On vertex operator algebras as sl_2-modules

Chongying Dong, Zongzhu Lin, Geoffrey Mason

1. Introduction

Suppose that V is a vertex operator algebra (VOA). One of the axioms for a VOA is that V is a module for the Virasoro algebra Vir, and in particular there are operators $L(0), L(1), L(-1)$ in Vir which generate a copy of the Lie algebra $sl_2(\mathbb{C})$ represented on V. The purpose of present paper is to study VOAs as modules for sl_2, and more generally, to study twisted V-modules as modules for sl_2.

Our interest in this was stimulated by two sources: the calculations of McKay and Strauss [MS] on the decomposition of an initial segment of the Moonshine module into simple Monster modules, and the results of Li [Li] on VOAs. These paper are related as follows. Among other things, Li establishes that the operator $L(-1)$ is almost an injection. (See Section 2.4 for a precise statement; the results are probably well-known to experts.) Li's result applies in particular to the Moonshine module $V^\natural$ [FLM] with its usual decomposition into $L(0)$-eigenspaces (called "head characters" of the Monster $\mathbb{M}$ in [MS]), $V^\natural = V_0^\natural \oplus V_2^\natural \oplus \cdots$. It implies that there are $\mathbb{M}$-invariant injections $V_n^\natural \to V_{n+1}^\natural$ $(n \neq 0)$ which are induced by $L(-1)$. One can plainly see this effect in the decompositions of [MS].

In this paper we will show dually that $L(1)$ is almost surjective. When applied to $V^\natural$, for example, we obtain decompositions

$$V_{n+1}^\natural = L(-1)V_n^\natural \oplus \ker(L(1)|V_{n+1}^\natural)$$

of $\mathbb{M}$-modules valid for all n. These decompositions seem to be not generally known to group-theorists.

We also obtain analogues for twisted V-modules. Thus, for example, one expects decompositions similar to the above for the action of the Baby Monster on a certain $\mathbb{Z}_2$-twisted $V^\natural$-module, though the module in question has at this time not been constructed rigorously.

The properties of $L(-1)$ and $L(1)$ mentioned above have consequences for the action of $\langle L(0), L(1), L(-1)\rangle = sl_2$ on V and its twisted modules. In fact, any twisted V-module is a direct sum of indecomposable sl_2-modules $M(\mu)$ (Verma modules), $P(\mu)$ (projective modules), $M(0)'$ (the dual module of $M(0)$) and $\mathbb{C}$ (the trivial module) in the category $\mathcal{O}$ defined by Bernstein-Gelfand-Gelfand. These are explored in Sections 2 and 3.

In Section 4, we study vertex operator algebras with different Virasoro elements and we also construct a family of simple vertex operator algebras V with the properties that V_0 is not one-dimensional and $V_n \neq 0$ for some negative n.

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2. Injectivity of $L(-1)$

2.1. Vertex operator algebras. For the readers' convenience, we briefly recall the definition of a vertex operator algebra and certain properties. (See [B], [FLM] and [FHL] for details.)

A *vertex operator algebra* is a $\mathbb{Z}$-graded vector space:

$$V = \coprod_{n\in\mathbb{Z}} V_n; \quad \text{for} \quad v \in V_n, \quad n = \operatorname{wt} v; \text{ and}$$

$$\dim V_n < \infty \text{ for all } n \text{ and } V_n = 0 \text{ for } n << 0 \tag{1}$$

equipped with a linear map

$$\begin{aligned} & V \to (\operatorname{End} V)[[z, z^{-1}]] \\ & v \mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \qquad (v_n \in \operatorname{End} V) \end{aligned}$$

and with two distinguished vectors $\mathbf{1} \in V_0,\ \omega \in V_2$ satisfying the following conditions for $u, v \in V$:

$$\begin{aligned} & u_n v = 0 \qquad \text{for } n \text{ sufficiently large;} \\ & Y(\mathbf{1}, z) = 1; \\ & Y(v,z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z\to 0} Y(v,z)\mathbf{1} = v; \end{aligned}$$

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1)$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v, z_2) \tag{2}$$

(Jacobi identity) where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ is the algebraic formulation of δ-function at 1 and all binomial expressions, for instance, $(z_1 - z_2)^n$ $(n \in \mathbb{Z})$ are to be expanded in nonnegative integral powers of second variable z_2. The Jacobi identity is interpreted algebraically as follows: if this identity is applied to a single vector of V then the coefficient of each monomial in z_0, z_1, z_2 is a finite sum in V;

