse polynomials is the same as the socle filtration pattern of $H^0(\lambda)$ we calculated here. So we conclude

Theorem *If λ is a p-regular weight such that $0 < \langle \lambda + \rho, \alpha^\vee \rangle \leq p(p-h+2)$ for all $\alpha \in R^+$, then the socle filtration of $H^0(\lambda)$ can be determined by the coefficients of the inverse Kazhdan-Lusztig polynomials. Furthermore, the socle filtration of $H^0(\lambda)$ satisfies the same sum formula as the dual filtration of the Jantzen filtration in the Weyl module $V(-w_0(\lambda))$ does.*

Remark This gives more evidence for the conjecture that the Jantzen filtration should be the radical filtration for all Weyl modules.

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Figure 1: Redrawing of the new chamber wall