Extensions between Simple Modules for Frobenius Kernels

Short running title: Extensions

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Introduction Let G be a simply connected and connected semisimple group over an algebraically closed field k of characteristic $p > 0$. $T \subset G$ is a maximal torus and R is the root system relative to T. $X(T)$ is the weight lattice. Let $B \supset T$ be a Borel subgroup corresponding to the negative roots $R^- = R^+$. Denote by G_r the r-th Frobenius kernel of G. The socle and radical structures of the cohomology groups of line bundles on the flag variety G/B are determined by those structures of the G_rT-modules $Z_r(\lambda) = \mathrm{Ind}_{B_rT}^{G_rT}\lambda$ (cf. [12]). So the study of the G_rT-structure of these modules turns out to be more interesting. Calculation of the extensions between simple modules plays an important rule in determining the socle structure. In this paper, we calculate the socle series of the Weyl modules with p-singular highest weight for the group of type G_2 by studying the extensions between simple modules for Frobenius kernel.

In the first section of this paper, we study the properties of $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))$, which turns out to be semisimple and to have a good filtration for large p and p^r-restricted weights μ and λ. Some of the vanishing properties of these modules are also studied. Then we use these properties to calculate $\mathrm{Ext}^1_{G_r}(L(\mu), L(\lambda))$, which will lead a calculation of the extensions between simple G_rT-modules. The results in this section will be used in Section 3 to calculate the socle series of $Z_1(\lambda)$ with p-singular weights λ for the group of type G_2.

The author ([13] ch3) used the method of Doty and Sullivan [7] and calculated the socle series of $H^0(\lambda)$ for the p-singular weights λ in the bottom p^2-alcove for the groups of type A_2 and B_2. However when the multiplicities of simple modules in

$H^0(\lambda)$ are larger than one, the calculation there seems to be unsuccessful. So, instead of studying the intertwining homomorphisms between G-modules $H^{\ell(w)}(w\cdot\lambda)$, we study the intertwining of G_1T-modules in Section 2. That will give an estimate for the socle series of $Z_1(\lambda)$. Assuming the Lusztig conjecture, the socle series of $Z_1(\lambda)$ for p-regular weights λ can be calculated from the K-L polynomials [4]. We expect the socle series of $Z_1(\lambda)$ for p-singular weights λ to be calculated by translation functors. This has been proved to be true for the groups of type B_2 and A_2 ([13] ch 3). Section 3 is to verify this for the group of type G_2.

In Section 4 we prove that if the translation functors preserve the extensions of simple G_1T-modules, they also preserve the extensions of simple G-modules (the highest weights are not necessarily in the bottom p^2-alcove). As a by-product, the G-module structure of $\mathrm{Ext}^1_{G_1}(L(\mu), L(\lambda))$ is determined in Section 4.2 for the group of type G_2. Therefore the extensions of any two simple G-modules in this situation can be calculated from Andersen's results in [2]. This calculation has been done for $SL(3,K)$ by Yehia [18] and for $Sp(4,K)$ by Ye [17].

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Notations. Let $W = N(T)/T$ be the Weyl group and $\langle\ ,\ \rangle$ be a W-invariant inner product on $X(T)\otimes\mathbf{R}$. For each $\alpha \in R$, $\alpha^\vee$ is the coroot of α. W is generated by the set of reflections s_α ($\alpha \in R$) such that $s_\alpha(\lambda) = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$ (for all $\lambda \in X(T)$). Let S be the set of simple roots in R^+ and $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha = \sum_{\alpha\in S}\omega_\alpha$, where $\{\omega_\alpha\}_{\alpha\in S}$ is the set of fundamental weights. The dot action of W on $X(T)$ is defined by $w\cdot\lambda = w(\lambda+\rho)-\rho$. We say that $\lambda \geq \mu$ for $\lambda, \mu \in X(T)$ if $\lambda - \mu$ is a sum of positive roots.

For a positive integer r, $X_r(T) = \{\lambda \in X(T) \mid 0 < \langle\lambda+\rho, \alpha^\vee\rangle \leq p^r, \forall\alpha \in S\}$ is the set of the p^r-restricted weights. Each weight $\lambda \in X(T)$ has a unique decomposition $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda^0 \in X_r(T)$. Another new W-action (which is called alcove transition by Doty and Sullivan [7]) on $X(T)$ is defined by $w_r \cdot \lambda = \lambda^0 + p^r w \cdot \lambda^1$. If $\nu = p^r\eta - \rho$ is a special point, we set $(w_r)_\nu \cdot \lambda = w_r \cdot (\lambda - p^r\eta) + p^r\eta$.

The affine Weyl group W_p is generated by the reflections $s_{\alpha,np}$ ($\forall\alpha \in R^+, n \in \mathbf{Z}$) with $s_{\alpha,np}\cdot\lambda = s_\alpha\cdot\lambda + np\alpha$. A facet F is defined by a set of integers $\{n_\alpha \mid \alpha \in R^+\}$

and a partition $R^+ = R_0^+ \dot\cup R_1^+$ as follows:

$$F = \left\{ \lambda \in X(T) \;\middle|\; \begin{array}{ll} \langle \lambda+\rho, \alpha^\vee \rangle = n_\alpha p & \forall \alpha \in R_0^+ \\ (n_\alpha - 1)p < \langle \lambda+\rho, \alpha^\vee \rangle < n_\alpha p & \forall \alpha \in R_1^+ \end{array} \right\}.$$

If $R_0^+ = \emptyset$, we will call F an alcove. The closure $\bar{F}$ (or the upper closure $\hat{F}$) of a facet F is defined by changing both (or only the one on the right hand side) $<$ into $\leq$. For each alcove A, $\bar{A}$ is a fundamental domain of W_p in $X(T)$, so that any alcove is an image of the alcove $C = \{\lambda \in X(T) \mid 0 < \langle \lambda+\rho, \alpha^\vee \rangle < p, \forall \alpha \in R^+\}$ by an element of W_p. An r-box is a translate of $X_r(T)$ by an element $p^r\nu$ of $p^r X(T)$; this r-box has the upper vertex $p^r\nu + (p^r - 1)\rho$ and lower vertex $p^r\nu - \rho$.

Since $Z_r(\lambda) = \mathrm{Ind}_{B_rT}^{G_rT}\lambda \cong \mathrm{Ind}_B^{G_rB}\lambda \cong \mathrm{Ind}_{B_r}^{G_r}\lambda$ as G_r-modules (as G_rT-modules for the first isomorphism), we will use $Z_r(\lambda)$ for all of them provided the group is clear. The same setup is assumed for the simple G_rT-modules (G_rB-modules) $L_r(\lambda)$. M_λ will always denote the weight space of λ in a T-module M. Further, if M is a G-module, $M^{(n)}$ is the Frobenius twist for $n \in \mathbf{Z}$. Whenever M is a module of finite length and L a simple module in an understood category, $[M : L]$ denotes the multiplicity of L in M as composition factor.

1 Extensions over Frobenius Subgroups

Let $\lambda \in X_r(T)$. The simple G_r-module $L_r(\lambda)$ is isomorphic to $L(\lambda)$ as G_r-modules. It has been proved that $\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu)) \neq 0$ implies $\mu \in W_p \cdot \lambda + p^r X(T)$ and $\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu)) = \mathrm{Ext}^i_{G_r}(L_r(\mu), L_r(\lambda))$ ([10] II 9.16). $\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu))$ has a $G/G_r \cong G$-module structure if $\lambda, \mu \in X_r(T)$. Thus there is a G-module structure on $\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu))^{(-r)}$ such that its Frobenius twist is $\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu))$.

1.1. Note that $L_r(\lambda) \otimes p^r\nu = L_r(\lambda + p^r\nu)$ for $\lambda, \nu \in X(T)$. Following ([10] I 6.9), one has

$$\mathrm{Ext}^i_{G_rT}(L_r(\lambda+p^r\nu), L_r(\mu)) = \mathrm{Ext}^i_{G_rT}(L_r(\lambda), L_r(\mu - p^r\nu)) = \mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu))_{p^r\nu} \quad (1)$$

for all $\mu \in X(T)$. Here the right hand side is the weight space of the weight $p^r\nu$ in $\mathrm{Ext}^i_{G_r}(L_r(\lambda), L_r(\mu))$.