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

and

$$\begin{aligned}
&\operatorname{rank} V \in \mathbb{Q};\\
&L(0)v = nv = (\operatorname{wt} v)v \quad \text{for} \quad v \in V_n \ (n \in \mathbb{Z});\\
&\frac{d}{dz} Y(v, z) = Y(L(-1)v, z).
\end{aligned}$$

This completes the definition. We denote the vertex operator algebra just defined by $(V, Y, \mathbf{1}, \omega)$ (or briefly, by V). The series $Y(v, z)$ are called vertex operators.

REMARK. The finiteness condition (1) is essential for our main result, although many of the consequences do not need this condition.

2.2. TWISTED MODULES. An *automorphism* g of the vertex operator algebra V is a linear automorphism of V preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v, z)$ on V are compatible in the sense that

$$gY(v, z)g^{-1} = Y(gv, z)$$

for $v \in V$. Then $gV_n \subset V_n$ for $n \in \mathbb{Z}$. Furthermore, if g acts diagonally on each V_n with eigenvalues in the unit circle S^1 :

$$V_n = \coprod_{r\in[0,1)} V_n^r$$

where V_n^r is the eigenspace of g in V_n with eigenvalue $e^{2\pi i r}$, we call g *unitary*. In this case $V = \coprod_{r\in[0,1)} V^r$ with an obvious notation V^r. Note that any automorphism of V of finite order is always unitary.

We also have the notion of g-*twisted module* generalizing that defined in [FFR] and [D2] where g is of finite order. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and g be a unitary automorphism of V. A g-twisted *module* M for V is a $\mathbb{C}$-graded vector space:

$$M = \coprod_{n\in\mathbb{C}} M_n; \quad \text{for} \quad w \in M_n, \quad n = \mathrm{wt}\, w;$$

$$\begin{aligned}
&\dim M_n < \infty \qquad \text{for} \quad n \in \mathbb{C}, \text{ and,} \\
&M_{n+\lambda} = 0 \qquad \text{for each fixed } \lambda \in \mathbb{C} \ \text{ and } \quad n \in \mathbb{Z} \quad \text{sufficiently small;}
\end{aligned}$$

equipped with a linear map

$$\begin{aligned}
&V \to (\mathrm{End}\, M)\{z\} \\
&v \mapsto Y_M(v,z) = \sum_{n\in\mathbb{R}} v_n z^{-n-1} \quad (v_n \in \mathrm{End}\, M)
\end{aligned}$$

(where for any vector space W, we define $W\{z\}$ to be the vector space of W-valued formal series in z, with arbitrary real powers of z) satisfying the following conditions for $u, v \in V$, $w \in M$, and $r \in [0,1)$:

$$\begin{aligned}
&Y_M(v,z) = \sum_{n\in r+\mathbb{Z}} v_n z^{-n-1} \quad \text{ for } \ v \in V^r; \\
&v_l w = 0 \quad \text{for} \quad l \in \mathbb{R} \ \text{ sufficiently large;} \\
&Y_M(\mathbf{1}, z) = 1;
\end{aligned}$$

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) Y_M(v,z_2)Y_M(u,z_1)$$

$$= z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-r} \delta\left(\frac{z_1-z_0}{z_2}\right) Y_M(Y(u,z_0)v,z_2) \tag{3}$$

if $u \in V^r$;

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3-m)\delta_{m+n,0}(\mathrm{rank}\, V)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2};$$

$$L(0)w = nw = (\mathrm{wt}\, w)w \quad \text{for} \quad w \in M_n \ (n \in \mathbb{C}); \tag{4}$$

$$\frac{d}{dz} Y_M(v,z) = Y_M(L(-1)v, z). \tag{5}$$

This completes the definition. We denote this module by (M, Y_M) (or briefly by M).

The following are consequences of the definition

$$[L(-1), Y_M(v,z)] = Y_M(L(-1)v, z); \tag{6}$$

$$[L(0), Y_M(v,z)] = Y_M(L(0)v, z) + zY_M(L(-1)v, z). \tag{7}$$

REMARK. From (3), (4), (5) and (7) we find that if $v \in V$ is homogeneous, then v_n has weight $\mathrm{wt}\, v - n - 1$ as an operator, that is, $v_n : M_\lambda \to M_{\lambda + \mathrm{wt}\, v - n - 1}$. In particular, $L(n)$ has weight $-n$.

REMARK. If $g = 1$, we obtain the ordinary module notion for the vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ in the precise sense of [FLM]. In the rest of the paper, when we say twisted modules, we will include the ordinary modules.

Clearly, V is a V-module and it is called the *adjoint module*. A simple V-module M is defined to be a V-module in which 0 and M are the only submodules. A vertex operator algebra is simple if it is a simple module for itself.

2.3. CONTRAGRADIENT TWISTED MODULES. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and (M, Y_M) be a g-twisted V-module for a unitary automorphism g. The *contragradient module* M' is defined as follows: For each $n \in \mathbb{C}$, let M_n' be the dual vector space of M_n and $M' = \oplus_{n \in \mathbb{C}} M_n'$. Consider the natural pairing $\langle\ ,\ \rangle : M' \otimes M \to \mathbb{C}$ defined by $\langle w', w \rangle = w'(w)$ for any $w \in M$ and $w' \in M'$. Define the linear map $Y_{M'} : V \to (\mathrm{End}\, M')\{z\}$ such that

$$\langle Y_{M'}(v,z)w', w \rangle = \langle w', Y_M(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})w \rangle$$

for all $v \in V$, $w \in M$, and $w' \in M'$. Then $(M', Y_{M'})$ carries a g^{-1}-twisted V-module structure (cf. [FHL], [X]), and we have

$$\langle L(n)w', w \rangle = \langle w', L(-n)w \rangle$$

for all $n \in \mathbb{Z}$, $w \in M$, and $w' \in M'$. Since $L(n)$ is an operator on the graded vector space M of degree $-n$, the linear map $L(-n) : M_\lambda' \to M_{\lambda+n}'$ is exactly the dual linear map of the linear map $L(n) : M_{\lambda+n} \to M_\lambda$.

For an ordinary V-module M, it is proved in [FHL] that the natural identification $M \to (M')'$ is actually a V-module isomorphism. This result is also true for g-twisted modules with the same argument.

2.4. Let g be a unitary automorphism of V and M be a g-twisted V-module. An element $w \in M$ is called *vacuum-like* if

$$v_n(w) = 0 \quad \text{for all } v \in V \text{ and } 0 \le n \in \mathbb{R}. \tag{8}$$

In particular, all vacuum-like elements are in M_0 and killed by $L(-1)$ as one sees by considering the element $v = \omega$ and $n = 1$ and 0. The following lemma was proved in [Li] for vacuum-like elements in ordinary modules.

Lemma. *Let M be a twisted V-module. An element $w \in M$ is vacuum-like if and only if $L(-1)w = 0$.*

Proof:— The proof is basically the same as in [Li]. Suppose that $L(-1)w = 0$. For each $r \in [0,1)$, take any element $v \in V^r$ (if there are any) such that $Y(v,z)w \neq 0$. Then there exists $k \in r + \mathbb{Z}$ such that $v_k w \neq 0$ and $v_n w = 0$ for all $n \in \mathbb{R}$ such that $n > k$. Using the derivative property $\frac{d}{dz}Y(v,z) = Y(L(-1)v,z)$ and (6), we can get $[L(-1), v_{k+1}] = -(k+1)v_k$. This shows that $v_k w = 0$ if $k \neq -1$ since $L(-1)w = 0$ and $v_{k+1}w = 0$. In particular, $v_n w = 0$ for all $n \in \mathbb{R}$ and $n \geq 0$. $\square$

Remark. If $r \neq 0$, then $k + 1 \neq 0$ for all $k \in r + \mathbb{Z}$. The proof of the lemma shows that $Y_M(v,z)w = 0$ for all $v \in V^r$ with $r \neq 0$ and any vacuum-like element w.

Proposition. *Let M be a g-twisted module. Then*
(i) the linear map $L(-1) : M_\lambda \to M_{\lambda+1}$ is injective for all $\lambda \in \mathbb{C}$ unless $\lambda = 0$;
(ii) the linear map $L(1) : M_{\lambda+1} \to M_\lambda$ is surjective for all $\lambda \in \mathbb{C}$ unless $\lambda = 0$.

Proof:— (i) is exactly the above lemma since $\ker(L(-1)) \subseteq M_0$. To show (ii), we consider the contragradient twist module module M'. Applying (i) to M', the linear map $L(-1) : M'_\lambda \to M'_{\lambda+1}$ is injective. Thus its dual map $L(-1)' : (M'_{\lambda+1})' \to (M_\lambda)'$ is surjective. By the finite dimensionality of M_λ's, we have $M_\lambda = (M'_\lambda)'$. By the remark in 2.3, $L(1)$ is exactly $L(-1)'$. Thus (ii) follows. $\square$

Corollary. *Let V be a vertex operator algebra. Then $L(-1)$ gives an* $\mathrm{Aut}(V)$*-module embedding from V_n to V_{n+1} for $n \neq 0$.* $\square$

Remark. Recall from [FLM] that the moonshine module $V^\natural$ is a vertex operator algebra whose automorphism group is the Monster simple group $\mathbb{M}$. The corollary above asserts that $V^\natural_n$ is an $\mathbb{M}$-submodule of $V^\natural_{n+1}$ if $n \neq 0$. This is consistent with the computation [MS] for the first few characters of $V^\natural$. We shall obtain a refinement of this result in the next section.

3. sl_2-structure of modules for a vertex operator algebra

3.1. Note that $\{L(-1), L(0), L(1)\}$ generate a three dimensional Lie algebra isomorphic to $sl_2 = \{X, H, Y\}$ by setting $X = -L(1), H = -2L(0)$, and $Y = L(-1)$. Under this correspondence, the $L(0)$ weight λ corresponds to the H-weight -2λ for any $\lambda \in \mathbb{C}$. The simple root and the fundamental weight of sl_2 corresponds to the $L(0)$-weight -1 and $-\frac{1}{2}$ respectively. For consistency, we will use the $L(0)$ weight throughout this paper. For an sl_2-module M, M_λ means the $L(0)$ weight space corresponding to the weight λ. However, when we say highest weight we mean the highest H-weight with respect to the root system of sl_2.

For the Lie algebra sl_2, the category $\mathcal{O}$, as defined by Bernstein-Gelfand-

Gelfand in [BGG], consists of all sl_2-modules N such that a) N is finitely generated over the universal enveloping algebra $U(sl_2)$, b) N is a direct sum of H-eigenspaces and each eigenspace is finite dimensional, and c) $L(1)$ acts on every element of N nilpotently. For each $\lambda \in \mathbb{C}$, the Verma module $M(\lambda) = \mathbb{C}[L(-1)]v^+$ is such that $L(0)v^+ = \lambda v^+$ and $L(1)v^+ = 0$. $M(\lambda)$ is in $\mathcal{O}$ and any module generated by an element v with $L(0)v = \lambda v$ and $L(1)v = 0$ is a quotient of $M(\lambda)$. $M(\lambda)$ has a unique irreducible quotient $V(\lambda)$ and every simple module in $\mathcal{O}$ is isomorphic to one of the $V(\lambda)$'s. Let Z be the center of the enveloping algebra $U(sl_2)$. Z acts on $V(\lambda)$ via a central character $\chi_\lambda : Z \to \mathbb{C}$. $\chi_\lambda = \chi_\mu$ if and only if $\mu = -\lambda + 1$ by the Harish-Chandra Theorem [H1, 23.3] . If two simple modules $V(\lambda)$ and $V(\mu)$ appear in the same indecomposable module as composition factors, then $\chi_\lambda = \chi_\mu$ and $\lambda - \mu \in \mathbb{Z}$. For all $\lambda \in \mathbb{C}$, we have $V(\lambda) = M(\lambda)$ unless $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda \leq 0$ (corresponding to the dominant H-weights). For the exceptional λ, $M(\lambda)$ has exactly two composition factors $V(\lambda)$, which is finite dimensional, and $V(-\lambda+1)$. We will use $P(\lambda)$ to denote the projective cover of $V(\lambda)$. We have $P(\lambda) = M(\lambda)$ unless $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda > \frac{1}{2}$. For the exceptional λ, $P(\lambda)$ has three composition factors with $V(-\lambda+1)$ being in the middle and $V(\lambda)$ as the simple quotient and simple submodule. There is an exact contravariant functor (called duality functor [CI]) $D : \mathcal{O} \to \mathcal{O}$ such that $D(V(\lambda)) = V(\lambda)$ (such modules are called self-dual). $P(\lambda)$ is self-dual unless $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda \leq 0$. For more properties about the category $\mathcal{O}$, we refer the reader to [H2, I]. For the exceptional λ's, we have $P(\lambda) = M(\lambda) \neq V(\lambda)$.

Each twisted module M for a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ is automatically a module for the Lie algebra sl_2. Note that M satisfies the conditions b) and c) but fails to satisfy the finite generation condition a). For one can see that, apart from the trivial module, all other "highest weight" simple modules of the Virasoro algebra are not finitely generated over $U(sl_2)$ since the dimension of the weight space of weight n approaches infinity as n approaches infinity. However, for each given $\lambda \in \mathbb{C}$, the sl_2-submodule of M generated by M_λ is in the category $\mathcal{O}$. Although the module M is far from being finitely generated over $U(sl_2)$ and thus not in $\mathcal{O}$, M is a union of submodules which are in $\mathcal{O}$.

Lemma. *As sl_2-module, M is a direct sum of (infinitely many) indecomposable modules in the category $\mathcal{O}$.*

Proof:— Since the module M is not finitely generated, one cannot simply use the Krull-Schmidt theorem. We shall see that the proof for the category $\mathcal{O}$ still works. Let Z be the center of the universal enveloping algebra $U(sl_2)$. M can be regarded as a Z-module. For each central character $\chi : Z \to \mathbb{C}$, we define

$$M^\chi = \{w \in M \mid \text{ for any } z \in Z, (z - \chi(z))^k w = 0 \text{ for some positive integer } k\}.$$

Then M^χ is an sl_2-submodule of M and $M = \oplus_\chi M^\chi$. The latter holds since M_λ is Z-invariant and finite-dimensional for each $\lambda \in \mathbb{C}$. Next we will show that each M^χ is in the category $\mathcal{O}$, so that we can apply the Krull-Schmidt theorem to refine the direct sum decomposition of M with each component being in $\mathcal{O}$. Note that each module in $\mathcal{O}$ has a composition series.

For a given central character χ, suppose λ and $-\lambda+1$ both define χ. Let $T(\lambda)$ be the sl_2-submodule of M^χ generated by $M^\chi_\lambda + M^\chi_{-\lambda+1}$, which is finite-dimensional. Thus $T(\lambda)$ is in $\mathcal{O}$. Note that M^χ is a union of modules in $\mathcal{O}$. Let N be any sl_2-submodule of M^χ such that N is in $\mathcal{O}$. Then $N+T(\lambda)$ is a submodule of M^χ and also in $\mathcal{O}$. Therefore, any composition factor of $N+T(\lambda)$ has to be isomorphic to either $V(\lambda)$ or $V(-\lambda+1)$. However, $(N+T(\lambda))/T(\lambda)$ has no weights λ and $-\lambda+1$, and thus has no composition factor isomorphic to either $V(\lambda)$ or $V(-\lambda+1)$. Hence $(N+T(\lambda))/T(\lambda)$ has to be 0. This shows that $N \subseteq T(\lambda)$, whence $M^\chi = T(\lambda)$ is in $\mathcal{O}$. □

3.2. We now discuss the sl_2-indecomposable components of M. Since $L(-1)$ is injective and $L(1)$ is surjective at M_λ with $\lambda \neq 0$, then any sl_2-direct summand of M will also have the same property. In particular, this is the case for each indecomposable component N of M.

PROPOSITION. *Let N be an indecomposable sl_2-module in the category $\mathcal{O}$ such that $L(-1)$ is injective at each non-zero $L(0)$-weight and $L(1)$ is onto to N_λ for each $\lambda \neq 0$. Then the following hold:*

(i) If N has a weight not in $\frac{1}{2}\mathbb{Z}$ or all weights of N are in $\frac{1}{2}\mathbb{Z}$ and $\frac{1}{2}$ is the smallest one, then N is isomorphic to a simple Verma module $M(\mu)$ with $\mu \notin \frac{1}{2}\mathbb{Z}$ or $\mu = \frac{1}{2}$;

(ii) If N has a weight $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda < 0$, then N is isomorphic to the projective cover $P(\mu)$ of a simple Verma module $M(\mu)$ with $\mu \in \frac{1}{2}\mathbb{Z}$ and $\mu > \frac{1}{2}$;

(iii) If all the weights of N are in $\frac{1}{2}\mathbb{Z}$ and all weights are at least 1, then N is isomorphic to a simple Verma module $M(\lambda)$ for some $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda \geq 1$, which is not projective;

(iv) If all the weights of N are in $\frac{1}{2}\mathbb{Z}$ with 0 being the smallest weight and $L(1)N_1 \neq 0$, then N is isomorphic to the projective module $P(1)$ if there exists $v \in N_1$ such that $L(-1)L(1)v \neq 0$ or isomorphic to the dual of the Verma module $M(0)$ otherwise;

(v) If all the weights of N are in $\frac{1}{2}\mathbb{Z}$ and the smallest weight is 0 and $L(1)N_1 = 0$, then N is isomorphic to either the Verma module $M(0)$ or the trivial module.

PROOF:— Since N is in the category $\mathcal{O}$ and indecomposable, there exists λ such that $N_{\lambda-n} = 0$ for $n \in \mathbb{C}$ unless $n \in \mathbb{Z}$. In fact N has to belong to a central character χ_λ for some λ. Furthermore, we can choose λ such that $N_\lambda \neq 0$ and $N_{\lambda-n} = 0$ for all $n \in \mathbb{Z}$ and $n > 0$. We discuss the possibilities of λ:

(i) $\lambda \notin \frac{1}{2}\mathbb{N}$ or $\lambda = \frac{1}{2}$. In this case, the Verma modules $M(\lambda)$ and $M(-\lambda+1)$ are the only two irreducible composition factors of N. Both of them are projective modules as well by BGG-reciprocity. Thus any non-zero quotient map from N to a simple module splits and gives a direct sum decomposition. Thus N is isomorphic to $M(\lambda)$ or $M(-\lambda+1)$.

(ii) $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda < 0$. The finite-dimensional simple module $V(\lambda)$ and the simple Verma module $M(-\lambda+1)$ are the only two simple modules in the category $\mathcal{O}$ with the central character χ_λ. They are therefore the only two possible composition factors of N. Note that $M(-\lambda+1)_0 = 0$ and the operator $L(-1)$ kills no non-zero vector in $V(\lambda)_0$. This shows that $L(-1)$ acts on N injectively by assumption. Therefore, the sl_2-submodule $T(\lambda)$ generated by N_λ has to be isomorphic to a direct sum of copies of the Verma module $M(\lambda)$ with quotient being a direct sum of copies of the simple Verma module $M(-\lambda+1)$. By properties of the Verma module $M(\lambda)$, we have $L(1)T(\lambda)_{-\lambda+1} = 0$. However, $-\lambda \neq 0$, and $L(1)$ maps $N_{-\lambda+1}$ onto $N_{-\lambda}$. This shows that there exists an element $v \in N_{-\lambda+1}$ such that $L(1)v \neq 0$ and v generates a submodule isomorphic to the projective cover $P(-\lambda+1)$ of the simple Verma module $M(-\lambda+1)$. This projective module is self-dual in the sense of [CI] and therefore is injective. Thus N has a direct summand isomorphic to $P(-\lambda+1)$. This shows that N is isomorphic to $P(-\lambda+1)$.

(iii) $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda \geq 1$. There are only two irreducible modules $V(\lambda)$ and $V(-\lambda+1)$ in the category $\mathcal{O}$ with central character χ_λ. Since $L(-1)$ acts on N injectively, the sl_2-submodule generated by N_λ is a direct sum of the Verma module $M(\lambda)$ and the quotient has no composition factor with the central character χ_λ. Thus N has to be isomorphic to a direct sum of Verma modules and therefore is isomorphic to the simple Verma module $M(\lambda)$.

(iv) $\lambda = 0$. In this case the only possible composition factors of N are the trivial module $V(0)$ and the simple Verma module $M(1)$. For any subquotient Q of N as sl_2-module and $v \in Q_\mu$ with $\mu > 1$, we have $L(1)v \neq 0$. This means that $L(1): Q_{\mu+1} \to Q_\mu$ has to be injective for $\mu > 0$. In particular, $L(1): N_{\mu+1} \to N_\mu$ is a linear isomorphism for $\mu > 0$. By comparing dimensions and using the injectivity of $L(-1)$ on N_μ with $\mu > 0$, we conclude that $L(-1): N_\mu \to N_{\mu+1}$ is an isomorphism for all $\mu > 0$. If $L(1)N_1 \neq 0$, we fix a $v \in N_1$ such that $L(1)v \neq 0$. Let T be the sl_2-submodule of N generated by v. We claim that T is an injective module in the category $\mathcal{O}$. Therefore, we have $N = T$. Indeed, if $L(-1)L(1)v \neq 0$, then the argument is the same as in case (ii) and T is isomorphic to the projective cover $P(1)$ of the simple Verma module $M(1)$. Then $P(1)$ is self-dual and injective. If $L(-1)L(1)v = 0$, $L(1)v$ generates the one-dimensional trivial submodule and its quotient by T gives the simple Verma module $M(1)$. Thus T is isomorphic to the dual of the Verma module $M(0)$, which is projective. Thus T is injective and N is isomorphic to T.

(v) If the condition is the same as in (iv) except that we have $L(1)N_1 = 0$, then we set $T_0 = \ker(L(-1)|N_0)$. Take T_1 such that $N_0 = T_0 \oplus T_1$ gives a vector space decomposition. Set $K_0 = L(-1)T_1 = L(-1)N_0 \subset N_1$ and a vector space decomposition $N_1 = K_0 \oplus K_1$ for a vector subspace K_1 of N_1. T_0 is already an sl_2-submodule (trivial). Let $\langle T_1 \rangle$ and $\langle K_1 \rangle$ be the sl_2-submodules generated by T_1 and K_1 respectively. Then $L(-1)$ acts on both $\langle T_1 \rangle$ and $\langle K_1 \rangle$ injectively. Since any possible H-weights of $N/(\langle T_1 \rangle + \langle K_1 \rangle + T_0)$ are larger than 1, this quotient has to be zero since it belongs to the central character χ_0. Thus we have $N = \langle T_1 \rangle + \langle K_1 \rangle + T_0$. We now show that this is a direct sum. It is enough to show that it is a direct sum on each weight space. It is a direct sum of the weight spaces N_0 and N_1 by definition since all the generators are maximal vectors (i.e., killed by $L(1)$). The direct sum decomposition follows from the fact that $L(-1): N_\mu \to N_{\mu+1}$ is an linear isomorphism. Since $N_0 \neq 0$, we have either $T_0 \neq 0$ or $\langle T_1 \rangle \neq 0$. Now the indecomposibility of N shows that $K_1 = 0$ and N is isomorphic to either $\langle T_1 \rangle$, which is a single copy of the Verma module $M(0)$, or T_0, which is the one-dimensional trivial module. □

Corollary. *As a module for sl_2, a twisted V-module M is a direct sum of $M(\mu)$ ($\mu \notin \frac{1}{2}\mathbb{Z}$ or $\mu \geq 0$), $P(\mu)$ ($\mu \in \frac{1}{2}\mathbb{Z}$ and $\mu \geq 0$), $M(0)^*$, and the trivial module $\mathbb{C}$.* □

3.3. An element w of a twisted V-module M is called *quasi-primary* if $L(1)w = 0$. Let $QP(M)$ be the set of all quasi-primary vectors in M. Then $\mathbb{C}[L(-1)]QP(M)$ is an sl_2-submodule of M. In [Li], a vertex operator algebra V is called *QP-generated* if $V = \mathbb{C}[L(-1)]QP(V)$. He proved that any QP-generated simple vertex operator algebra V has no negative weights, i.e., $V_n = 0$ for all negative integers n. We will adopt his terminology and call a twisted V-module M QP-generated if $\mathbb{C}[L(-1)]QP(M) = M$.

Theorem. *For a vertex operator algebra V, a twisted V-module M is QP-generated if and only if $M_n = 0$ for all $n \in \frac{1}{2}\mathbb{Z}$ with $n < 0$ and $L(1)M_1 = 0$.*

Proof:— Note that M is QP-generated if and only if each sl_2-indecomposable component is QP-generated. Suppose that M is QP-generated. Let us fix a direct sum decomposition of M into indecomposable sl_2-modules. Assume that there exists $0 > n \in \frac{1}{2}\mathbb{Z}$ such that $M_n \neq 0$. We can take n to be the minimal such number. Then there exists an indecomposable component N of M such that $N_n \neq 0$ and $N_m = 0$ if $m < n$. Note that the weights of N are in $n + \mathbb{Z}$. By the proof of Proposition 3.2(ii), N is isomorphic to $P(-n+1)$. