FREENESS OF THE QUANTUM COORDINATE ALGEBRAS

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1. Introduction

The main purpose of this note is to prove that the quantum coordinate algebra $\mathcal{A}[U]$ is free over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. In [L1], Lusztig defined the quantum coordinate algebra over $\mathbb{Q}[v, v^{-1}]$ and prove that it is free as $\mathbb{Q}[v, v^{-1}]$-module. In [APW], Andersen, Polo, and Wen defined the quantum coordinate algebra over the ring $\mathcal{A}_{(p,v-1)}$ and proved that the coordinate algebra is free. The main idea was to specialize v to 1 from $\mathcal{A}_{(p,v-1)}$ and then use the result from the representation theory of algebraic groups in characteristics $p > 0$. The two important consequences of specifying v to 1 from $\mathcal{A}_{(p,v-1)}$ are that the set of weights of an integrable U-module is W-invariant and the universal highest weight modules $D(\lambda)$ are free and finitely generated over $\mathcal{A}_{(p,v-1)}$ with characters given by Weyl's character formula using Kempf's vanishing theorem in characteristic p. Using the ring $\mathcal{A}_{(p,v-1)}$ gives restriction on ℓ when one wants to consider cases when q is an ℓth root of 1. In [AW], Andersen and Wen extended the freeness to the ring $\mathcal{A}_1$, which is still larger than $\mathcal{A}$, by studying the representations at mixed case. In this paper we will use Lusztig's canonical basis to replace Kempf's vanishing theorem (or its analogy at mixed cases as in [AW]). But it is still not clear that whether the set of weights of an integrable U-module will be W-invariant (even in the rank 1 case).

In Section 2, we introduce the quantum analogue of Joseph's induction [Jo]. It is not clear that Joseph's induction functor can be defined on all integrable $U^{\geq 0}$-modules of type **1**. In Section 3, the freeness of the universal highest weight module $D(\lambda)$ defined by the Joseph induction over $\mathcal{A}$ is proved using the canonical basis constructed by Lusztig [L2]. We will treat the parabolic cases as well. As a consequence, we prove that integrable modules of type **1** are locally $\mathcal{A}$-finite for the parabolic subalgebras. Then using the freeness of the universal highest weight modules, we follow more or less the argument as in [APW] to prove the freeness of the coordinate algebras of the quantum parabolic algebras over $\mathcal{A}$ in Section 4. In Section 5, the exactness of certain induction functors is proved using the exactness of the quantum coordinate algebras over $\mathcal{A}$. As application,

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we prove the quantum analogues of several classical results over $\mathcal{A}$, such as Demazure's vanishing theorem, Kempf's vanishing theorem. It should be mentioned that Ryom-Hansen has proved Kempf's vanishing and Demazure's vanishing theorems over any field of characteristic p using Kashiwara's crystal basis and Andersen-Wen's result in the mixed case [AW].

Throughout this paper, we will use the notations of [APW, AW] with the exception that algebras U, $U^{\leq 0}$, $U^{\geq 0}$, etc., are the algebras over the ring $\mathcal{A}$. All tensor product will be taken over the ring $\mathcal{A}$ unless the base ring is specified.

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2. Joseph's induction

2.1. In this paper we fix a root system R and a set S of simple roots. Let $(a_{ij})_{1\leq i,j\leq n}$ be the Cartan matrix of R (i.e., $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ji}$) and $d_1, \dots, d_n$ be the smallest positive integers such that the matrix $(d_i a_{ij})$ is symmetric. $\mathcal{A}'$ is the quotient field of $\mathcal{A}$ and U' the quantum enveloping algebra over $\mathcal{A}'$ generated by E_i, F_i, K_i, K_i^{-1} $(i = 1, \dots, n)$. Here we use Lusztig's version as in [L1]. The $\mathcal{A}$-algebra U generated by $E_i^{(r)} = E_i^r/[r]_{d_i}^!$, $F_i^{(r)} = F_i^r/[r]_{d_i}^!$, $(r \geq 0)$ and K_i, K_i^{-1} $(i = 1, \dots, n)$ has a triangular decomposition $U = U^- \otimes U^0 \otimes U^+$. We will use a different notation for the algebras $U^{\leq 0} = U^- U^0$ and $U^{\geq 0} = U^0 U^+$ (instead of the notation in [APW]).

Since we will be working over the ring $\mathcal{A}$, let us state some of the properties of $\mathcal{A}$, which we will use in this paper without giving specific references every-time.

(a). $\mathbb{Z}[v, v^{-1}]$ is a Noetherian domain of global dimension 2 (see [GS, 18.9, 20.3]);

(b). Any projective module over $\mathcal{A}$ is free (a special case of Seshadri's theorem [B]).

For each $\lambda \in X = \mathbb{Z}^n$ and $\sigma \in \{\pm 1\}^n$, let $\chi_{\sigma,\lambda} : U^0 \to \mathcal{A}$ be the algebra homomorphism defined in the same way as in [APW, 1.1]. For each U^0-module M, $M_{\chi_{\sigma,\lambda}} = \{x \in M \mid ux = \chi_{\sigma,\lambda}(u)x, \forall u \in U^0\}$ is called the $\chi_{\sigma,\lambda}$-weight space of M.

Lemma 2.2. *For any U^0-module M, $M_{\chi_{\sigma,\lambda}} \cap M_{\chi_{\delta,\mu}} = 0$ if $\lambda \neq \mu$.*

Proof. Suppose that there are $x \in M_{\chi_{\sigma,\lambda}} \cap M_{\chi_{\delta,\mu}}$, then $(\chi_{\sigma,\lambda}(u) - \chi_{\delta,\mu}(u))x = 0$ for all $u \in U^0$. We will show that the ideal J of $\mathcal{A}$ generated by $\{(\chi_{\sigma,\lambda}(u) - \chi_{\delta,\mu}(u)) \mid u \in U^0\}$ is $\mathcal{A}$. Since $\lambda \neq \mu$, there exists i such that $\lambda_i \neq \mu_i$ (say, $\lambda_i > \mu_i$). By setting $c = -\mu_i$, $t = \lambda_i - \mu_i$, and $u = \left[\begin{smallmatrix} K_i;c \\ t \end{smallmatrix} \right] \in U^0$, we have, by [APW, 1.1],

$$(\chi_{\sigma,\lambda}(u) - \chi_{\delta,\mu}(u)) = \sigma_i^t \left[\begin{smallmatrix} \lambda_i+c \\ t \end{smallmatrix} \right] - \delta_i^t \left[\begin{smallmatrix} \mu_i+c \\ t \end{smallmatrix} \right] = \sigma_i^t. \quad \square$$

Remark 2.3. There exists U^0-module M such that $M_{\chi_{\sigma,\lambda}} \cap M_{\chi_{\delta,\lambda}} \neq 0$ with $\sigma \neq \delta$. In fact the ideal J in $\mathcal{A}$ as in the proof of the lemma is $\mathcal{A}2$. This can be easily verified by using the basis for U^0 constructed by Lusztig in [L1]. However in this paper we only consider U^0-modules of type **1**, i.e., the modules M with $M_{\chi_{\sigma,\lambda}} \neq 0$ only if $\sigma = (1, \dots, 1)$. In this case we simply write $\chi_\lambda = \chi_{\sigma,\lambda}$ and $M_{\chi_{\sigma,\lambda}} = M_\lambda$. We will denote by $\Pi(M)$ the set of weights of M, i.e., the set $\{\lambda \in X \mid M_\lambda \neq 0\}$.

2.4. For each U-module M, $\sum_{\lambda\in X} M_\lambda$ is a U-submodule. By Lemma 2.2, we have $\sum_{\lambda\in X} M_\lambda = \oplus_{\lambda\in X} M_\lambda$. In fact, if $v_{\lambda^0} + v_{\lambda^1} + \cdots + v_{\lambda^r} = 0$ is a shortest expression with distinct λ^i's and nonzero components $v_{\lambda^i} \in M_{\lambda^i}$, then we apply u and χ_{λ^0} to the above expression and then make a difference we would get a shorter expression by choosing some $u \in U^0$. u exist since the ideals of $\mathcal{A}$ generated by $(\chi_{\lambda^i}(u) - \chi_{\lambda^0}(u))$ $(u \in U^0)$ is $\mathcal{A}$. We denote by $\mathcal{F}(M)$ the U-submodule $\{x \in \oplus_{\lambda\in X} M_\lambda \mid E_i^{(r)}x = F_i^{(r)}x = 0, 1 \leq i \leq n, r >> 0\}$. A U-module M is called integrable of type **1** if $\mathcal{F}(M) = M$. We denote by $\mathcal{C}$ the category of all integrable U-modules of type **1**.

Lemma 2.5 (APW, 1.11)**.** *Suppose $M \in \mathcal{C}$.*

(a). *If $0 \neq v \in M_\nu$, and $\lambda_i = \max\{r \mid E_i^{(r)}v \neq 0\}$, $\mu_i = \max\{r \mid F_i^{(r)}v \neq 0\}$, then $\nu_i = \mu_i - \lambda_i$ for all i.*

(b). *If M is $\mathcal{A}$-torsion free, then the subset $\Pi(M)$ of X is W-invariant under the usual action of W on X.*

Proof. The proof of (a) is essentially in [APW, 1.11]. To prove (b), we set $y = E_i^{(\lambda_i)}v$. Then $y \neq 0$ and $F_i^{(r)}y \neq 0$ for $r = 0, 1, \dots, \lambda_i + \mu_i$. In fact,

$$E_i^{(r)}F_i^{(r)}y = \sum_{t=0}^{r} F_i^{(r-t)} \begin{bmatrix} K_i; 2t-2r \\ t \end{bmatrix} E_i^{(r-t)}y = \begin{bmatrix} K_i \\ r \end{bmatrix} y = \begin{bmatrix} \nu_i+2\lambda_i \\ r \end{bmatrix}_{d_i} y \neq 0$$

since M is $\mathcal{A}$-torsion free. Thus $\nu+(\lambda_i-r)\alpha_i$ is a weight of M for all $r = 0, 1, \dots, \lambda_i+\mu_i$. Since $s_{\alpha_i}(\nu) = \nu - \nu_i\alpha_i = \nu + (\lambda_i - r)\alpha_i$ for $r = \mu_i$, we have $s_{\alpha_i}(\nu) \in \Pi(M)$. Now (b) follows from the fact that W is generated by elements s_{α_i} $(i = 1, \dots, n)$. □

Lemma 2.6. *Let M be a U-module such that $M = \oplus_{\lambda\in X} M_\lambda$ and $\Pi(M) \subseteq \cup_{r=1}^{s}(\chi_i - \mathbb{N}R^+)$ for a finite set $\{\chi_1, \dots, \chi_s\} \subseteq X$. Then there exists a unique minimal submodule K such that M/K is in $\mathcal{C}$.*

Proof. Note that M/K is in $\mathcal{C}$ if and only if K satisfies the following: for any $x \in M$, there exists $s_i > 0$ for each i such that $F_i^{(r)}x \in K$ and $E_i^{(r)}x \in K$ for all $r > s_i$. By the assumption on $\Pi(M)$, for each $x \in M$, there exists s_i for each i such that $E_i^{(r)}x = 0$ for all $r > s_i$. Now let $\{K_\alpha \mid \alpha \in \Lambda\}$ be a collection of submodules of M such that M/K_α

is in $\mathcal{C}$ for all $\alpha \in \Lambda$. Let $K = \cap_{\alpha\in\Lambda} K_\alpha$. We will show that M/K is also in $\mathcal{C}$. For fixed $\nu \in X$ and $x \in M_\nu$, let $\lambda^\alpha = (\lambda_1^\alpha, \cdots, \lambda_n^\alpha)$ and $\mu^\alpha = (\mu_1^\alpha, \cdots, \mu_n^\alpha)$ be the sequences of non-negative integers such $\lambda_i^\alpha = \max\{r \mid E_i^{(r)}x \notin K_\alpha\}$ and $\mu_i^\alpha = \max\{r \mid F_i^{(r)}x \notin K_\alpha\}$. Then we have $\nu_i = \mu_i^\alpha - \lambda_i^\alpha$ for all i by Lemma 2.2(a). However, we must have $\lambda_i^\alpha \leq s_i$ for all $\alpha \in \Lambda$. Take $r_i = \nu_i + s_i$. Thus we have that $\mu_i^\alpha \leq r_i$ for all i for all $\alpha \in \Lambda$. Therefore $F_i^{(r)}x \in K$ for all $r > r_i$ and all i. This proves the lemma. □

2.7. Let us consider the categories $\mathcal{C}^{\geq 0}$ and $\mathcal{C}^{\leq 0}$ of $U^{\geq 0}$ and $U^{\leq 0}$-modules respectively, such that a $U^{\geq 0}$-module (or $U^{\leq 0}$-module) M is in $\mathcal{C}^{\geq 0}$ (or in $\mathcal{C}^{\leq 0}$) if and only if $M = \oplus_{\mu\in X} M_\mu$ and for each $v \in M$, $E_i^{(r)}v = 0$ (or $F_i^{(r)}v = 0$) for all $r >> 0$.

Let N be a $U^{\geq 0}$-module in $\mathcal{C}^{\geq 0}$ and finitely generated over $\mathcal{A}$. Then N is a direct sum of weight spaces and the set $\Pi(N)$ is finite. Consider the U-module $M = U \otimes_{U^{\geq 0}} N$ which is a direct sum of weight spaces and $\Pi(M) \subseteq \cup_{\chi\in\Pi(N)}(\chi - \mathbb{N}R^+)$. By Lemma 2.6, there is a unique minimal U-submodule K of M such that M/K is a U-module in $\mathcal{C}$. We denote such M/K by $D(N)$. It can be easily verified that D defines a functor from the category of all $\mathcal{A}$-finite modules in $\mathcal{C}^{\geq 0}$ to $\mathcal{C}$.

For each $\mathcal{A}$-finite module N in $\mathcal{C}^{\geq 0}$, the natural embedding $N \to U \otimes_{U^{\geq 0}} N$ $(x \mapsto 1 \otimes x)$ induces a $U^{\geq 0}$-module homomorphism $N \to D(N)$. The following properties of the functor D can be verified using a similar argument as in [APW, 1.16-1.19]:

(a). D is right exact;

(b). (**Frobenius reciprocity**) For each E in $\mathcal{C}$ and $\mathcal{A}$-finite N in $\mathcal{C}^{\geq 0}$, the map $\phi : \mathrm{Hom}_U(D(N), E) \to \mathrm{Hom}_{U^{\geq 0}}(N, E)$, defined by $\phi(f) = f \circ \sigma$ for all $f \in \mathrm{Hom}_U(D(N)$, is an isomorphism of $\mathcal{A}$-modules;

(c). For each $\mathcal{A}$-finite N in $\mathcal{C}^{\geq 0}$ and finite free $\mathcal{A}$-module E in $\mathcal{C}$, the maps $D(E \otimes N) \to E \otimes D(N)$ and $D(E \otimes N) \to E \otimes D(N)$, induced by the maps $1 \otimes \sigma : E \otimes N \to E \otimes D(N)$ and $\sigma \otimes 1 : N \otimes E \to D(N) \otimes E$ via Frobenius reciprocity, are isomorphisms of U-modules;

(d). $D(N)$ is generated by $\sigma(N)$ as U^--module.

3. Freeness of $D(\lambda)$.

3.1. For each $\lambda \in X$, $\mathcal{A}_\lambda$ is the $U^{\geq 0}$-module such that U^0 acts on it via the character χ_λ and $E_i^{(r)}$ as zero for all $r > 0$. We will write $D(\lambda)$ for $D(\mathcal{A}_\lambda)$ if there is no confusion. Note that $D(\lambda)$ is generated by $v_\lambda = \sigma(1_\lambda)$ as U^--module. Here $v_\lambda \in D(\lambda)_\lambda$. If $D(\lambda) \neq 0$, then $v_\lambda \neq 0$. Since $D(\lambda)$ is in $\mathcal{C}$, the numbers $r_i = \max\{r \mid E_i^{(r)}v_\lambda \neq 0\}$ and $s_i = \max\{s \mid F_i^{(s)}v_\lambda \neq 0\}$ are well defined and finite. By Lemma 2.5, we have $\lambda_i = s_i - r_i$.

However, $r_i = 0$ for all $i = 1, \dots, n$. Thus $\lambda_i \geq 0$ and $\lambda \in X^+$. This shows that $D(\lambda) = 0$ if $\lambda \notin X^+$.

For $\lambda \in X^+$, let $I(\lambda)$ be the left ideal of U generated by $\mathrm{Ann}_{U^{\geq 0}}(\mathcal{A}_\lambda)$. Then $I(\lambda)$ is generated by $\mathrm{Ker}(\chi_\lambda) \subseteq U^0$ and $\{E_i^{(r)} \mid r > 0, 1 \leq i \leq n\}$. We thus have an isomorphism of U-modules $U/I(\lambda) \to M(\lambda) = U \otimes_{U^{\geq 0}} \mathcal{A}_\lambda$.

For $\lambda \in X^+$, let $J^-(\lambda)$ be the left ideal of U^- generated by $\{F_i^{(r_i)} \mid r_i > \lambda_i, 1 \leq i \leq n\}$. Using a similar argument as in [APW, 1.20], one can show that $J(\lambda) = J^-(\lambda) + I(\lambda)$ is a left ideal of U

Theorem 3.2. *For $\lambda \in X^+$, $U/J(\lambda)$ is a U-module in $\mathcal{C}$ and is finitely generated and free over $\mathcal{A}$. The character of $U/J(\lambda)$ is given by Weyl's character formula.*

Proof. Let v_λ be the image of 1 in $U/J(\lambda)$. As a U^--module, $U/J(\lambda)$ is isomorphic to $U^-/J^-(\lambda)$. So we only need to show that $U^-/J^-(\lambda)$ is finitely generated and free over $\mathcal{A}$.

Let $\mathbf{B}$ be the canonical basis for U^- constructed by Lusztig [L2]. We will show that $\mathbf{B} \cap J^-(\lambda)$ is an $\mathcal{A}$-basis of $J^-(\lambda)$. Let $J^-(\lambda)'$ be the left ideal of $(U^-)' = U^- \otimes_{\mathcal{A}} \mathcal{A}'$ generated by the set $\{F_i^{\lambda_i} \mid 1 \leq i \leq n\}$. Lusztig showed that $\mathbf{B} \cap J^-(\lambda)'$ is an $\mathcal{A}'$-basis for $J^-(\lambda)'$ [L2, 11.8]. Thus $\mathbf{B} \cap J^-(\lambda)'$ is an $\mathcal{A}$ basis for $J^-(\lambda)' \cap U^-$ since any finite sum $\sum_{b \in \mathbf{B}} a_b b$ with $a_b \in \mathcal{A}'$ is in U^- if and only if $a_b \in \mathcal{A}$. It is clear that $J^-(\lambda) \subseteq J^-(\lambda)' \cap U^-$. We only need to show that $\mathbf{B} \cap J^-(\lambda)' \subseteq J^-(\lambda)$.

By [L2, 10.16], there is an isomorphism $\lambda_\Omega : U^- \to \mathcal{K}_\Omega$ of $\mathcal{A}$-algebras such that the canonical basis $\mathbf{B}$ is the inverse image of the perverse sheaves in $\mathcal{P}_{\mathbf{V},\Omega}$. In particular, we have $\lambda_\Omega(F_i^{(r)}) = L_{i,r}$. Here $L_{i,r}$ is the only simple perverse sheaf in $\mathcal{P}_{\mathbf{W},\Omega}$ with $\dim W_i = r$ and $W_j = 0$ for $j \neq i$. As in [L2, 11.6], for each $b \in \mathbf{B}$ such that $\lambda_\Omega(b) = L$ is a simple perverse sheaf, we have $s_i(b) = s_i(L)$ as defined in [L2, 6.5, 11.6]. If $b \in J^-(\lambda) \cap \mathbf{B}$, then $b \in (U^-)' F_i^{(\lambda_i+1)}$ for some i by [L2, 11.7, 11.8]. Thus we have $s_i(b) \geq \lambda_i + 1$. By [L2, 6.6], the isomorphism λ_Ω, and the anti-automorphism $\Psi : U^- \to U^-$ $(F_i^{(r)} \mapsto F_i^{(r)})$, we have $b = \sum_{r \geq s_i(b)} z_r F_i^{(r)}$ with $z_r \in U^-$. Thus we have $b \in J^-(\lambda)$. This proves that the image of $\mathbf{B} \setminus (\mathbf{B} \cap J^-(\lambda)')$ in $U^-/J^-(\lambda)$ is an $\mathcal{A}$-basis. Therefore $U/J(\lambda)$ is finitely generated and free over $\mathcal{A}$ and its character is give by Weyl's character formula since there is a bijection between the basis constructed above and the canonical basis in the simple modules for the complex semisimple Lie algebra of the same highest weight. □

Corollary 3.3. *For $\lambda \in X$, $D(\lambda) = 0$ unless $\lambda \in X^+$ and for $\lambda \in X^+$, $D(\lambda)$ is finitely generated and free over $\mathcal{A}$ and its character is given by Weyl's character formula.*

Proof. Note that $D(\lambda)$ is generated by the element $v_\lambda = \sigma(1_\lambda)$ over U and U^0 acts on $\mathcal{A}v_\lambda$ via the weight λ. Since $E_i^{(r)} v_\lambda = 0$ for all $r > 0$ and i, Lemma 2.5 shows that $D(\lambda) = 0$

unless $\lambda_i \geq 0$ for all i. For $\lambda \in X^+$, Lemma 2.5 also shows that $F_i^{(r)} v_\lambda = 0$ for all $r > \lambda_i$ and all i. Thus $J(\lambda) \subseteq \mathrm{Ann}_U(v_\lambda)$ and we have a surjective map $U/J(\lambda) \to D(\lambda)$. Since $U/J(\lambda) \cong M(\lambda)/(J^-(\lambda)M(\lambda))$ is a quotient of $M(\lambda)$ and in $\mathcal{C}$, then we must have an isomorphism $U/J(\lambda) \cong D(\lambda)$. The rest of the corollary follows from Theorem 3.2. □

For each $\mu \in X^+$, let $J^+(\mu)$ be the left ideal of U^+ generated by the set $\{E_i^{(r)} \mid r > \mu_i, 1 \leq i \leq n\}$. Let w_0 be the longest element in the Weyl group W.

Corollary 3.4. *Let $\lambda \in X^+$.*

(a). *For any $w \in W$, $D(\lambda)_{w(\lambda)}$ is a rank 1 free $\mathcal{A}$-module;*

(b). *If $v_w \in D(\lambda)_{w(\lambda)}$ is a basis element, then $Uv_w = D(\lambda)$;*

(c). *$D(\lambda) = U^+ v_{w_0} \cong U^+/J^+(-w_0(\lambda))$. Here the last isomorphism is as U^+-modules.*

Proof. By the construction of the canonical basis, $\mathbf{B} \cap D(\lambda)_\nu$ is the basis of $D(\lambda)_\nu$. Now (a) follows from Weyl's character formula. To prove (b), we use induction on the length $l(w)$. Suppose $Uv_w = D(\lambda)$, let α_i be a simple root such that $l(s_{\alpha_i} w) = l(w) + 1$. By Weyl's character formula, we have $D(\lambda)_{w(\lambda)+r\alpha_i} = 0$ for all $r > 0$. Thus $E_i^{(r)} v_w = 0$ for all $r > 0$. Let $r_i = \max\{r \mid F_i^{(r)} v_w \neq 0\}$. Set $\mu = w(\lambda)$. Then we have $\mu_i = r_i$ by Lemma 2.5. Thus $s_{\alpha_i}(\mu) = \mu - r_i\alpha_i$ and $y = F_i^{(r_i)} v_w \in D(\lambda)_{w(\lambda)-r_i\alpha_i}$. By using the same formula as 2.5, we have $E_i^{(r_i)} y = \begin{bmatrix} K_i \\ r_i \end{bmatrix} = \begin{bmatrix} r_i \\ r_i \end{bmatrix} v_w = v_w$. This shows that $v_w \in Uy$. Thus $D(\lambda) = Uy$ by induction. This also shows that y is an $\mathcal{A}$-basis element of $D(\lambda)_{s_{\alpha_i}(\mu)}$.

In (c), we take $w = w_0$, the longest element of the Weyl group W. Denote by $\mu = -w_0(\lambda)$. Then $\mu \in X^+$. By Weyl's character formula, $w_0(\lambda)$ is the lowest weight of $D(\lambda)$ and we have $F_i^{(r)} v_{w_0} = 0$ for all $r > 0$ and $1 \leq i \leq n$. Thus we have $D(\lambda) = U^+ v_{w_0}$ by using the triangular decomposition $U = U^+U^0U^-$. By Weyl's character formula, $E_i^{(r)} v_{w_0} = 0$ for all $r > \mu_i$ and $1 \leq i \leq n$. Thus we have surjective map $U^+/J^+(\mu) \to D(\lambda)$ of U^+-modules. Once again, by using the canonical basis for U^+, $U^+/J^+(\mu)$ is also free of the same rank as $D(\lambda)$. Since $D(\lambda)$ is free over $\mathcal{A}$, the above surjective map split as $\mathcal{A}$-module map and we have $U^+/J^+(\mu) = D(\lambda) \oplus N$ for some $\mathcal{A}$-module N. Since N is projective and therefore free as $\mathcal{A}$-modules by a result of Seshadri [B], we then have $N = 0$ by comparing the ranks of the free $\mathcal{A}$-modules. □

Remark 3.5. If $\lambda \in X^+$, then $D(\lambda)$ is the universal highest weight module in $\mathcal{C}$ and the universal lowest weight module in $\mathcal{C}$ with lowest weight $w_0(\lambda)$.

3.6. Let $I \subseteq \{1, 2, \ldots, n\}$. Denote by U_I^+ the subalgebra of U generated over $\mathcal{A}$ by $\{E_i^{(r)} \mid i \in I, r \geq 0\}$. We will consider the subalgebra $P_I = U^-U^0U_I^+$ of U. P_I is a Hopf subalgebra of U and is free over $\mathcal{A}$. Let $\mathcal{C}^I$ be the category of P_I-modules M such

that $M = \oplus M_\mu$ as U^0-modules and for each $x \in M$, $E_i^{(r)}x = F_j^{(r)}x = 0$ for all $i \in I$, $j \in \{1, 2 \dots, n\}$, and $r >> 0$.

For each $\lambda \in X^+$, we define $J_I^+(-\lambda)$ to be the ideal of U_I^+ generated by $\{E_i^{(r)} \mid i \in I, r > \lambda_i\}$. By Lusztig's canonical basis for U_I^+, $U_I^+/J_I^+(-\lambda)$ is finitely generated and free as $\mathcal{A}$-module.

Let $M_I(-\lambda) = P_I \otimes_{U^{\leq 0}} (\mathcal{A})_{-\lambda}$, which is a P_I-module generated by $x_{-\lambda} = (1 \otimes 1)$ over P_I. Then the left ideal $I_I(-\lambda) = \mathrm{Ann}_{P_I}(x_{-\lambda})$ is the kernel of the natural surjective map $P_I \to M_I(-\lambda)$. Here $M_I(-\lambda) \cong U_I^+$ as U_I^+-module. Set $N_I(-\lambda) = J_I^+(-\lambda)x_{-\lambda}$. We claim that $N_I(-\lambda)$ is a P_I-submodule of $M_I(-\lambda)$. Indeed, for each $i \in I$ and $r > \lambda_i$ and any $s \geq 0$, we have

$$F_i^{(s)}E_i^{(r)}x_{-\lambda} = \begin{cases} E_i^{(r-s)} \begin{bmatrix} \lambda_i+s-r \\ s \end{bmatrix} x_{-\lambda} & \text{if } r \geq s, \\ 0 & \text{if } r < s. \end{cases}$$

If $0 \leq r - s \leq \lambda_i$, then $\begin{bmatrix} \lambda_i+s-r \\ s \end{bmatrix} = 0$. If $r - s > \lambda_i$ then $F_i^{(s)}E_i^{(r)}x_{-\lambda} \in N_I(-\lambda)$. Now the result follows from the commuting relations between $F_j^{(s)}$ and $E_i^{(r)}$. Thus the kernel of the composition map $P_I \to M_I(-\lambda) \to M_I(-\lambda)/N_I(-\lambda) \cong U_I^+/J_I^+(-\lambda)$ is $J_I^+(-\lambda) + I_I(-\lambda)$, which is therefore a left ideal of P_I. We denote by $D_I'(-\lambda)$ the P_I-module $M_I(-\lambda)/N_I(-\lambda)$, which is finitely generated and free over $\mathcal{A}$.

We can also consider the P_I-module $P_I \otimes_{U^0U_I^+} (\mathcal{A})_\lambda$, which is isomorphic to U^- as U^--module. Similarly, we can prove that $J^-(\lambda)(1 \otimes 1)$ is a P_I-submodule of $P_I \otimes_{U^0U_I^+} (\mathcal{A})_\lambda$ with quotient isomorphic to $U^-/J^-(\lambda)$, which is free and finitely generated over $\mathcal{A}$. We will denote this quotient by $D(\lambda)$. This does not abuse the notation too much since this quotient is exactly the restriction to P_I of the U-module $D(\lambda)$ we defined by Joseph's induction. It should be pointed out that, as P_I-module, $D(\lambda)$ can not be defined as the largest quotient of $P_I \otimes_{U^0U_I^+} (\mathcal{A})_\lambda$ since it does not have a unique maximal quotient in $\mathcal{C}^I$ unless $I = \{1, 2, \dots, n\}$.

3.7. Although, $D(\lambda)$ can not be defined via Joseph's induction as P_I-module, the module $D_I'(-\lambda)$ can be defined by a similar procedure as in Section 2. For each $U^{\leq 0}$-module N in $\mathcal{C}^{\leq 0}$ which is finitely generated over $\mathcal{A}$, $\Pi(N)$ is finite and weights of $P_I \otimes_{U^{\leq 0}} N$ are in $\cup_{\mu \in \Pi(N)}(\mu + \mathbb{N}R^+)$. Following a similar argument as in the proof of Lemma 2.6, we can show that $P_I \otimes_{U^{\leq 0}} N$ has a unique minimal P_I-submodule K such that $(P_I \otimes_{U^{\leq 0}} N)/K$ is in $\mathcal{C}^I$. We denote by $D_I'(N)$ the quotient $(P_I \otimes_{U^{\leq 0}} N)/K$. By the definition, one easily show that functor D_I' defined on the category of $\mathcal{A}$-finite modules in $\mathcal{C}^{\leq 0}$ satisfies that analogous Frobenius reciprocity as in 2.7. Following a similar argument as in 3.3, we see that the P_I-module $D_I'(\lambda)$ we defined in 3.6 is indeed the module $D_I'((\mathcal{A})_\lambda)$. Now one

can argue similarly as in [APW, 1.19] to get the tensor identity

$$D'_I(N \otimes E) \cong D'_I(N) \otimes E$$

for all $\mathcal{A}$-finite module N in $\mathcal{C}^{\leq 0}$ and any P_I-module in $\mathcal{C}^I$ which is finitely generated and free over $\mathcal{A}$.

4. Freeness of Quantum Coordinate Algebra

4.1. In this section, we still fix a subset I is $\{1, 2, \dots, n\}$ and the subalgebra P_I of U. For any $\lambda, \mu \in X^+$, we will consider the P_I-module $D'_I(-\lambda) \otimes D(\mu)$. Let $x_\mu \in D(\mu)_\mu$ and $y_{-\lambda} \in D'_I(-\lambda)_{-\lambda}$ be the generators as $\mathcal{A}$-modules. x_μ and $y_{-\lambda}$ are the generators of the P_I-modules $D(\mu)$ and $D'_I(-\lambda)$ respectively. Then we have $J^-(\mu) = \mathrm{Ann}_{U^-}(x_\mu)$ and $J_I^+(-\lambda) = \mathrm{Ann}_{U_I^+}(y_{-\lambda})$. Let $x = y_{-\lambda} \otimes x_\mu \in D'_I(-\lambda) \otimes D(\mu)$ and $J_I(\lambda, \mu)$ be the left ideal of P_I generated by $J^-(\mu)$, $J_I^+(-\lambda)$, and $\ker(\chi_\nu) \subseteq U^0$. Here $\nu = \mu - \lambda$. Clearly we have $J_I(\lambda, \mu) \subseteq \mathrm{Ann}_{P_I}(x)$.

Lemma 4.2. (a). *$D'_I(-\lambda) \otimes D(\mu)$ is generated by x over P_I;*

(b). *The natural surjective map $P_I/J_I(\lambda, \mu) \to D'_I(-\lambda) \otimes D(\mu)$ induced by x is an isomorphism of P_I-modules.*

Proof. (a) is similar to the proof of [APW, 1.26], but one has to be a little careful when using the tensor identity since the functor D is not defined for all $\mathcal{A}$-finite $U^0U_I^+$-modules. Since $U^{\leq 0}$ acts on $\mathcal{A}y_{-\lambda}$ via the character $\chi_{-\lambda}$, we have $U^-x = y_{-\lambda} \otimes D(\mu)$. Now one applies the functor D'_I and uses the tensor identity as pointed out in 3.7. Then everything goes similarly as in [APW, 1.26]. To prove (b), we write $Q = P_I/J_I(\lambda, \mu)$ and write $y = \bar{1}$, the image of 1 in Q. Then the natural map $\pi : Q \to D'_I(-\lambda) \otimes D(\mu)$ takes y to x. Since y generates Q over P_I and $y \in \mathcal{F}_I(Q) = \{v \in Q \mid E_i^{(r)}v = F_j^{(r)}v = 0$ for all $i \in I$ and $j = 1, 2, \dots, n$ and as $r >> 0\}$ (cf. 1.2), which is a P_I-submodule of Q, we must have Q in $\mathcal{C}^I$. Let $N = U^-y \subseteq Q$. We have an isomorphism $N \to y_{-\lambda} \otimes D(\mu)$ of $U^{\leq 0}$-modules induced by π since $\mathrm{Ann}_{U^-}(y) = \mathrm{Ann}_{U^-}(x)$. Let $\psi : y_{-\lambda} \otimes D(\mu) \to N \subseteq Q$ be the inverse map of the above isomorphism of $U^{\leq 0}$-modules. We have a surjective map $\psi' : D'_I(y_{-\lambda} \otimes D(\mu)) \to Q$ induced by ψ via the Frobenius reciprocity and Q is generated by N over P_I (using the fact that $P_I = U_I^+U^0U^-$. Note that we have isomorphism of P_I-modules $D'_I(y_{-\lambda}) \otimes D(\mu)) \to D'_I(-\lambda) \otimes D(\mu)$ which sends the image of x to x. Thus we have the following homomorphisms

$$D'_I(-\lambda) \otimes D(\mu) \xrightarrow{\psi'} Q \xrightarrow{\pi} D'_I(-\lambda) \otimes D(\mu)$$

such that both ψ' and π are surjective and $(\pi \circ \psi')(x) = x$. Thus $(\pi \circ \psi') = id$ since $D'_I(-\lambda) \otimes D(\mu)$ is generated by x over P_I. This shows that ψ' has to be injective and therefore an isomorphism of U-modules since ψ' is surjective. □

Corollary 4.3. (a). *Any P_I-module M in $\mathcal{C}^I$ is locally $\mathcal{A}$-finite;*

(b). *Any $\mathcal{A}$-finite P_I-module M in $\mathcal{C}^I$ has an $\mathcal{A}$-finite free cover in $\mathcal{C}^I$, i.e., there is a P_I-module F in $\mathcal{C}^I$ such that F is $\mathcal{A}$-free of finite rank and M is a quotient of F;*

(c). *If $0 \neq M$ is in $\mathcal{C}^I$. Then there exists $\mu \in X$ such that $M_\mu \neq 0$ and $\mu_i \geq 0$ for all $i \in I$.*

Proof. Let $x \in M_\nu$. Set $r_i = \max\{r \mid E_i^{(r)} x \neq 0\}$ for each $i \in I$ and $s_j = \max\{r \mid F_j^{(r)} x \neq 0\}$ for each $j = 1, 2, \dots, n$. Then r_i and s_j are finite since M is in $\mathcal{C}^I$. We can choose $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ in X^+ such that $\nu = \mu - \lambda$ and $\lambda_i \geq r_i$ and $\mu_j \geq s_j$ for all $i \in I$ and $j = 1, 2, \dots, n$. Thus we have $J_I(\lambda, \mu) \subseteq \mathrm{Ann}_{P_I}(x)$ and a surjective map $P_I/J_I(\lambda, \mu) \to P_I x$. By lemma 4.2, $P_I/J_I(\lambda, \mu)$ is $\mathcal{A}$-free of finite rank. Thus $P_I x$ is $\mathcal{A}$-finite.

(b) Let $\{x_1, \dots, x_m\}$ be a set of $\mathcal{A}$-generators of M. We can assume that each x_s is a weight vector. Thus the proof of (a) shows that there exists F_s in $\mathcal{C}^I$ such that F_s is $\mathcal{A}$-free of finite rank and $F_s \to P_I x_s$ is surjective. Now we have $\oplus_{s=1}^m F_s \to M$ being a surjective P_I-map. Clearly, $\oplus_{s=1}^m F_s$ is in $\mathcal{C}^I$ and $\mathcal{A}$-free of finite rank.

(c) Take $0 \neq y \in M$. The P_I-submodule $N = P_I y \subseteq M$ is finitely generated over $\mathcal{A}$ by (a). Thus the set $\Pi(N) = \{\lambda \in X \mid N_\lambda \neq 0\}$ of all weights of N is none empty and finite. There exists a maximal element with respect to the usual ordering in X. Take μ being a maximal element of $\Pi(N)$ and $0 \neq v \in N_\mu$. By 2.5(a), we have $\mu_i \geq 0$ for each $i \in I$ since both $F_i^{(s)}$ and $E_i^{(r)}$ act on N. □

4.4. Similar to [APW, 1.8], one can define the quantum coordinate algebra $\mathcal{A}[P_I]$ by

$$\mathcal{A}[P_I] = \{f \in \mathrm{Hom}_{\mathcal{A}}(P_I, \mathcal{A}) \mid f(I) = 0 \text{ for some } I \in \mathcal{I}\},$$

where $\mathcal{I}$ is the set of ideals I of P_I such that P_I/I is a finitely generated over $\mathcal{A}$ and $I \cap U^0$ contains a finite intersection of ideals $\ker(\chi_\lambda)$ in U^0 with $\lambda \in X$. In particular, when $I = \{1, 2, \dots, n\}$, $\mathcal{A}[U] = \mathcal{A}[P_I]$ is defined by Lusztig in [L1].

In [APW, 1.31], it is proved that the coordinate algebra $\mathcal{A}_{(p,v-1)}[U]$ is free over $\mathcal{A}_{(p,v-1)}$. This has been extended to a smaller ring $\mathcal{A}_1$ which is still larger than $\mathcal{A}$ in [AW]. In the following theorem we show the freeness over $\mathcal{A}$.

Theorem 4.5. *For each $I \subseteq \{1, 2, \dots, n\}$ The algebra $\mathcal{A}[P_I]$ is a free $\mathcal{A}$-modules.*

Proof. The proof is essentially the same as in [APW]. To prove the theorem, we first recall that $\mathcal{A}[P_I]$ is a P_I-module in $\mathcal{C}^I$ with respect to the module structure defined by $(uf)(x) = f(xu)$ for all $x, u \in P_I$ and $f \in \mathcal{A}[P_I]$ (see [APW, 1.28–29]. In particular, $\mathcal{A}[P_I] = \oplus_{\nu \in X} \mathcal{A}[P_I]_\nu$ as U^0-module. We will only need to show that each $\mathcal{A}[P_I]_\nu$ is free over $\mathcal{A}$.

We consider $\mathrm{Hom}_{\mathcal{A}}(P_I, \mathcal{A})_\nu$ first. Let $K(\nu)$ be the left ideal of P_I generated by $\ker(\chi_\nu) \subseteq U^0$. Then $\mathrm{Hom}_{\mathcal{A}}(P_I, \mathcal{A})_\nu \cong \mathrm{Hom}_{\mathcal{A}}(P_I/K(\nu), \mathcal{A})$. Let $\Omega(\nu) = \{(\lambda, \mu) \in X^+ \times X^+ \mid \mu - \lambda = \nu\}$. Recall that the left ideal $J_I(\lambda, \mu)$ of P_I is generated by $J_I^+(-\lambda), J^-(\mu)$, and $\ker(\chi_\nu)$. Since $P_I/J_I(\lambda, \mu) \cong D_I'(-\lambda) \otimes D(\mu)$, which is free and finitely generated over $\mathcal{A}$ by 4.2(b), we have $D_I(\lambda, \mu) = \mathrm{Hom}_{\mathcal{A}}(P_I/J_I(\lambda, \mu), \mathcal{A})$ is also free and finitely generated over $\mathcal{A}$. Since $D_I(\lambda, \mu) \subseteq \mathrm{Hom}_{\mathcal{A}}(P_I, \mathcal{A})_\nu$ as $K(\nu) \subseteq J_I(\lambda, \mu)$, the natural embedding $D_I(\lambda, \mu) \subseteq \mathrm{Hom}_{\mathcal{A}}(P_I, \mathcal{A})$ has image in $\mathcal{A}[P_I]_\nu$. Note that $D_I(\lambda, \mu)$ has a natural right P_I-module structure defined by $(fu)(x) = f(ux)$ for any $u, x \in P_I$ and $f \in D_I(\lambda, \mu)$. Let $I_I(\lambda, \mu) = \mathrm{Ann}_{P_I}(D_I(\lambda, \mu))$. Then $P_I/I_I(\lambda, \mu)$ is finitely generated over $\mathcal{A}$ since $D_I(\lambda, \mu)$ is finitely generated and free over $\mathcal{A}$. Since $D_I(\lambda, \mu)$ is a direct sum of finitely many weight spaces (with U^0-module as the restriction to U^0 from the right P_I-module structure), $I_I(\lambda, \mu)$ contains the intersection of finitely many $\ker(\chi_\sigma)$'s and $D_I(\lambda, \mu) \subseteq \mathcal{A}[P_I]_\nu$. Let $f \in \mathcal{A}[P_I]_\nu$. By the proof of Corollary 4.3, there exists $\lambda, \mu \in X^+$ such that $\nu = \mu - \lambda$ and $f(J_I(\lambda, \mu)) = 0$. Then $f \in D_I(\lambda, \mu)$. Thus we have $\mathcal{A}[P_I]_\nu = \cup_{(\lambda,\mu)\in\Omega(\nu)} D_I(\lambda, \mu)$. Fix any $(\lambda^0, \mu^0) \in \Omega(\nu)$ and define $(\lambda^m, \mu^m) = (\lambda^0 + m\rho, \mu^0 + m\rho)$ for all $m \geq 0$. Here ρ is the half-sum of positive roots. Note that, for each $(\lambda, \mu) \in \Omega(\nu)$, we have $J_I(\lambda, \mu), \mathcal{A}) \subseteq J_I(\lambda^m, \mu^m)$ for some m. Thus $\mathcal{A}[P_I]_\nu = \cup_{m\geq 0} D_I(\lambda^m, \mu^m)$. Now the rest of the proof is exactly the same as the proof of [APW, 1.31] using the freeness of projective modules over $\mathcal{A}$ by a Theorem of Seshadri [B]. □

Remark 4.6. One can replace P_I by $P_I^+ = U_I^- U^0 U^+$ which is also a Hopf subalgebra of U. By an analogous argument as for P_I, one can show that the coordinate algebra $\mathcal{A}[U_I^- U^0 U^+]$ is also free as $\mathcal{A}$-module.

5. Duality and the vanishing theorems.

5.1. We still fix $I \subseteq \{1, 2, \ldots, n\}$. For each P_I-module M, let $\mathcal{F}_I(M) = \{m \in \oplus_\nu M_\nu \mid E_i^{(r)} m = F_j^{(r)} m = 0 \text{ for } i \in I, j = 1, 2, \ldots, n, \text{ as } r >> 0\}$, which is the unique maximal submodule of M in $\mathcal{C}^I$. For any $N \in \mathcal{C}^{\leq 0}$, we define the induced module $\mathrm{Ind}_{U^{\leq 0}}^{P_I}(N) = \mathcal{F}_I(\mathrm{Hom}_{U^{\leq 0}}(P_I, N))$. Here the P_I-module structure is defined by

$(uf)(x) = f(xu)$ for $x, u \in P_I$ and $f \in \mathrm{Ind}_{U^{\leq 0}}^{P_I}(N)$. Since $\mathcal{C}^{\leq 0}$ has enough injective modules, we can define the right derived functors of $\mathrm{Ind}_{U^{\leq 0}}^{P_I}$ to get P_I-modules $R^j \mathrm{Ind}_{U^{\leq 0}}^{P_I} M$, denoted by $H_I^j(M)$, in $\mathcal{C}^I$ for each module M in $\mathcal{C}^{\leq 0}$.

It is expected that $H_I^j(N) = 0$ for all $j > 0$ if all weights μ of M are I-dominant, i.e., $\mu \in X$ such that $\mu_i \geq 0$ for all $i \in I$. In the classical case, where one can interpret these modules as the sheaf cohomology groups of the sheaves on the quotient scheme P_I/B, which is isomorphic to the flag variety of a reductive group with smaller rank, and then use Kempf's vanishing theorem. Even in the special case when $I = \{i\}$, the classical argument still needs to be justified. Such attempt have been made in [Lin], where one needs to replace $\mathcal{A}$ by a field.

We define for each $\mathcal{A}$-module N, the induced module $\mathrm{Ind}_{\mathcal{A}}^{P_I} N = \mathcal{F}_I(\mathrm{Hom}_{\mathcal{A}}(P_I, N))$. This defines the induction functor $\mathrm{Ind}_{\mathcal{A}}^{P_I}$ from the category of all $\mathcal{A}$-modules to $\mathcal{C}^I$. We can also define, for each $\mathcal{A}$-module N, a P_I-module $\mathcal{A}[P_I] \otimes N$ with P_I acting on N trivially.

Lemma 5.2. *The induction functor* $\mathrm{Ind}_{\mathcal{A}}^{P_I}$ *is exact. Furthermore,* $\mathrm{Ind}_{\mathcal{A}}^{P_I}(N) \cong \mathcal{A}[P_I] \otimes N$ *as* P_I*-modules.*

Proof. The proof of the lemma is basically the same as [APW, 1.30]. Here one would like to use the freeness of $D_I(\lambda, \mu)$ over $\mathcal{A}$. □

5.3. For each subalgebra H of P_I, besides the left P_I-module structure as in 5.2, $\mathcal{A}[P_I]$ also has a left H-module structure defined by $(uf)(x) = f(\gamma(u)x)$ for all $u \in H$, $f \in \mathcal{A}[P_I]$ and $x \in P_I$. These two module structures commute each other. If H is further a Hopf subalgebra of P_I, $\mathcal{A}[P_I] \otimes N$ has a P_I-H-bimodule structure for each H-module N with the tensor product H-module structure. Thus the fixed point set $(\mathcal{A}[P_I] \otimes N)^H$ turns out to be a P_I-module.

On the other hand, we can also consider the induction functor $\mathrm{Ind}_{U^0}^{P_I}$ which is defined by $\mathrm{Ind}_{U^0}^{P_I} N = \mathcal{F}_I(\mathrm{Hom}_{U^0}(P_I, N))$ for each U^0-module N in $\mathcal{C}^0$. Note that $\mathrm{Ind}_{U^0}^{P_I}(N)$ is a P_I-submodule of $\mathrm{Ind}_{\mathcal{A}}^{P_I}(N)$.

Lemma 5.4. *Under the isomorphism of Lemma 5.2, we have* $\mathrm{Ind}_{U^0}^{P_I} N \cong (\mathcal{A}[P_I] \otimes N)^{U^0}$ *for each* N *in* $\mathcal{C}^0$*. In particular, the induction functor* $\mathrm{Ind}_{U^0}^{P_I}$ *is exact and* $\mathrm{Ind}_{U^0}^{U^{\leq 0}} N$ *is* $\mathrm{Ind}_{U^{\leq 0}}^{P_I}$*-acyclic.*

Proof.

$$\begin{aligned}\mathrm{Ind}_{U^0}^{P_I}(N) &= \mathcal{F}_I(\mathrm{Hom}_{U^0}(P_I, N)) \\ &= \mathcal{F}_I(\mathrm{Hom}_{\mathcal{A}}(P_I, N)^{U^0}) \\ &= (\mathcal{F}_I \,\mathrm{Hom}_{\mathcal{A}}(P_I, N))^{U^0} \\ &= (\mathcal{A}[P_I] \otimes N)^{U^0}.\end{aligned}$$

Here the second equality uses [Lin, 1.3] and the last equality follows from 5.2. The exactness of the induction functor $\mathrm{Ind}_{U^0}^{P_I}$ follows from fact that $\mathcal{A}[P_I]$ is free over $\mathcal{A}$ and $(\mathcal{A}[P_I] \otimes N)^{U^0} = \oplus_{\mu \in X}(\mathcal{A}[P_I]_\mu \otimes N_{-\mu})$. Finally, the exactness of $\mathrm{Ind}_{U^0}^{U^{\leq 0}}$ yields an isomorphism $R^j \mathrm{Ind}_{U^{\leq 0}}^{P_I}(\mathrm{Ind}_{U^0}^{U^{\leq 0}} N) \cong R^j \mathrm{Ind}_{U^0}^{P_I} N$ from a Grothendieck spectral sequences. Now the last result follows from the exactness of $\mathrm{Ind}_{U^0}^{P_I}$. □

Corollary 5.5. *For each U^0-module N in $\mathcal{C}^0$ we have, as $\mathcal{A}$-modules.*

$$\mathrm{Ind}_{U^0}^{U^{\leq 0}} N = \bigcup_{\lambda \in X^+} \mathrm{Hom}_{\mathcal{A}}(U^-/J^-(\lambda), N).$$

In particular, for each $\mu \in X$, $\mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_\mu = \varinjlim \mathrm{Hom}_{\mathcal{A}}(U^-/J^-(m\rho), \mathcal{A}_\mu)$. Furthermore, $\mathrm{Hom}_{\mathcal{A}}(U^-/J^-(m\rho), \mathcal{A}_\mu)$ is a direct summand of $\mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_\mu$ and $\mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_\mu$ is free as $\mathcal{A}$-module

Proof. This follows from the definition directly since each element f in $\mathrm{Ind}_{U^0}^{P_I} N$ is uniquely determined by values on U^- and vanishes at $J^-(\lambda)$ for some $\lambda \in X^+$. The direct limit follows from the fact that for any $\lambda \in X^+$, $J^-(\lambda) \supseteq J^-(m\rho)$ for some m. The last result follows from the fact that $J^-(m\rho) \supseteq J^-((m+1)\rho)$ and $U^-/J^-(m\rho)$ is free and finitely generated over $\mathcal{A}$. □

5.6. Consider the Hopf algebra anti-automorphism $\Phi : U \to U$ such that $\Phi(E_i) = -F_i$, $\Phi(F_i) = -E_i$, $\Phi(K_i) = K_i^{-1}$. Then $\Phi^2 = \mathrm{Id}$. It is easy to see that $\Phi(U^0) = U^0$, $\Phi(U_I^-) = U_I^+$, and $\Phi(P_I) = P_I^+$. Here $P_I^+ = U_I^- U^{\geq 0}$. For any $\lambda \in X$, let $\Phi(\lambda) \in X$ be the weight defining the $\mathcal{A}$-algebra homomorphism $\chi_\lambda \circ \Phi : U^0 \to \mathcal{A}$. Then $\Phi(\lambda) = -\lambda$.

For each right P_I^+-module M, $\Phi(M)$ is a left P_I-module via the anti-isomorphism $\Phi : P_I \to P_I^+$. If the category $\mathcal{C}_r(P_I^+)$ of right P_I^+-modules is defined similarly as the category $\mathcal{C}^I$ of left P_I-modules, then Φ induces a category isomorphism between $\mathcal{C}_r(P_I^+)$ and $\mathcal{C}^I$.

5.7. Let M be a module in $\mathcal{C}^{\geq 0}$, then $\mathrm{Hom}_{\mathcal{A}}(P_I^+ \otimes_{U^{\geq 0}} M, \mathcal{A})$ is an right P_I^+-module. As left P_I-modules, we have the following isomorphisms:

$$\begin{aligned}\Phi(\mathrm{Hom}_{\mathcal{A}}(P_I^+ \otimes_{U^{\geq 0}} M, \mathcal{A})) &\cong \Phi(\mathrm{Hom}_{U^{\geq 0}}(P_I^+, \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}))\\ &\cong \mathrm{Hom}_{\Phi(U^{\geq 0})}(\Phi(P_I^+), \Phi(\mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A})))\\ &\cong \mathrm{Hom}_{U^{\leq 0}}(P_I, \mathrm{Hom}_{\mathcal{A}}(\Phi(M), \mathcal{A})).\end{aligned}$$

Here $\mathrm{Hom}_{\mathcal{A}}(\Phi(M), \mathcal{A})$ is a left $U^{\leq 0}$-module arising from the right $U^{\leq 0}$-module $\Phi(M)$. If M in $\mathcal{C}^{\geq 0}$ is finitely generated and free over $\mathcal{A}$, then $\mathrm{Hom}_{\mathcal{A}}(\Phi(M), \mathcal{A}) = \Phi(M^*) = (\Phi(M))^*$ is in $\mathcal{C}^{\leq 0}$ and $\mathcal{A}$-free of finite rank.

Theorem 5.8. $\Phi(\mathrm{Hom}_{\mathcal{A}}(D_I(\lambda), \mathcal{A})) \cong \mathrm{Ind}_{U^{\leq 0}}^{P_I}(\lambda)$ *as P_I-modules for all $\lambda \in X$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I$.*

Proof. Recall the definition of $D_I(\lambda)$. Let π be the quotient map in the exact sequence

$$0 \to K \xrightarrow{j} P_I^+ \otimes_{U^{\geq 0}} \mathcal{A}_\lambda \xrightarrow{\pi} D_I(\lambda) \to 0.$$

Since $D_I(\lambda)$ is free over $\mathcal{A}$ and the above exact sequence splits over $\mathcal{A}$, , thus K is also free over $\mathcal{A}$ (projective modules are free over $\mathcal{A}$). This gives the following exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(D_I(\lambda), \mathcal{A}) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathcal{A}}(P_I^+ \otimes_{U^{\geq 0}} \mathcal{A}_\lambda, \mathcal{A}) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{A}}(K, \mathcal{A}) \to 0$$

of right P_I^+-modules. Then we apply the functor Φ to get an exact sequence of left P_I-modules. Note that $\mathrm{Hom}_{\mathcal{A}}(P_I^+ \otimes_{U^{\geq 0}} \mathcal{A}_\lambda, \mathcal{A})_\mu \cong \mathrm{Hom}_{\mathcal{A}}((P_I^+ \otimes_{U^{\geq 0}} \mathcal{A}_\lambda)_{-\mu}, \mathcal{A})$, as U^0-modules. The resulting sequences will still be exact if we replace each term by the direct sum of the weight spaces. We then apply the functor $\mathcal{F}_I$, to get exact sequence

$$0 \to \Phi(D_I(\lambda)^*) \xrightarrow{\pi^*} \mathrm{Ind}_{U^{\leq 0}}^{P_I}(\mathcal{A}_\lambda^*) \to \mathcal{F}_I \Phi(\mathrm{Hom}_{\mathcal{A}}(K, \mathcal{A})).$$

Consider the cokernel $\mathrm{Coker}(\pi^*)$, which is in $\mathcal{C}^I$ and thus locally $\mathcal{A}$-finite. For any given $\mathcal{A}$-finite P_I-submodule of $\mathrm{Coker}(\pi^*)$, let $\{f_1, \ldots, f_r\}$, consisting of weight vectors, be a $\mathcal{A}$-generating set. Then $f_i \in \mathrm{Hom}_{\mathcal{A}}(K, \mathcal{A})$ and $K_1 = \cap_{i=1}^r \mathrm{Ker}(f_i)$ is a P_I^+-submodule of K and K/K_1 is in $\mathcal{C}(P_I^+)$. Therefore, $(P_I^+ \otimes_{U^{\geq 0}} \mathcal{A}_\lambda)/K_1$, which is $\mathcal{A}$-finitely generated and a direct sum of weight spaces, is also in $\mathcal{C}(P_I^+)$. By the definition of $D_I(\lambda)$, we must have $K_1 = K$ and $\mathrm{Coker}(\pi^*) = 0$ and thus π^* is an isomorphism. □

Remark 5.9. The Proof of the Theorem 5.8 still works when $\mathcal{A}_\lambda$ is replaced by any $U^{\geq 0}$-module M provided $D_I(M)$ is finitely generated free over the ground ring $\mathcal{A}$. In particular, for any field k with a ring homomorphism $\mathcal{A} \to k$, if we consider the k-algebras $U_k = U \otimes_{\mathcal{A}} k$ and the corresponding subalgebras over k (by adding a subscript k), by repeating everything for $\mathcal{A}$ over k, we have $\Phi_k(D_{I,k}(M)^*) \cong \mathrm{Ind}_{U_k^{\leq 0}}^{U_k}(\Phi(M^*))$ for any finite dimensional module M in the category $\mathcal{C}_k^{\geq 0}$.

Corollary 5.10. *For any $\lambda \in X$, then*

(a). $\mathrm{Ind}_{U^{\leq 0}}^{P_I} \mathcal{A}_\lambda$ *is free and finitely generated over* A.

(b). $\mathrm{Ind}_{U^{\leq 0}}^{P_I} \mathcal{A}_\lambda \otimes_{\mathcal{A}} k \cong \mathrm{Ind}_{U_k^{\leq 0}}^{P_{I,k}} k_\lambda$ *for any field* k *with a ring homomorphism* $\mathcal{A} \to k$.

Proof. If λ is I-dominant, then (a) follows from Theorem 5.8. For other λ, by lemma 2.5(a), we have $D_I(\lambda) = 0$. Thus the proof of Theorem 5.8 still works by the above remark. Now to prove (b), one uses Remark 5.9 to get $\mathrm{Ind}_{U_k^{leq0}}^{P_{I,k}} k_\lambda \cong \Phi_k(D_{I,k}(k_\lambda)^*)$. However, we have $D_{I,k}(k_\lambda) \cong D_I(\mathcal{A}_\lambda) \otimes_{\mathcal{A}} k$ by the canonical basis. Therefore $\Phi_k(D_{I,k}(k_\lambda)^*) \cong \Phi(D_I(\mathcal{A}_\lambda)^*) \otimes_{\mathcal{A}} k$ since all the modules are free and finitely generated over $\mathcal{A}$. □

5.11. For $i \in \{1, 2, \dots, n\}$, $\{E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1} | m \in \mathbb{N}\}$ generates an $\mathcal{A}$-subalgebra of U, denoted by U_i. The subalgebras U_i^+, U_i^-, $U_i^{\leq 0}$, and $U_i^{\geq 0}$ of U_i are defined accordingly. We will also denote by $\mathcal{C}_i$, $\mathcal{C}_i^{\leq 0}$ and $\mathcal{C}_i^{\geq 0}$ the corresponding categories of integrable modules of type **1** for these subalgebras respectively. For each $\mathcal{A}$-finite $U_i^{\geq 0}$-module N in $\mathcal{C}_i^{\leq 0}$, let $D_i(N)$ be the unique maximal quotient of the U_i-module of $U_i \otimes_{U_i^{\geq 0}} N$ in $\mathcal{C}_i$. Note that each module in $\mathcal{C}^{\leq 0}$ ($\mathcal{C}^{\geq 0}$) is in $\mathcal{C}_i^{\leq 0}$ ($\mathcal{C}_i^{\geq 0}$) when restricted to $U_i^{\leq 0}(U_i^{\leq 0})$. A direct verification will show $D_i(N) = D_{\{i\}}(N)$ as U_i-modules for any $U^{\geq 0}$-module N in $\mathcal{C}^{\geq 0}$. Similarly, we have the restriction functors from $\mathcal{C}$ to $\mathcal{C}_i$. The proof of the following is almost in Andersen's notes [A1].

Lemma 5.12. *For each N in $\mathcal{C}^{\leq 0}$, and $\nu \in X$,*

$$(\mathrm{Ind}_{U^{\leq 0}}^{P_i} N)_\nu = \Big\{ f \in \mathrm{Hom}_{\mathcal{A}}(U_i^+, N) \mid f(E_i^{(s)}) \in N_{\nu + s\alpha_i},\ \ f(E_i^{(s)}) = 0 \text{ for } s >> 0, \text{ and}$$

$$\sum_{0 \leq t \leq \min\{r,s\}} \begin{bmatrix} \nu_i + s - r \\ t \end{bmatrix} F_i^{(r-t)} f(E_i^{(s-t)}) = 0 \text{ for } 0 \leq s \leq r - \nu_i - 1 \text{ when } r >> 0 \Big\}.$$

In particular, $\mathrm{Ind}_{U^{\leq 0}}^{P_i} N = \mathrm{Ind}_{U_i^{\leq 0}}^{U_i} N$ *when considered as U_i-modules in $\mathcal{C}_i$.*

Proof. We only need to show that the right hand side is contained in the left hand side since $(E_i^{(s)} f)(1) = 0$ when $s >> 0$ and $(F_i^{(r)} f)(E_i^{(s)}) = 0$ for all $s \geq 0$ when $r >> 0$. Here one uses the commuting relation between $E_i^{(s)}$ and $F_i^{(r)}$ as used in the proof of lemma 2.5. Let f be an element on the right hand side. f defines an $U^{\leq 0}$-map $f : P_I \to N$. One clearly has $(E_i^{(r)} f)(E_i^{(s)}) = 0$ for all $s \geq 0$ when $r >> 0$. We still need to verify that $(F_i^{(r)} f)(E_i^{(s)}) = 0$ for each $s \geq 0$ when $r >> 0$. Note that

$$(F_i^{(r)} f)(E_i^{(s)}) = \sum_{0 \leq t \leq \min\{r,s\}} \begin{bmatrix} \nu_i + s - r \\ t \end{bmatrix} F_i^{(r-t)} f(E_i^{(s-t)}).$$

Thus we already have $(F_i^{(r)} f)(E_i^{(s)}) = 0$ for all $0 \leq s \leq r - \nu_i - 1$ when r is large. We must show that $(F_i^{(r)} f)(E_i^{(s)}) = 0$ for $s \geq r - \nu_i$. In this case, we have $\nu_i + s - r \geq 0$ and

$\begin{bmatrix} \nu_i+s-r \\ t \end{bmatrix} = 0$ unless $t \leq \nu_i + s - r$ which implies that $s - t \geq r - \nu_i$ and $f(E_i^{(s-t)}) = 0$ when r is large. (Here ν_i is fixed.) If $j \neq i$, then we have

$$(F_j^{(r)} f)(E_i^{(s)}) = f(E_i^{(r)} F_j^{(s)}) = f(F_j^{(r)} E_i^{(s)}) = F_j^{(r)} f(E_i^{(s)}) = 0$$

for all s when r is large since $f(U_i^+) \subseteq N$ is finitely generated over $\mathcal{A}$ and N is in $\mathcal{C}^{\leq 0}$. □

The proof of the following is similar to the proof in [AP].

Proposition 5.13. *$R^j \operatorname{Ind}_{U_i^{\leq 0}}^{U_i}(N|_{U_i^{\leq 0}}) \cong (R^j \operatorname{Ind}_{U^{\leq 0}}^{P_i} N)|_{U_i}$ for any module N in $\mathcal{C}^{\leq 0}$.*

Proof. The case when $j = 0$ has already been proved in Lemma 5.12. Each N in $\mathcal{C}^{\leq 0}$ admits a resolution $0 \to N \to I^\cdot$ such that each $I^j = \operatorname{Ind}_{U^0}^{U^{\leq 0}} Q_j$ with Q_j being a module in $\mathcal{C}^0$. If follows from the exactness of $\operatorname{Ind}_{U^0}^{U^{\leq 0}}$ and $\operatorname{Ind}_{U^0}^{P_i}$ and a Grothendieck spectral sequence arising from the composition $\operatorname{Ind}_{U^0}^{U^{\leq 0}} = \operatorname{Ind}_{U^{\leq 0}}^{P_i} \circ \operatorname{Ind}_{U^0}^{U^{\leq 0}}$ that $I^j = \operatorname{Ind}_{U^0}^{U^{\leq 0}} Q_j$ is $\operatorname{Ind}_{U^{\leq 0}}^{P_i}$-acyclic. The proposition will follow from 5.12 if we show that each I^j is $\operatorname{Ind}_{U_i^{\leq 0}}^{U^{\leq 0}}$-acyclic. We will consider the module $I = \operatorname{Ind}_{U^0}^{U^{\leq 0}} Q$. Since $Q = \oplus Q_\mu$ as U^0-modules, we can assume that $Q = Q_\mu$ for some μ. Furthermore, $\mathcal{A}$ has global dimension 2 [GS, 18.9, 20.3]. By using an $\mathcal{A}$-free resolution of Q of finite length, we can assume that $Q = \mathcal{A}_\mu$. This follows from the following lemma. □

Lemma 5.14. (a). *The directed limit of injective modules in the category $\mathcal{C}$ ($\mathcal{C}^{\leq 0}$, $\mathcal{C}_i^{\leq 0}$, etc.) is still injective.*

(b). *The induction functor $\operatorname{Ind}_{U^{\leq 0}}^{U}$ commutes with directed limit.*

(c). *For any $\mu \in X$, then $\operatorname{Ind}_{U^0}^{U^{\leq 0}}(\mathcal{A}_\mu) = \varinjlim \operatorname{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu}$.*

(d). *For any $\mu \in X$, $\operatorname{Ind}_{U^0}^{U^{\leq 0}}(\mathcal{A}_\mu)$ is $\operatorname{Ind}_{U_i^{\leq 0}}^{U_i}$-acyclic.*

Proof. (a) Note that the category $\mathcal{C}$ ($\mathcal{C}^{\leq 0}$, etc.) is $\mathcal{A}$-locally finite and $\mathcal{A}$ is Noetherian. Therefore the categories we are considering are locally Noetherian in the sense of [R]. Thus the directed limits of injective modules are injective [R, Thm 2].

(b) Let $\{(M_i, \phi_{ij}) \mid i, j \in I \text{ with } i \leq j\}$ be a directed system such that for any $i, j \in I$, there exists $k \in I$ with $k \geq i$ and $k \geq j$. Let $M = \varinjlim M_i$. We have the induced maps $\hat{\psi}_i : \operatorname{Ind}_{U^{\leq 0}}^{U}(M_i) \to \operatorname{Ind}_{U^{\leq 0}}^{U}(M)$ from the maps $\psi_i : M_i \to M$. If V is any module in $\mathcal{C}$ with maps $\xi_i : \operatorname{Ind}_{U^{\leq 0}}^{U}(M_i) \to V$ such that $\xi_i = \xi_j \circ \hat{\phi_{ij}}$. if $i \leq j$. We will construct a homomorphism $\zeta : \operatorname{Ind}_{U^{\leq 0}}^{U}(M) \to V$ such that $\xi_i = \zeta \circ \hat{\psi}_i$. For each $f \in \operatorname{Ind}_{U^{\leq 0}}^{U} M \subseteq \operatorname{Hom}_{\mathcal{A}}(U, M)$, $f(U)$ is a finitely generate $\mathcal{A}$-submodule of M. Let $\{m_1, \ldots, m_r\}$ is a generating set of $f(U)$ over $\mathcal{A}$. There exists $j \in I$ such that $f(U) \subseteq \psi_j(M_j) \subseteq M$. For any such j, there exists a $U^{\leq 0}$-module map $g : U \to M_j$ such that $\psi_j \circ g = f$ since U is a free $U^{\leq 0}$-module. In fact, we can choose g such that $g(U) \subseteq N$ with N being an $\mathcal{A}$-finitely generated $U^{\leq 0}$-submodule of M such that $\psi_j(N) = f(U)$.

Thus we have $g \in \mathrm{Ind}_{U^{\leq 0}}^{U}(M_j)$. We can now define $\zeta(f) = \xi_j(g)$. It is standard to verify that $\xi_j(g)$ is independent of the choice of j and g. If we take $V = \varinjlim \mathrm{Ind}_{U^{\leq 0}}^{U} M_i$, we get $\varinjlim \mathrm{Ind}_{U^{\leq 0}}^{U} M_i \cong \mathrm{Ind}_{U^{\leq 0}}^{U} \varinjlim M_i$. This isomorphism is natural. In particular, $\mathrm{Ind}_{U_i^{\leq 0}}^{U_i}$ commutes with directed limit.

(c) First we show that $\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu}$ is isomorphic to a $U^{\leq 0}$-submodule of $\mathrm{Ind}_{U^0}^{U^{\leq 0}}(\mathcal{A}_\mu)$. Note that $\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) = \Phi(D(m\rho)^*)$ is $\mathcal{A}$-free with character given by Weyl's character formula. The projection $\mathrm{pr}_{-m\rho} : \mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \twoheadrightarrow \mathcal{A}_{-m\rho}$ induces a projection map $\mathrm{pr}_\mu : \mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu} \twoheadrightarrow \mathcal{A}_\mu$. As a U^+-module, $D(m\rho)$ is generated by $D(m\rho)_{-m\rho}$, then any nonzero U^--submodule of $\Phi(D(m\rho)^*)$ must have an element h such that $h(D(m\rho)_{-m\rho}) \neq 0$. The Frobenius reciprocity induces a map $f : \mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu} \twoheadrightarrow \mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_\mu$, which is given by $f(x \otimes 1)(u) = \mathrm{pr}_{-m\rho}(ux)$ for all $u \in U^{\leq 0}$. The kernel of f is a U^--submodule. Thus f is injective.

The above discussion for the injectivity of f still works if $\mathcal{A}$ is replaced by any field or local domain k with an $\mathcal{A}$-algebra structure, since the weights and ranks (or dimensions) of $\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho)$ is independent of the choice of the ground ring $\mathcal{A}$ (See Corollary 5.10). Thus the resulting map f is still injective. This induces injective maps $f \otimes \mathrm{Id} : (\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu}) \otimes k \twoheadrightarrow (\mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_\mu) \otimes k$. For each weight ν of $\mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_\mu$, we have $f(\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu})_\nu = (\mathrm{Ind}_{U^0}^{U^{\leq 0}} \mathcal{A}_{mu})_\nu$ when m is sufficiently large. This is true when k is a field. Over the localizations of $\mathcal{A}$ at prime ideals one can use the Nakayama Lemma since each weight space is finitely generated over $\mathcal{A}$. The result over $\mathcal{A}$ follows from [AM, Prop. 3.9]. Thus (c) follows.

(d) Using (a) and (b), we have $R^j \mathrm{Ind}_{U_i^{\leq 0}}^{U_i} \varinjlim M_\alpha \cong \varinjlim (R^j \mathrm{Ind}_{U_i^{\leq 0}}^{U_i} M_\alpha)$ for all j by considering the Grothendieck spectral sequences since the direct limit is exact. By using (c), we have

$$\varinjlim R^j \mathrm{Ind}_{U_i^{\leq 0}}^{U_i}(\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu}) \cong R^j \mathrm{Ind}_{U_i^{\leq 0}}^{U_i}(\mathrm{Ind}_{U^0}^{U^{\leq 0}}(\mathcal{A}_\mu)).$$

The left hand side is zero for $j > 0$ since

$$R^j \mathrm{Ind}_{U_i^{\leq 0}}^{U_i}(\mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes \mathcal{A}_{m\rho+\mu}) \cong \mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho) \otimes R^j \mathrm{Ind}_{U_i^{\leq 0}}^{U_i}(\mathcal{A}_{m\rho+\mu}) = 0$$

for all $m+\mu_i \geq 0$ and $j > 0$ by the generalized tensor identity. (Note that the proof of the generalized tensor identity in [APW, 2.19] works over $\mathcal{A}$ using 5.4.) Hence, $\mathrm{Ind}_{U^0}^{U^{\leq 0}}(\mathcal{A}_\mu)$ is $\mathrm{Ind}_{U_i^{\leq 0}}^{U_i}$-acyclic. $\square$.

Let $w_0 = s_{i_N} \cdots s_{i_1}$ be a fixed reduced expression of the longest element w_0 of the Weyl group. Set $w_r = s_{i_r} \cdots s_{i_1}$. Then for any $\lambda \in X_+$, we can repeatedly apply the functors D_{i_j}'s to get the $U^{\geq 0}$-module $D_{i_r} \cdots D_{i_1} \mathcal{A}_\lambda$ (see 5.11). On the other the hand, the

quantum Demazure module is defined as $V_{w_r}(\lambda) = U^{\geq 0} v_{w_r(\lambda)}$. Here $\mathcal{A}v_{w_r(\lambda)} = D(\lambda)_{w_r(\lambda)}$ [La].

Theorem 5.15. *As $U^{\geq 0}$-module, we have $D_{i_r} \cdots D_{i_1}(\mathcal{A}_\lambda) \cong V_{w_r}(\lambda)$.*

Proof. Note that using the crystal bases or the Seshadri-Lakshmibai bases [La] that $V_{w_r}(\lambda)$ is an $\mathcal{A}$-direct summand of $D(\lambda)$ and thus free over $\mathcal{A}$. Furthermore, when restricted to $U^{\leq 0}_{i_{r+1}}$, $V_{w_r}(\lambda)$ admits a filtration with each subquotient being isomorphic to either a module $D_{i_{r+1}}(\mu)$ or isomorphic to $\mathcal{A}_\mu$ with $\mu \in X$ such that $\mu_{i_{r+1}} \geq 0$ (see [La]). We now use induction on r. When $r = 1$, this follows from 3.4-3.6. Suppose the lemma has been proved for r. The above discussion shows that $D_{i_r} \cdots D_{i_1}(A_\lambda) = V_{w_r}(\lambda)$ has a filtration as $U^{\geq 0}_{i_{r+1}}$-module with quotient isomorphic to either $D_{i_{r+1}}(\mu)$ or $\mathcal{A}_\mu$ with $\mu_{i_{r+1}} \geq 0$. Note that $V_{w_{r+1}}(\lambda)$ is generated by $V_{w_r}(\lambda)$ over $U_{i_{r+1}}$. Thus we have a surjective map $D_{i_{r+1}}(V_{w_r}(\lambda)) \xrightarrow{\pi} V_{w_{r+1}}(\lambda)$ of $U_{i_{r+1}}$-modules. Since $D_{i_{r+1}}(V_{w_r}(\lambda))$ is finitely generated over $\mathcal{A}$, which is a Noetherian ring, to show $\mathrm{Ker}(\pi) = 0$ it is enough to show the surjective map $D_{i_{r+1}}(V_{w_r}(\lambda)) \otimes_{\mathcal{A}} k \xrightarrow{\pi \otimes k} V_{w_{r+1}}(\lambda) \otimes_{\mathcal{A}} k$ is an isomorphism for every field k with a ring homomorphism $\mathcal{A} \to k$. By the definition of the functor $D_{i_{r+1}}$ for the algebra $U_{i_{r+1},k}$ from $U^{\geq 0}_{i_{r+1},k}$, we have the following surjective map $D_{i_{r+1}}(V_{w_r}(\lambda) \otimes_{\mathcal{A}} k) \to V_{w_{r+1}}(\lambda) \otimes_{\mathcal{A}} k$. Using Remark 5.9, we have the isomorphism $\Phi((D_{i_{r+1}}(V_{w_r}(\lambda) \otimes_{\mathcal{A}} k))^*) \cong \mathrm{Ind}^{U_{i_{r+1},k}}_{U^{\leq 0}_{i_{r+1},k}}(\Phi((V_{w_r}(\lambda) \otimes_{\mathcal{A}} k)^*))$. By considering the filtration of $\Phi((V_{w_r}(\lambda) \otimes_{\mathcal{A}} k)^*)$ induced from the above mentioned filtration of $V_{w_r}(\lambda)$ and Kempf's vanishing theorem for $\mathrm{Ind}^{U_{i_{r+1},k}}_{U^{\leq 0}_{i_{r+1},k}}$ proved by Andersen in [A1], one can compute the character of $\mathrm{Ind}^{U_{i,k}}_{U_{i_{r+1},k}}(\Phi((V_{w_r}(\lambda) \otimes_{\mathcal{A}} k)^*))$, which is independent of the choice of the field k since the filtration of $V_{w_r}(\lambda)$ has $\mathcal{A}$-free quotients. In particular, one can take k to be algebraically closed and specialize v to 1 to get (inductively) that this character coincides with Demazure's character formula [A3]. Now by [La] we have $\dim_k(\mathrm{Ind}^{U_{i_{r+1},k}}_{U^{\leq 0}_{i_{r+1},k}}(\Phi((V_{w_r}(\lambda) \otimes_{\mathcal{A}} k)^*))) = \dim_k(V_{w_{r+1}}(\lambda))$ and the isomorphism $\pi \otimes k$. □

Corollary 5.16. *For each $\lambda \in X_+$,*

(a). *the map $\phi_r : \mathrm{Ind}^U_{U^{\leq 0}}(\lambda) \to \mathrm{Ind}^{U_{i_r}}_{U^{\leq 0}_{i_r}} \cdots \mathrm{Ind}^{P_{i_1}}_{U^{\leq 0}_{i_1}}(\lambda)$, induced from the Frobenius reciprocity, is surjective for all r.*

(b). *$R^j \mathrm{Ind}^{U_{i_r}}_{U^{\leq 0}_{i_r}}(\mathrm{Ind}^{U_{i_{r-1}}}_{U^{\leq 0}_{i_{r-1}}} \cdots \mathrm{Ind}^{U_{i_1}}_{U^{\leq 0}_{i_1}}(\lambda)) = 0$ for all $j > 0$.*

(c). *if we we denote $I_r = \mathrm{Ind}^{U_{i_r}}_{U^{\leq 0}_{i_r}} \cdots \mathrm{Ind}^{P_{i_1}}_{U^{\leq 0}_{i_1}}$, then $R^j I_r(\lambda) = 0$ for all $j > 0$.*

Proof. Consider the embedding map $D_{i_r} \cdots D_{i_1}(\lambda) \to D(\lambda)$. Since both modules are finitely generated and free over $\mathcal{A}$, we can take the $\mathcal{A}$-dual and then apply the functor

Φ to get the surjective map $\Phi(D(\lambda)^*) \twoheadrightarrow \Phi((D_{i_r}\cdots D_{i_1}(\lambda))^*)$. By applying Remark 5.9 repeatedly, we have

$$\Phi((D_{i_r}\cdots D_{i_1}(\lambda))^*) \cong \mathrm{Ind}_{U^{\leq 0}_{i_r}}^{U_{i_r}} \cdots \mathrm{Ind}_{U^{\leq 0}_{i_1}}^{U_{i_1}}(\lambda). \tag{5.1}$$

Thus (a) follows from Theorem 5.8. To show (b), we use Eq. (5.1) and the filtration of $V_{w_r}(\lambda)$ used in the proof of 5.15. By applying the functor $\mathrm{Ind}_{U^{\leq 0}_{i_{r+1}}}^{U_{i_{r+1}}}$ to the filtration we can get a sequences of long exact sequences. Now (b) follows from Kempf's vanishing theorem for $\mathrm{Ind}_{U^{\leq 0}_{i_{r+1}}}^{U_{i_{r+1}}}$ proved in [A1] and the generalized tensor identity.

(c) We use induction on r. For $r = 1$, this is Kempf's vanishing theorem for the type A_1 proved in [A1]. Note that $I_{r+1} = \mathrm{Ind}_{U^{\leq 0}_{i_{r+1}}}^{U_{i_{r+1}}} \circ I_r$. Consider the $U^{\leq 0}$-modules $I(m,\mu) = \mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho)\otimes \mathcal{A}_{m\rho+\mu}$ such that $m\rho+\mu$ is dominant. By induction hypothesis, $R^j I_r(I(m,\mu)) \cong \mathrm{Ind}_{U^{\leq 0}}^{U}(m\rho)\otimes R^j I_r(\mathcal{A}_{m\rho+\mu}) = 0$. By (b), $I_r(I(m,\mu))$ is $\mathrm{Ind}_{U^{\leq 0}_{i_{r+1}}}^{U_{i_{r+1}}}$-acyclic. For any $U^{\leq 0}$-module N which is finitely generated and free over $\mathcal{A}$, one can easily construct a resolution $0 \to N \to I^{\cdot}$ such that I is a direct sum of the the modules of the form $I(m,\mu)$. By [APW, 5.2] one can use this resolution to calculate $R^j I_{r+1}(N)$ and to get a spectral sequence

$$R^p \mathrm{Ind}_{U^{\leq 0}_{i_{r+1}}}^{U_{i_{r+1}}}(R^q I_r(N) \Rightarrow R^{p+q} I_{r+1}(N).$$

In particular, when $N = \mathcal{A}_\lambda$, we can apply (b) and induction assumption to get (c). □

Theorem 5.17. *Let $\lambda \in X_+$, then*

(a). $R^j \mathrm{Ind}_{U^{\leq 0}}^{U}(\lambda) = 0$ *for* $j > 0$;

(b). $R^j \mathrm{Ind}_{U^{\leq 0}}^{U}(N)$ *is finitely generated over $\mathcal{A}$ if N is finitely generated over $\mathcal{A}$.*

Proof. Using a 5.16(a)(c) and 5.14, one can follow the argument as in [APW, 5.7] to get (a).

(b) If N is finitely generated and free over $\mathcal{A}$, we use (a) and the generalized tensor identity to the resolution of N used in the proof of 5.16(c) to get (b). For general M, we can first reduce to the case that M has a single weight and then use induction on the projective dimension of N over $\mathcal{A}$. □

Remark 5.18. 5.16(c) is the analogue of the Demazure vanishing theorem, 5.17(a) is the analogue of Kempf's vanishing, while 5.17(b) resembles Serre's theorem, which says that the cohomology groups of a coherent sheaf on a projective scheme over a Noetherian ring are finitely generated over the base ring [H, III 5.2]. Many other results in [APW] can also be generalized over $\mathcal{A}$.

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