Proposition. *Let $\lambda, \mu \in X(T)$ and $\nu = p^r\mu^1 + (p^r - 1)\rho$. Then, for all $w \in W$,*

$$\mathrm{Ext}^i_{G_rT}(L_r(\lambda), L_r(\mu)) = \mathrm{Ext}^i_{G_rT}(L_r((w_r)_\nu \cdot \lambda), L_r(\mu)) \quad (2)$$

Proof: Let us consider the G/G_r-module $\mathrm{Ext}^i_{G_r}(L_r(\lambda^0), L_r(\mu^0))$. The Weyl group W permutes the weights of this G-module. All the weight spaces of the weights in a W-orbit have the same dimension. So we have the isomorphism of vector spaces

$$\mathrm{Ext}^i_{G_r}(L_r(\lambda^0), L_r(\mu^0))_{p^r\eta} \cong \mathrm{Ext}^i_{G_r}(L_r(\lambda^0), L_r(\mu^0))_{w(p^r\eta)}$$

for all $w \in W$ and $\eta \in X(T)$. Using (1) we get

$$\mathrm{Ext}^i_{G_rT}(L_r(\lambda^0 + p^r\eta), L_r(\mu^0)) \cong \mathrm{Ext}^i_{G_rT}(L_r(\lambda^0 + p^r w(\eta)), L_r(\mu^0)).$$

Specially, we take $\eta = \lambda^1 - \mu^1$ to get

$$\begin{aligned} & \mathrm{Ext}^i_{G_rT}(L_r(\lambda), L_r(\mu)) = \mathrm{Ext}^i_{G_rT}(L_r(\lambda^0 + p^r w(\lambda^1 - \mu^1)), L_r(\mu^0)) \\ = \; & \mathrm{Ext}^i_{G_rT}(L_r(\lambda^0 + p^r(w \cdot (\lambda^1 - \mu^1 - \rho) + \rho + \mu^1)), L_r(\mu)) \\ = \; & \mathrm{Ext}^i_{G_rT}(L_r((w_r)_\nu \cdot \lambda), L_r(\mu)). \blacksquare \end{aligned}$$

Remarks (1) From the proof of the above proposition we can also get

$$\mathrm{Ext}^i_{G_rT}(L_r(\lambda), L_r(\mu)) \cong \mathrm{Ext}^i_{G_rT}(L_r(w_r \cdot \lambda), L_r(w_r \cdot \mu)). \tag{3}$$

(2) If $\lambda^1 = \mu^1$ in the above proposition, then $(w_r)_\nu \cdot \lambda = \lambda$ for all $w \in W$. Thus $\mathrm{Ext}^i_{G_rT}(L_r(\lambda), L_r(\mu))$ is the weight space of the weight 0 in $\mathrm{Ext}^i_{G_r}(L_r(\lambda^0), L_r(\mu^0))$.

1.2. Theorem. *Let $\lambda, \mu \in X(T)$. If $\mathrm{Ext}^i_{G_rT}(L_r(\mu), Z_r(\lambda)) \neq 0$ for some $i \geq 0$, then λ is strongly linked to μ. Moreover, $\mathrm{Ext}^i_{G_rT}(L_r(\lambda), Z_r(\lambda)) = 0$ for $i > 0$.*

Proof: Let $M = \mathrm{Coind}^{G_rT}_{B_r^+T}\mu$, which has the simple head $L_r(\mu)$. Applying the functor $\mathrm{Hom}_{G_rT}(?, Z_r(\lambda))$ we can get the exact sequence

$$\mathrm{Ext}^{i-1}_{G_rT}(\mathrm{Rad}_{G_rT}M, Z_r(\lambda)) \to \mathrm{Ext}^i_{G_rT}(L_r(\mu), Z_r(\lambda)) \to \mathrm{Ext}^i_{G_rT}(M, Z_r(\lambda)) \to . \tag{4}$$

It follows from ([10] II 11.1) that $\mathrm{Ext}^i_{G_rT}(M, Z_r(\lambda)) = 0$ for $i \geq 1$.

We use induction on i to prove the theorem. If $i = 0$, $\mathrm{Hom}_{G_rT}(L_r(\mu), Z_r(\lambda)) \neq 0$ if and only if $\lambda = \mu$. Suppose the theorem is true for $i - 1$ with $i > 0$. Then $\mathrm{Ext}^i_{G_rT}(L_r(\mu), Z_r(\lambda)) \neq 0$ implies $\mathrm{Ext}^{i-1}_{G_rT}(\mathrm{Rad}_{G_rT}M, Z_r(\lambda)) \neq 0$ by (4). Therefore there is a composition factor $L_r(\mu')$ of $\mathrm{Rad}_{G_rT}M$ such that $\mathrm{Ext}^{i-1}_{G_rT}(L_r(\mu'), Z_r(\lambda)) \neq 0$. So λ is strongly linked to μ'. By the strong linkage principle ([10] II ch6), μ' is strongly linked to μ. Hence λ is strongly linked to μ. The second assertion follows from the fact that $\mu' < \mu$ for $i > 0$. $\blacksquare$

1.3. It has been proved that (see [10] II 2.14) for $\lambda, \mu \in X_+(T)$ with $\mu \not> \lambda$, $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) = \mathrm{Hom}_G(L(\mu), H^0(\lambda)/L(\lambda))$. The same property holds for the simple G_rT-modules.

Corollary. *Let $\lambda, \mu \in X(T)$ with $\mu \not> \lambda$. Then*

$$\mathrm{Ext}^1_{G_rT}(L_r(\mu), L_r(\lambda)) \cong \mathrm{Hom}_{G_rT}(L_r(\mu), Z_r(\lambda)/L_r(\lambda)). \tag{5}$$

Proof: Applying the functor $\mathrm{Hom}_{G_rT}(L_r(\mu), \ ?\ )$ to the exact sequence $0 \to L_r(\lambda) \to Z_r(\lambda) \to Z_r(\lambda)/L_r(\lambda) \to 0$ we get the following exact sequence of vector spaces:

$$0 \to \mathrm{Hom}_{G_rT}(L_r(\mu), L_r(\lambda)) \to \mathrm{Hom}_{G_rT}(L_r(\mu), Z_r(\lambda)) \to \mathrm{Hom}_{G_rT}(L_r(\mu), Z_r(\lambda)/L_r(\lambda))$$
$$\to \mathrm{Ext}^1_{G_rT}(L_r(\mu), L_r(\lambda)) \to \mathrm{Ext}^1_{G_rT}(L_r(\mu), Z_r(\lambda)).$$

Note that $L_r(\lambda)$ is the socle of $Z_r(\lambda)$ as G_rT-module. So the first two terms on the left side are always isomorphic. It follows from Theorem 1.2 that the last term is zero. So the corollary follows from the above exact sequence. ∎

1.4. It follows from Theorem 1.2 that $\mathrm{Ext}^i_{G_rT}(L_r(\mu), Z_r(\lambda)) = 0$ if λ is not strongly linked to μ. Note that the induction functor $\mathrm{Ind}_{B_rT}^{G_rT}$ is exact. By using Frobenius reciprocity and a spectral sequence we can get

$$\mathrm{Ext}^i_{B_rT}(L_r(\mu), \lambda) = \mathrm{Ext}^i_{G_rT}(L_r(\mu), Z_r(\lambda)) = 0. \tag{6}$$

On the other hand, it has been proved that $\mathrm{Ext}^i_{B_r}(L_r(\mu), \lambda)$ is a B/B_r-module and $\mathrm{Ext}^i_{B_rT}(L_r(\mu), \lambda) = \mathrm{Ext}^i_{B_r}(L_r(\mu), \lambda)^{T/T_r}$. Therefore, if λ is not strongly linked to μ, we have

$$\mathrm{Ext}^i_{B_r}(L_r(\mu), \lambda)_0 = 0. \tag{7}$$

Recall that the r-th Steinberg module is defined as $St_r = H^0((p^r - 1)\rho)$ which is a simple G-module and injective G_rT-module.

Corollary. *Let $\mu, \lambda \in X_r(T)$. If ν is a weight of $\mathrm{Ext}^i_{B_r}(L_r(\mu), \lambda)^{(-r)}$, then $\lambda - p^r\nu$ is strongly linked to μ. Furthermore, if $i = 1$, then $\lambda - \mu - p^r\nu + (p^r - 1)\rho$ is a weight of St_r.*

Proof: Let ν be a weight of $\mathrm{Ext}^r_{B_r}(L_r(\mu), \lambda)^{(-r)}$. It follows from ([10] I 6.9(5)) that $\mathrm{Ext}^i_{B_r}(L(\mu), \lambda)_{p^r\nu} = \mathrm{Ext}^i_{B_rT}(L(\mu), \lambda - p^r\nu)$. Now the first part follows from (6) and Theorem 1.2. The second part is a result of Andersen ([2] 3.2). ∎

1.5. It was ([10] II ch12) conjectured that $\mathrm{Ext}^j_{G_r}(L(\mu), H^0(\lambda))$ with $\mu \in X_r(T)$ and $\lambda \in X_+(T)$ possesses a good filtration. Therefore the following spectral sequence

$$\mathrm{Ext}^i_G(H^0(\nu)^*, \mathrm{Ext}^j_{G_r}(L(\mu), H^0(\lambda))^{(-r)}) \Rightarrow \mathrm{Ext}^{i+j}_G((H^0(\nu)^*)^{(r)} \otimes L(\mu), H^0(\lambda)) \tag{8}$$

would degenerate and yield the following isomorphism

$$\mathrm{Hom}_G(H^0(\nu)^*, \mathrm{Ext}^j_{G_r}(L(\mu), H^0(\lambda))^{(-r)}) \cong \mathrm{Ext}^j_G((H^0(\nu)^*)^{(r)} \otimes L(\mu), H^0(\lambda)). \tag{9}$$

Lemma. *Let $\mu \in X_r(T)$ and $\lambda, \nu \in X_+(T)$. Then (9) is true for $j = 1$.*

Proof: Let $\lambda = \lambda^0 + p^r\lambda^1$ with $\lambda^0 \in X_r(T)$. Then $\mathrm{Soc}_{G_r} H^0(\lambda) = L(\lambda^0) \otimes H^0(\lambda^1)^{(r)}$ by [2] (4.2). So we have

$$\mathrm{Hom}_{G_r}(L(\mu), H^0(\lambda))^{(-r)} = \begin{cases} H^0(\lambda^1) & \text{if } \mu = \lambda^0, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Therefore $\mathrm{Ext}^i_G(H^0(\nu)^*, \mathrm{Hom}_{G_r}(L(\mu), H^0(\lambda))^{(-r)}) = 0$ for all $i > 0$ (see [10] II 4.13). So the lemma follows from the five-term exact sequence corresponding to the spectral sequence (8). ∎

1.6. Lemma. *Let $\mu, \lambda \in X_r(T)$. Then any weight ν of $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}$ satisfies $\lambda - 2(p^r - 1)\rho \leq \mu + p^r\nu \leq w_0(\lambda) + 2(p^r - 1)\rho$.*

Proof: Note that λ is the lowest weight of the B-module $(p^r-1)\rho \otimes L(w_0(\lambda) + (p^r-1)\rho)$. By using the left exactness of the induction functor Ind_B^G and the tensor identity one shows that $H^0(\lambda)$ is a G-submodule of $St_r \otimes L(w_0(\lambda) + (p^r - 1)\rho)$. Now one can use the similar argument in the proof of [2] 5.1 to get the lemma. ∎

1.7. Theorem. *Let $\mu, \lambda \in X_r(T)$ and $p \geq 3(h-1)$. Then*

(i) $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}$ *is semisimple as G-module;*

(ii) $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}$ *possesses a good filtration;*

(iii) $[\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)} : L(\nu)] = \dim \mathrm{Ext}^1_G(L(\mu + p^r\nu), H^0(\lambda))$;

(iv) *If $L(\nu)$ is composition factor of $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}$, then λ is strongly linked to $\mu + p^r\nu$.*

Proof: Let ν be a dominant weight of $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}$. We will show that ν is in $\hat{C}$. Then (i) will follow from the strong linkage principle and (ii) from Bott's theorem ([10] II 5.5). In fact, by Lemma 1.6 we have $\mu + p^r\nu \leq w_0(\lambda) + 2(p^r - 1)\rho$. Let α_0 be the highest short root. Then $p^r\langle \nu + \rho, \alpha_0^\vee\rangle \leq 3p^r(h-1) - 2(h-1)$ and $\nu \in C$. (iii) is a consequence of (i) and Lemma 1.5. (iv) can be got by (iii) and [10](II 6.20). ∎

Corollary. *Let $\mu, \lambda \in X_r(T)$. If λ is not strongly linked to μ and $p \geq 3(h-1)$, then $\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}$ has no trivial composition factor.* ∎

Remark By Jantzen [10] (II 12.8) we have a natural isomorphism of G-modules:

$$\mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)} \cong \mathrm{Ind}_B^G(\mathrm{Ext}^1_{B_r}(L(\mu), \lambda)^{(-r)}). \tag{11}$$

So this corollary can also be proved by using Corollary 1.4 and Sullivan [16](Prop. 1.1).

1.8. Let $\mu \in X_r(T)$ and $\lambda \in X_+(T)$. It follows from [10] (I 6.6(1)) that, for each $\nu \in X(T)$, there is a spectral sequence with the E_2 terms

$$E_2^{n,m} = \mathrm{Ext}^n_{B/B_r}(p^r\nu, \mathrm{Ext}^m_{B_r}(L(\mu), \lambda)) \Rightarrow \mathrm{Ext}^{n+m}_B(L(\mu), \lambda - p^r\nu).$$

The corresponding five-term exact sequence is

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{B/B_r}(p^r\nu, \mathrm{Hom}_{B_r}(L(\mu), \lambda)) &\to \mathrm{Ext}^1_B(L(\mu), \lambda - p^r\nu) \to \\ \mathrm{Hom}_{B/B_r}(p^r\nu, \mathrm{Ext}^1_{B_r}(L(\mu), \lambda)) &\to \mathrm{Ext}^2_{B/B_r}(p^r\nu, \mathrm{Hom}_{B_r}(L(\mu), \lambda)). \end{aligned} \tag{12}$$

Now we assume that ν is a weight of $\mathrm{Ext}^1_{B_r}(L(\mu), \lambda)^{(-r)}$. Thus $\mu > \lambda - p^r\nu$ by Corollary 1.4. It follows from Lemma 3.1 of [2] that

$$\mathrm{Hom}_{B_r}(L(\mu), \lambda) = \begin{cases} p^r\eta & \text{if } \lambda = \mu + p^r\eta, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Now we claim that the Ext^2-term in (12) is zero. In fact we only need to consider the case when $\lambda = \mu + p^r\eta$. Thus the last term turns out to be

$$\mathrm{Ext}^2_{B/B_r}(p^r\nu, p^r\eta) = \mathrm{Ext}^2_B(\nu, \eta) = H^2(B, \eta - \nu).$$

Note that $p^r(\eta - \nu) + (p^r - 1)\rho = \lambda - \mu - p^r\nu + (p^r - 1)\rho$ is a weight of st_r. Now $H^2(B, \eta - \nu) = 0$ follows from 12.5(b) of [10].

Theorem. *If $\lambda, \mu \in X_1(T)$ and $\lambda \geq \mu$, then* $\mathrm{Ext}^1_{G_1}(L(\mu), H^0(\lambda)) = 0$.

Proof: Let ν be a weight of $\mathrm{Ext}^1_{B_r}(L(\mu), \lambda)^{(-r)}$, then $p^r\nu > \lambda - \mu$ by Corollary 1.4. So that $\nu > 0$. Since $\lambda \in X_r(T)$, $\lambda - p^r\nu \notin X_+(T)$.

Now let us use the fact $r = 1$ to show that the first two terms are isomorphic in (12) for the above chosen ν. It follows from (2.3) of [2] that

$$\mathrm{Ext}^1_B(L(\mu), \lambda - p\nu) = \begin{cases} k & \text{if } \lambda - \mu - p\nu = -p\alpha \text{ for some } \alpha \in S, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathrm{Ext}^1_B(L(\mu), \lambda - p\nu) \neq 0$, then $\lambda - \mu = p(\nu - \alpha)$. Thus

$$\mathrm{Ext}^1_{B/B_1}(p\nu, \mathrm{Hom}_{B_1}(L(\mu), \lambda)) = H^1(B, (\nu - \alpha) - \nu) = H^1(B, -\alpha) = k$$

by (2.4) of [2]. Now it follows from (12) that $\mathrm{Ext}^1_{B_1}(L(\mu), \lambda)$ is zero because there is no weight in its socle as B/B_1-module. So the theorem follows from (11). ∎

Corollary. *Let $\mu \in X_1(T)$ and $\lambda \in X_+(T)$ with $\lambda = \lambda^0 + p\lambda^1$. If $\lambda^0 \geq \mu$, then* $\mathrm{Ext}^1_{B_1}(L(\mu), \lambda) = 0$ *and* $\mathrm{Ext}^1_{G_1}(L(\mu), H^0(\lambda)) = 0$.

Proof: We have the isomorphism $\lambda \cong \lambda^0$ as B_1-modules. Therefore $\mathrm{Ext}^1_{B_1}(L(\mu), \lambda) \cong \mathrm{Ext}^1_{B_1}(L(\mu), \lambda^0)$ as vector spaces and the first assertion follows from the proof of the theorem. The second assertion is a consequence of the first one and the remark of ([10] II 12.8). ∎

1.9. Note that $G_r \subseteq G_rT \subseteq G_rB \subseteq G$ as group schemes and G_r is normal in the other three. If M is a G-module, then $\mathrm{Soc}^i_{G_r}M = \mathrm{Soc}^i_{G_rT}M$ are G-submodules of M for all $i \geq 1$. On the other hand, each simple G-module is semisimple if it is restricted to G_r or G_rT. Thus $\mathrm{Soc}^i_{G_r}M \supseteq \mathrm{Soc}^1_G M$.

Proposition. *Let $\lambda \in X_+(T)$. If all dominant highest weights μ of composition factors of $Z_r(\lambda)$ satisfy $\mu^1 \in \hat{C}_p$, then* $\mathrm{Soc}^i_{G_r}H^0(\lambda) = \mathrm{Soc}^i_G H^0(\lambda)$ *for all $i \geq 1$.*

Proof: Recall that $H^0(\lambda) = H^0(G/G_rB, Z_r(\lambda))$. It is proved by Bai-Wang-Wen [5] and Sullivan [16] that $\mathrm{Soc}^i_G H^0(\lambda) = H^0(G/G_rB, \mathrm{Soc}^i_{G_r}Z_r(\lambda))$ for each $i > 0$.

Let us consider the evaluation map $ev : H^0(\lambda) \to Z_r(\lambda)$ to get the restriction map $ev : \mathrm{Soc}^i_{G_r} H^0(\lambda) \to \mathrm{Soc}^i_{G_r} Z_r(\lambda)$ which is also a G_rB-map. Now we apply the induction functor $H^0(G/G_rB, \,?\,)$ to get the map

$$H^0(ev)_i : \mathrm{Soc}^i_{G_r} H^0(\lambda) \to H^0(G/G_rB, \mathrm{Soc}^i_{G_r} Z_r(\lambda)) = \mathrm{Soc}^i_G H^0(\lambda).$$

Here we have used the tensor identity. On the other hand, the naturality of the map $H^0(ev)_i$ implies that $H^0(ev)_i$ is the restriction of the map $H^0(ev) = id : H^0(\lambda) \to H^0(\lambda)$. Hence, $\mathrm{Soc}^i_{G_r} H^0(\lambda) \subseteq \mathrm{Soc}^i_G H^0(\lambda)$ and the theorem follows. ∎

1.10. Now let us consider the extensions between simple G_r-modules.

Proposition. *If $\lambda, \mu \in X_r(T)$, then we have the following exact sequence of G-modules*

$$0 \to \bigoplus_{\nu \in X_+(T)} L(\nu)^{\oplus m_\nu} \to \mathrm{Ext}^1_{G_r}(L(\mu), L(\lambda))^{(-r)} \to \mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda))^{(-r)}. \qquad (14)$$

Here $m_\nu = [\mathrm{Soc}^2_G H^0(\lambda)/L(\lambda) : L(\mu + p^r\nu)]$.

Proof: Applying the left exact functor $\mathrm{Hom}_{G_r}(L(\mu), \,?\,)$ to the exact sequence $0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0$ we can get the exact sequence of G-modules

$$0 \to \mathrm{Hom}_{G_r}(L(\mu), H^0(\lambda)/L(\lambda)) \to \mathrm{Ext}^1_{G_rT}(L(\mu), L(\lambda)) \to \mathrm{Ext}^1_{G_r}(L(\mu), H^0(\lambda)). \quad (15)$$

Here we have used the fact that $\mathrm{Hom}_{G_r}(L(\mu), L(\lambda)) = \mathrm{Hom}_{G_r}(L(\mu), H^0(\lambda))$ since $\mathrm{Soc}_{G_r} H^0(\lambda) = L(\lambda)$ by Proposition 1.9.

Note that $\mathrm{Hom}_{G_r}(L(\mu), H^0(\lambda)/L(\lambda)) = \mathrm{Hom}_{G_r}(L(\mu), \mathrm{Soc}^2_G H^0(\lambda)/L(\lambda))$ by Proposition 1.9. By using Steinberg's tensor product theorem the second G-socle layer can be decomposed to be

$$\mathrm{Soc}^2_G H^0(\lambda)/L(\lambda) \cong \bigoplus_{\xi \in X_r(T)} (L(\xi) \otimes (\oplus_{\nu \in X_+(T)} (L(\nu)^{(r)})^{\oplus m_\nu}))$$

with $m_\nu = [\mathrm{Soc}^2_G H^0(\lambda)/L(\lambda) : L(\xi) \otimes L(\nu)^{(r)}]$. Therefore the first term in (15) can be calculated from this decomposition. Now the proposition follows by taking the Frobenius twist on the exact sequence (15). ∎

2 Intertwining Homomorphisms

2.1. Let $w \in W$ and $\dot{w} \in N_G(T)$ be a representative of w. We define the twisted Borel subgroup by $B^w = \dot{w}^{-1}B\dot{w}$. Set $Z_r(w, \lambda) = \mathrm{Ind}_{B_r^w T}^{G_r T}\lambda$. It is proved in [6] (2.1) that $Z_r(w, \lambda) \cong Z_r(w(\lambda))^w$. Here M^w, for a G_rT-module M, is the G_rT-module with the same vector space M and new action defined by $g * m = \dot{w}g\dot{w}^{-1}m$ for all $g \in G_rT$ and $m \in M$.

Let $L_r(\lambda)$ be a simple G_rT-module. Then $L_r(\lambda)^w = L_r(\lambda^0 + p^r w^{-1}(\lambda^1)) = L_r(w_r^{-1}(\lambda))$. Denote by $S_r[M]$, for a G_rT-module M, the socle series pattern in terms of the multiplicity and the highest weight of each composition factor in each socle layer. Any action of W on such a pattern is considered to be the action of W on the highest weights only.

Theorem. *Let $\lambda \in X(T)$ and $w \in W$. Then* $\mathrm{ch}Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0) = \mathrm{ch}Z_r(\lambda)$ *and*

$$S_r[Z_r(w, \lambda + (p^r - 1)w^{-1} \cdot 0)] = w_r^{-1} \cdot S_r[Z_r(w \cdot \lambda)]. \tag{16}$$

Proof: The first assertion is due to Doty [6] (6.1). The second assertion follows from the proof of Theorem 4.1 in [12]. ∎

2.2. From now on we assume $r = 1$ (we will still denote the action $w_r \cdot \lambda$ instead of $w_1 \cdot \lambda$). Let $\alpha \in S$ be a simple root, and P the minimal parabolic subgroup of G containing B with the only positive root α. Set $Z_1^\alpha(\lambda) = \mathrm{Ind}_{B_1T}^{P_1T}\lambda$ and $Z_1^\alpha(s_\alpha, \lambda) = \mathrm{Ind}_{B_1^{s_\alpha}T}^{P_1T}\lambda$. It is well known [6] (Section 4) that $Z_1^\alpha(\lambda) = L_1^\alpha(\lambda)$ is simple if $\langle \lambda + \rho, \alpha^\vee \rangle = np$ for some n; otherwise $0 \to L_1^\alpha(\lambda) \to Z_1^\alpha(\lambda) \to L_1^\alpha((s_\alpha)_* \cdot \lambda) \to 0$ is a non-split exact sequence of P_1T-modules. Here $(s_\alpha)_*$ is the reflection in the lower α-hyperplane.

Applying Theorem 2.1 to a rank one group we can get that either $Z_1^\alpha(s_\alpha, \lambda) = L_1^\alpha((s_\alpha)_r \cdot (s_\alpha \cdot \lambda))$ is simple or

$$0 \to L_1^\alpha((s_\alpha)_r \cdot (s_\alpha \cdot \lambda)) \to Z_1^\alpha(s_\alpha, \lambda + (p-1)s_\alpha \cdot 0) \to L_1^\alpha((s_\alpha)_r \cdot (s_\alpha)_* \cdot (s_\alpha \cdot \lambda)) \to 0 \tag{17}$$

is exact and non-split. For $r = 1$, it can be calculated that

$$(s_\alpha)_r \cdot (s_\alpha \cdot \lambda) = (s_\alpha)_* \cdot \lambda \quad , \quad (s_\alpha)_r \cdot (s_\alpha)_* \cdot (s_\alpha \cdot \lambda) = \lambda. \tag{18}$$

Lemma. (Andersen and Kaneda [4]) *Let $\alpha \in S$ and $\lambda \in X(T)$.*

(i) $Z_1(\lambda) \cong Z_1(s_\alpha, \lambda + (p-1)s_\alpha \cdot 0)$ *if* $\langle \lambda + \rho, \alpha^\vee \rangle = np$;

(ii) *If* $\langle \lambda + \rho, \alpha^\vee \rangle \not\equiv 0 \pmod p$, *then there is a unique (up to a scalar multiple) nonzero map* $\varphi : Z_1(\lambda) \to Z_1(s_\alpha, \lambda + (p-1)s_\alpha \cdot 0)$ *with* $\mathrm{Ker}(\varphi) = \mathrm{Ind}_{P_1T}^{G_1T} L_1^\alpha(\lambda)$. ∎

2.3. In general we have the following

Proposition. (Andersen and Kaneda [4]) *Let* $\alpha \in S$, $\lambda \in X(T)$ *and* $w \in W$.

(i) *If* $\langle \lambda + \rho, w^{-1}(\alpha^\vee) \rangle \equiv 0 \pmod p$, *then* $Z_1(w, \lambda + (p-1)w^{-1} \cdot 0) \cong Z_1(s_\alpha w, \lambda + (p-1)w^{-1}s_\alpha \cdot 0)$;

(ii) *If* $\langle \lambda + \rho, w^{-1}(\alpha^\vee) \rangle \equiv d \pmod p$, *then there is a unique (up to a scalar multiple) nonzero map* $\varphi : Z_1(w, \lambda + (p-1)w^{-1} \cdot 0) \to Z_1(s_\alpha w, \lambda + (p-1)w^{-1}s_\alpha \cdot 0)$ *with*

$$\mathrm{ch}(\mathrm{Ker}(\varphi)) = \sum_{i>0} (\mathrm{ch} Z_1(\lambda - ipw^{-1}(\alpha)) - \mathrm{ch} Z_1(\lambda' - ipw^{-1}(\alpha))). \tag{19}$$

Here $\lambda' = \lambda - dw^{-1}(\alpha)$. ∎

2.4. Let $w \in W$. Set $R^+(w) = \{\alpha \in R^+ \mid w(\alpha) \in R^-\}$. For $\lambda \in X(T)$ we define $\ell_\lambda(w) = \#(R^+(w) \cap R_\lambda)$. Here $R_\lambda = \{\alpha \in R \mid \langle \lambda + \rho, \alpha^\vee \rangle \equiv 0 \pmod p\}$. Let $W[\lambda]$ be the subset of W such that $w \in W[\lambda]$ if

(i). $\ell(w) - \ell_\lambda(w) = 1$;

(ii). $\ell(s_\alpha w) - \ell_\lambda(s_\alpha w) = 0$ for any $\alpha \in S$ with $\ell(s_\alpha w) = \ell(w) - 1$.

Lemma. *Let* $w = s_{\alpha_n} \cdots s_{\alpha_1}$ *be a reduced expression. Set* $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ *for* $i = 1, \cdots, n$. *Then* $w \in W[\lambda]$ *if and only if* $\beta_i \in R_\lambda$ *for all* $i < n$ *and* $\beta_n \notin R_\lambda$.

Proof: It is well known that $R^+(w) = \{\beta_1, \cdots, \beta_n\}$. So $\ell(w) - \ell_\lambda(w) = 1$ if and only if there is a unique i_0 such that $1 \le i_0 \le n$ and $\beta_{i_0} \notin R_\lambda$. If $i_0 < n$, then $\beta_n \in R_\lambda$. However, $\ell(s_{\alpha_n} w) = \ell(w) - 1$ and $R^+(s_{\alpha_n} w) = \{\beta_1, \cdots, \beta_{n-1}\}$. This leads to $\ell(s_{\alpha_n} w) - \ell_\lambda(s_{\alpha_n} w) = 1$. The other direction is easily checked. ∎

2.5. Let $\lambda \in X(T)$. For each $w \in W[\lambda]$ with a reduced expression $w = s_{\alpha_n} \cdots s_{\alpha_1}$, we define φ_w to be the composite of the following maps:

$$Z_1(\lambda) \to Z_1(s_{\alpha_1}, \lambda + (p-1)s_{\alpha_1} \cdot 0) \to \cdots \to Z_1(w, \lambda + (p-1)w^{-1} \cdot 0).$$

Here the first $n-1$ maps are isomorphisms by Lemma 2.4 and Proposition 2.3. Therefore the character of $\mathrm{Ker}(\varphi_w)$ can be calculated from the character of the kernel of the last map. Now using Proposition 2.3 we get

$$\mathrm{ch}(\mathrm{Ker}\varphi_w) = \sum_{i>0} \left(\mathrm{ch}(Z_1(\lambda + ipw^{-1}(\alpha_n))) - \mathrm{ch}(Z_1(\lambda' + ipw^{-1}(\alpha_n)))\right). \tag{20}$$

Here $\lambda' = \lambda + dw^{-1}(\alpha_n)$ with $\langle \lambda + \rho, w^{-1}(-\alpha_n)\rangle \equiv d \pmod p$ and $0 < d < p$.

For each φ_w defined above, one can see that $\varphi_w(\mathrm{Soc}_{G_1T} Z_1(\lambda)) = 0$. Thus

$$\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda) \subseteq \bigcap_{w\in W[\lambda]} \varphi_w^{-1}(\mathrm{Soc}^i_{G_1T} Z_1(w, \lambda + (p-1)w^{-1}\cdot 0)). \tag{21}$$

The intersection on the right side is not easy to calculate when a composition factor has multiplicity larger than one in $Z_1(\lambda)$, even if the composition factors of each inverse image were known. However we will estimate the right side. Define the following formal filtration patterns:

$$[B_i(\lambda)] = \{(\mu, m) \mid m = \min_{w\in W[\lambda]} \{[\varphi_w^{-1}(\mathrm{Soc}^i_{G_1T} Z_1(w, \lambda + (p-1)w^{-1}\cdot 0)) : L_1(\mu)]\}\}; \tag{22}$$

$$[\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda)] = \{(\mu, m) \mid m = [\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda) : L_1(\mu)]\}. \tag{23}$$

So the inclusion in (21) implies

$$[\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda)] \le [B_i(\lambda)]. \tag{24}$$

Here $\le$ means that $m \le n$ for each $\mu \in X(T)$ with $(\mu, m) \in [\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda)]$ and $(\mu, n) \in [B_i(\lambda)]$.

Since $\mathrm{ch}\varphi_w^{-1}(\mathrm{Soc}^i_{G_1T} Z_1(w, \lambda + (p-1)w^{-1}\cdot 0)) = \mathrm{ch}(\mathrm{Ker}\varphi_w) + \mathrm{ch}(\mathrm{Soc}^i_{G_1T} Z_1(w, \lambda + (p-1)w^{-1}\cdot 0))$, $[B_i(\lambda)]$ will be computable if $\mathrm{Soc}^i_{G_1T} Z_1(w\cdot\lambda)$ is known for all $w \in W[\lambda]$ (cf. Theorem 2.1).

2.6. Let $\lambda \in X(T)$. There is a unique alcove $A(\lambda)$ such that $\lambda \in \hat{A}(\lambda)$. Let $\lambda' \in A(\lambda)$ (assume $p > (h-1)$). Let us consider the translation functor $T^{\lambda}_{\lambda'}$. If $L_1(\mu')$ is a composition factor of $Z_1(\lambda')$, then

$$T^{\lambda}_{\lambda'} L_1(\mu') = \begin{cases} L_1(\mu) & \text{if } \mu \in W_p\cdot\lambda \cap \hat{A}(\mu'), \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Therefore $T_{\lambda'}^{\lambda}$ sends semisimple modules to semisimple modules. On the other hand, $T_{\lambda'}^{\lambda} Z_1(\lambda') = Z_1(\lambda)$. So

$$T_{\lambda'}^{\lambda} \mathrm{Soc}_{G_1T}^{i} Z_1(\lambda') \subseteq \mathrm{Soc}_{G_1T}^{i} Z_1(\lambda). \tag{26}$$

It is conjectured in [13] that the equality should hold in general.

Suppose one knows the G_1T-socle series of $Z_1(\lambda')$ for p-regular weights λ' (for instance, the socle series of $Z_1(\lambda')$ can be calculated from the generic inverse K-L polynomials if Lusztig's conjecture is true [4]). For a p-singular weight λ, the socle series of $Z_1(\lambda)$ can be estimated by

$$[T_{\lambda'}^{\lambda} \mathrm{Soc}_{G_1T}^{i+1} Z_1(\lambda')] \leq [\mathrm{Soc}_{G_1T}^{i+1} Z_1(\lambda)] \leq [B_i(\lambda)] \tag{27}$$

In the next section, we use the above estimation to calculate the socle series of $Z_1(\lambda)$ for p-singular weights λ from the socle series of $Z_1(\lambda')$ with p-regular highest weights λ' for the exceptional group of type G_2.

3 G_1T-Socle Series of $Z_1(\lambda)$ for p-singular λ

Throughout this section we consider only the group of type G_2. Let λ and λ' be as in 2.6. The socle series of $Z_1(\lambda')$ in this case can be calculated from the generic inverse K-L polynomials as proved by Andersen and Kaneda in [4] since the Lusztig conjecture has been proved in this case. Let $\{\alpha, \beta\}$ be the set of simple roots with α short. ω_α and ω_β are the two fundamental weights. There are twelve alcoves, eighteen 1-dimensional facets and six 0-dimensional fecets in the restricted region. The 1-dimensional facet types are numbered as in Figure 1

3.1. The G-Socle Series of $H^0(\lambda)$ for Restricted Weights. Let $\lambda \in X_1(T)$ be a p-singular weight and λ' is taken as in 2.6. The socle series of $H^0(\lambda')$ has been determined in [14] by the author. Therefore $T_{\lambda'}^{\lambda} \mathrm{Soc}_G^i H^0(\lambda')$ defines a filtration in $H^0(\lambda)$ with semisimple quotients. If $T_{\lambda'}^{\lambda} \mathrm{Soc}_G^2 H^0(\lambda') = H^0(\lambda)$, the socle filtration of $H^0(\lambda)$ is exactly the above filtration since $T_{\lambda'}^{\lambda} \mathrm{Soc}_G H^0(\lambda') = \mathrm{Soc}_G H^0(\lambda)$. The above condition is true if λ is not in the 1-dimensional facets of types 11, 13, 17. In these situations, $T_{\lambda'}^{\lambda} \mathrm{Soc}_G^3 H^0(\lambda') / T_{\lambda'}^{\lambda} \mathrm{Soc}_G^2 H^0(\lambda') = L(\nu)$ is simple. Next we will show that in these three situations $L(\nu)$ is not in the second socle layer. Therefore the socle filtration of $H^0(\lambda)$ will be the filtration determined by the translation functor $T_{\lambda'}^{\lambda}$.

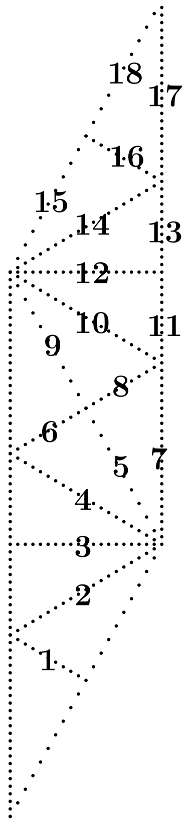

Figure 1: The 1-dimensional Facet Types

If λ is of type 13, let μ be the weight in $\bar{F} \setminus \hat{F}$. Here F is the facet containing λ. If $H^0(\lambda)/L(\lambda)$ is semisimple, then $H^0(\mu)$ is semisimple too because of the translation functor T_λ^μ. However $H^0(\mu)$ has two composition factors and a simple socle.

Let λ be of type 11 or 17. Set $\mu = (s_{\alpha+\beta})_* \cdot \lambda$ and ν as before. Then μ is a maximal weight of multiplicity one in $H^0(\lambda)/L(\lambda)$. It follows from Frobenius reciprocity that $\mathrm{Hom}_G(H^0(\lambda)/L(\lambda), H^0(\mu)) = k$. On the other hand, any G-map from $H^0(\lambda)$ to $H^0(\mu)$ kills the socle $L(\lambda)$. Therefore $\dim \mathrm{Hom}_G(H^0(\lambda), H^0(\mu)) = 1$. Let $\varphi : H^0(\lambda) \to H^0(\mu)$ be a nonzero map. If we can show that $L(\nu)$ is in the image of φ, we can conclude that $L(\nu)$ is not in the second socle layer of $H^0(\lambda)$.

Let $\lambda_1 = s_\beta \cdot \lambda$ and $\mu_1 = s_\beta \cdot \mu$. Since both μ and λ are in the restricted region, we have $H^0(\lambda) \cong H^1(\lambda_1)$ and $H^0(\mu) \cong H^1(\mu_1)$. We will construct a nonzero map $\varphi : H^1(\lambda_1) \to H^1(\mu_1)$ such that $L(\nu)$ is in its image.

In these two situations we can use the vanishing behavior of $s_\alpha \cdot \lambda_1$ described by Andersen in [1] to get

$$0 \to H^1(C_\alpha(\lambda_1)) \to H^2(s_\alpha \cdot \lambda_1) \to H^1(\lambda_1) \xrightarrow{\psi_1} H^1(C_\alpha(\lambda_1)).$$

Here $\mathrm{ch}H^1(C_\alpha(\lambda_1)) = -\chi(C_\alpha(\lambda_1))$ and $[H^1(C_\alpha(\lambda_1)) : L(\nu)] = 1$. If we show that $[H^2(s_\alpha \cdot \lambda_1) : L(\nu)] \leq 1$, we can conclude from the above exact sequence that $[\mathrm{Im}(\psi_1) : L(\nu)] = 1$. However the assertion can be derived through the following inequality:

$$[H^\ell(G/G_1B, M) : L(\nu)] \leq \sum_{w \in W, \ell(w)=\ell} [M : L_1(w_r \cdot \nu)]$$

for a G_1B-module M with highest weights of all its composition factors in $X_r + p^r W \cdot \hat{C}$.

Next we consider the exact sequence of B-modules $0 \to I \to C_\alpha(\lambda_1) \to \mu_1 \to 0$. It induces an exact sequence of G-modules

$$0 \to H^1(I) \to H^1(C_\alpha(\lambda_1)) \xrightarrow{\psi_2} H^1(\mu_1) \to H^2(I) \to .$$

It follows from the strong linkage principle that $[H^i(I) : L(\nu)] = 0$ since all weights of I are strictly smaller than ν. This yields $[\mathrm{Im}(\psi_2) : L(\nu)] = 1$. Therefore the composite $\varphi = \psi_2 \circ \psi_1$ is a map from $H^1(\lambda_1) \to H^1(\mu)$ such that $L(\nu)$ is a composition factor in its image. Thus we conclude

Theorem. *Let $\lambda \in X_1(T)$ and $\lambda' \in A(\lambda)$. Then $T_{\lambda'}^{\lambda}\mathrm{Soc}_G^i H^0(\lambda') = \mathrm{Soc}_G^i H^0(\lambda)$ for all $i \geq 1$.* ∎

3.2. The Second Socle Let us first calculate the second socle of $Z_1(\lambda)$. Then we can use Corollary 1.3 to calculate the extensions between simple modules in order to calculate the higher socles. The calculation is divided into three steps. For notational simplicity we denote $U_i(\lambda) = T_{\lambda'}^{\lambda}\mathrm{Soc}_{G_1T}^{i+1} Z_1(\lambda')$ for each $i \geq 0$.

Step 1. Calculate the patterns $[U_i(\lambda)]$ and $[B_1(\lambda)]$ from 2.5 and 2.6. If $(\mu, m_i) \in [U_i]$, then the number i appears $m_i - m_{i-1}$ times in the facet containing μ. For the formal filtration pattern $[B_i]$, we can calculate only $[B_1(\lambda)]$ since we do not know the higher socles of $Z_1(w, \lambda + (p-1)w^{-1} \cdot 0)$ yet at this moment.

Step 2. Compare the patterns we got in Step 1. If μ is a weight such that $(\mu, m(\mu)) \in [B_1(\lambda)]$, set $m = \min\{m((w_r)_v \cdot \mu) \mid w \in W\}$. Here $v = p\lambda^1 + (p-1)\rho$ is the upper vertex of the box containing λ. If $(\mu, m) \in [U_1(\lambda)]$, we can conclude, by

Table 1: Exceptional Cases for Second Socle

λ	μ	$m(\mu)$	$[U_1(\lambda) : L_1(\mu)]$
6	1	1	0
16	8	1	0
17	4	2	1
17	7	1	0
17	10	1	0
17	14	1	0
18	3	1	0
18	5	1	0
13	1	2	0
13	6	2	0
11	8	1	0

2.6(27), Proposition 1.1 and Corollary 1.3, that $[\mathrm{Soc}^2_{G_1T} Z_1(\lambda) : L_1(\mu)] = m$. It turns out that in all the situations we have $(\mu, m) \in [U_1(\lambda)]$ except for some weight μ which is in the box containing λ. In these exceptional cases $(w_r)_v \cdot \mu = \mu$ for all $w \in W$ and $m(\mu) > [U_1(\lambda) : L_1(\mu)]$. These exceptional weight pairs (λ, μ) are listed in Table 1. In this table we use facet type of the facet containing the weight to represent the weight since each pair of weights is in the same box.

Step 3. In the exceptional cases in Step 2, we can assume $\lambda, \mu \in X_1(T)$ and then apply Theorem 1.10 to calculate the extensions between $L_1(\lambda)$ and $L_1(\mu)$. The first term in (14) has no trivial composition factor by Proposition 3.1 in these cases, while the last term has no trivial composition factor by Corollary 1.7. Therefore $\mathrm{Ext}_{G_1}(L(\mu^0), L(\lambda^0))$ has no trivial composition factor and $\dim \mathrm{Ext}_{G_1T}(L_1(\mu), L_1(\lambda)) = \dim \mathrm{Ext}_{G_1}(L(\mu^0), L(\lambda^0))_0$ can be uniquely determined by $\dim \mathrm{Ext}_{G_1T}(L_1(\mu+p\nu), L_1(\lambda))$ for all $\nu \in X_+(T)$ with $\nu \neq 0$ since the character of $L(\nu)$ can be calculated. It follows from the above calculation that $\dim \mathrm{Ext}_{G_1T}(L_1(\mu), L_1(\lambda)) = [U_1(\lambda) : L_1(\mu)]$. By using Corollary 1.3 we get $\mathrm{Soc}^2_{G_1T} Z_1(\lambda) = U_1(\lambda)$.

3.3. The Higher Socles It follows from 3.2 and Corollary 1.3 that the extensions between two simple modules can be calculated. The format for calculating the higher socles of $Z_1(\lambda)$ is to use induction on i. Suppose we have calculated $\mathrm{Soc}^i_{G_1T} Z_1(\lambda)$ for

Table 2: Exceptional μ

λ	μ
(0,0;8)	$(-1,0;6),(0,-1;1,6),(-2,0;1,6),(-3,0;1),(-1,-1;1)$
(0,0;9)	$(-1,0;5),(-2,0;5),(0.-1;3,5)$
(0,0;10)	$(-1,0;4),(0,-1;2,4),(1,-1;4)$
(0,0;11)	$(0,-1;1)$
(0,0;12)	$(-1,0;3),-2,0;3,5),(0,-1;3)$
(0,0;14)	$(-1,0;2,4),(-2,0;2,4),(0,-1;2,4),(1,-1;2,4),(2,-2;2)$
(0,0;15)	$(-1,0;3)$
(0,0;16)	$(-1,0;1,13),(-2,0;1),(0,-1;1,6),(1,-1;1),(2,-2;1)$
(0,0;17)	$(-1,0;4,7),(0,0;4,7),(0,-1;4),(1,-1;4,7)$
(0,0;18)	$(-1,0;12)$

all λ. Therefore $\mathrm{Soc}^i_{G_1T} Z_1(w, \lambda + (p-1)w^{-1} \cdot 0)$ can be calculated from Theorem 2.1 and $[B_i(\lambda)]$ can be calculated as well. Then we will show that $\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda) = U_i(\lambda)$. For $i = 1$, we have done this in 3.2. Note that $[U_i(\lambda)] \leq [B_i(\lambda)]$ by (27).

Let $\mu \in X(T)$ such that $(\mu, m) \in [U_i(\lambda)]$ and $(\mu, n) \in [B_i(\lambda))]$. If $m = n$, then $[\mathrm{Soc}^{i+1}_{G_1T} Z_1(\lambda) : L_1(\mu)] = [U_i(\lambda) : L_1(\mu)]$ by (27). It turns out that $m < n$ appears only when $i = 2$. (For $i > 2$ we have $[U_i(\lambda)] = [B_i(\lambda)]$ if we believe $\mathrm{Soc}^i_{G_1T} Z_1(\lambda) = U_{i-1}(\lambda)$.) In Table 2 we list all μ for each type of λ such that $m < n$ for $i = 2$.

For each fixed λ, let μ be as listed in Table 2. The calculation as mentioned above shows that $\mathrm{Ext}^1_{G_1T}(L_1(\nu), L_1(\mu)) = 0$ for ν such that $[\mathrm{Soc}^2_{G_1T} Z_1(\lambda)/L_1(\lambda) : L_1(\nu)] \neq 0$. This implies that $[\mathrm{Soc}^3_{G_1T} Z_1(\lambda) : L_1(\mu)] = [\mathrm{Soc}^2_{G_1T} Z_1(\lambda) : L_1(\mu)]$. Comparing the positions of $L_1(\mu)$ in the filtration $\{U_i(\lambda)\}$ we can get $\mathrm{Soc}^3_{G_1T} Z_1(\lambda) = U_2(\lambda)$.

The induction has proved that $\mathrm{Soc}^i_{G_1T} Z_1(\lambda) = U_{i-1}(\lambda)$. Further, let $d(A_1, A_2)$ denote the distance from alcove A_1 to alcove A_2 as defined in Lusztig [15], which is a signed sum of the number of hyperplanes separating A_1 and A_2, with the positive direction of each hyperplane defined by the positive roots. Since the socle series of $Z_1(\lambda')$ can be calculated from the inverse K-L polynomials, the calculation of the polynomials shows the following:

Theorem. *Let F be a facet with $\lambda' \in F$ and $\lambda \in \bar{F}$. Then*

$$T_{\lambda'}^{\lambda}\mathrm{Soc}_{G_1T}^{i} Z_1(\lambda') = \mathrm{Soc}_{G_1T}^{i-d(A(\lambda),A(\lambda'))} Z_1(\lambda). \blacksquare \tag{28}$$

3.4. The Socle Series of $H^0(\lambda)$ In this subsection we assume that $\lambda \in X_+(T)$ satisfies the condition of Lusztig's conjecture. Now we can use a similar argument to that used by the author in [14] for p-regular weights to show that $H^j(\mathrm{Soc}_{G_1T}^{i} Z_1(\lambda)) = 0$ for $j > 0$ and $i > 0$. Therefore the socle series of $H^0(\lambda)$ can be calculated by the cancellation principle. One can consider the effects of the translation functors to the socle series of H^0's to get the first part of the following theorem. The second part follows from [14].

Theorem. *Let $\lambda \in X_+(T)$ be such that the Lusztig condition is satisfied.*

(i). *Let F be a facet with $\lambda' \in F$ and $\lambda \in \bar{F}$. Then*

$$T_{\lambda'}^{\lambda}\mathrm{Soc}_{G}^{i} H^0(\lambda') = \mathrm{Soc}_{G}^{i-d(A(\lambda),A(\lambda'))} H^0(\lambda); \tag{29}$$

(ii). *Let $N(\lambda)$ be the Loewy length of $H^0(\lambda)$. Then*

$$\sum_{i<N(\lambda)} \mathrm{chSoc}_{G}^{i} H^0(\lambda) = \sum_{\alpha \in R^- \setminus R_\lambda} C_\alpha(\lambda). \blacksquare \tag{30}$$

4 Extensions between Simple G-Modules

4.1. It follows from Corollary 1.3 and Theorem 3.3 that the extensions of simple G_1T-modules with p-singular highest weights can be calculated from the extensions of simple modules with p-regular highest weights for the group of type G_2. So combining with the result in ([13] ch3) we can see that the following conjecture is true for all groups of rank two.

Conjecture. *Let F_1, F_2 be two facets and $\lambda_i \in F_i$, $\mu_i \in \hat{F}_i$ for $i = 1, 2$. Suppose $\lambda_2 \in W_p \cdot \lambda_1$ and $\mu_2 \in W_p \cdot \mu_1$. Then* $\mathrm{Ext}_{G_1T}^{1}(L_1(\lambda_1), L_1(\lambda_2)) \cong \mathrm{Ext}_{G_1T}^{1}(L_1(\mu_1), L_1(\mu_2))$.$\blacksquare$

4.2. Andersen ([2] Theorem 5.2 and 5.3) has proved that $\mathrm{Ext}_{G}^{1}(L(\mu), L(\lambda))$ can be computed for all $\lambda, \mu \in X_+(T)$ provided $\mathrm{Ext}_{G_1}^{1}(L(\mu^0), L(\lambda^0))$ known for all $\mu^0, \lambda^0 \in X_1(T)$. If the socle series of $Z_1(\lambda)$ is known, the extensions $\mathrm{Ext}_{G_1T}^{1}(L_1(\mu), L_1(\lambda))$ can be calculated from Corollary 1.3. Thus the character of $\mathrm{Ext}_{G_1}^{1}(L(\mu^0), L(\lambda^0))^{(-1)}$ can be calculated by Proposition 1.1. Further, if the characters of simple modules are known, composition factors and G-module structure of $\mathrm{Ext}^1_{G_1}(L(\mu^0), L(\lambda^0))$ can be got from Theorem 5.5 of [2] for large p. If one assumes the Lusztig conjecture, for example, the above assumptions are true for p-regular weights. For the p-singular weights we have

Theorem. *Assume Conjecture* 4.1 *and* $p \geq 3(h-1)$. *Let* F_1, F_2 *be two facets contained in the restricted region and* $\lambda_i \in F_i$ *and* $\mu_i \in \hat{F}_i$ *for* $i = 1, 2$. *Suppose* $\lambda_2 \in pX(T) + W_p \cdot \lambda_1$ *and* $\mu_2 \in pX(T) + W_p \cdot \mu_1$. *Then* $\mathrm{Ext}^1_{G_1}(L(\mu_1), L(\mu_2)) \cong \mathrm{Ext}^1_{G_1}(L(\lambda_1), L(\lambda_2))$ *as* G*-modules.*

Proof: Note that both G-modules are semisimple by Theorem 5.5 of [2]. The theorem follows from Conjecture 4.1 since they have the same character. ∎

In Table 3, we list, for the group of type G$_2$, the G-modules $\mathrm{Ext}^1_{G_1}(L(\mu^0), L(\lambda^0))^{(-1)}$ with p-regular weights λ^0 and μ^0 such that $\mu^0 \in pX(T) + W_p \cdot \lambda^0$. We use roman numerals to denote the twelve alcove types following the strong linkage order with I in the bottom and the alcove $s_{2\beta+3\alpha,p} \cdot C_0$ is of type VI. For p-singular weights the extensions can be calculated through the above theorem.

4.3. Now let us consider the extensions of simple G-modules with p-singular highest weights. Here the weights do not have to be in the bottom p^2-alcove.

Theorem. *Assume Conjecture* 4.1. *Let* F_1, F_2 *be two facets in the dominant chamber and* $\lambda_i \in F_i$, $\mu_i \in \hat{F}_i$ *for* $i = 1, 2$. *Suppose* $\lambda_2 \in W_p \cdot \lambda_1$ *and* $\mu_2 \in W_p \cdot \mu_1$. *Then* $\mathrm{Ext}^1_G(L(\mu_1), L(\mu_2)) \cong \mathrm{Ext}^1_G(L(\lambda_1), L(\lambda_2))$.

Proof: We write $\lambda_i = \lambda_i^0 + p\lambda_i^1$ and $\mu_i = \mu_i^0 + p\mu_i^1$ with $\mu_i^0, \lambda_i^0 \in X_1(T)$ for $i = 1, 2$. By the assumption we have $\lambda_i^1 = \mu_i^1$ for $i = 1, 2$.

Following 5.2(a) and 1.4 of [2] we have the following:

$$\begin{aligned} &\mathrm{Ext}^1_G(L(\lambda_1), L(\lambda_2)) = \mathrm{Hom}_{G/G_1}(L(p\lambda_1^1), \mathrm{Ext}^1_{G_1}(L(\lambda_1^0), L(\lambda_2^0)) \otimes L(\lambda_2^1)) \\ &= \mathrm{Hom}_{G/G_1}(L(p\lambda_1^1), \mathrm{Ext}^1_{G_1}(L(\mu_1^0), L(\mu_2^0)) \otimes L(p\lambda_2^1)) = \mathrm{Ext}^1_G(L(\mu_1), L(\mu_2)). \end{aligned}$$

Here the equality in the middle follows from Theorem 4.2. ∎

Table 3: The G-module $\mathrm{Ext}^1_{G_1}(L(\mu^0), L(\lambda^0))^{(-1)}$

λ^0	μ^0	$\mathrm{Ext}^1_{G_1}(L(\mu^0), L(\lambda^0))^{(-1)}$	λ^0	μ^0	$\mathrm{Ext}^1_{G_1}(L(\mu^0), L(\lambda^0))^{(-1)}$
XII	XI	k	IX	III	$L(\omega_\alpha) \oplus k$
XII	VI	$L(\omega_\alpha)$	IX	I	$L(\omega_\alpha)$
XII	IV	$L(\omega_\alpha)$	VIII	VII	k
XII	II	$L(\omega_\beta) \oplus k$	VIII	VI	k
XI	X	k	VIII	IV	$L(\omega_\alpha) \oplus k$
XI	V	$L(\omega_\alpha)$	VIII	II	$L(\omega_\alpha)$
XI	III	k	VII	V	k
XI	I	$L(\omega_\beta) \oplus k$	VII	I	$L(\omega_\alpha)$
X	IX	k	VI	V	k
X	VII	$L(\omega_\alpha)$	VI	III	$L(\omega_\alpha)$
X	IV	$L(\omega_\alpha) \oplus k$	V	IV	k
X	II	k	IV	III	k
IX	VIII	k	III	II	k
IX	V	$L(\omega_\alpha) \oplus k$	II	I	k

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Added in proof: The argument in 3.1 for λ in the facet of type 13 is not correct since the integral weight μ claimed there does not exist unless $p = 2$. However, another argument can be given as follows. Let $\mu = (s_{3\alpha+2\beta})_* \cdot \lambda$. Similar to type 11 and 17, there is a (unique up to a scalar multiple) non-zero map $\psi : H^0(\lambda) \to H^0(\mu)$. One only needs to show that $[\mathrm{Im}\psi : L(\nu)] \neq 0$. Set $\mu' = (s_{3\alpha+2\beta})_* \cdot \lambda'$ and $\nu' \in A(\nu) \cap W_p \cdot \lambda'$. There is a (unique up to a scalar multiple) nonzero map $\phi : H^0(\lambda') \to H^0(\mu')$. It suffices to show $[\mathrm{Im}\phi : L(\nu')] \neq 0$ since $T_{\lambda'}^{\lambda}\mathrm{Im}\phi = \mathrm{Im}\psi$ by the exactness of $T_{\lambda'}^{\lambda}$. Choose an integral weight χ in the facet of type 12 (χ always exists) and apply the exact functor $T_{\lambda'}^{\chi}$ to the map ϕ. Thus $H^0(\chi) = \mathrm{Im}\phi = T_{\lambda'}^{\chi}H^0(\lambda')$. One can see that $[H^0(\chi) : T_{\lambda'}^{\chi}L(\nu')] = 1$. This shows that $[\mathrm{Im}\phi : L(\nu')] \neq 0$.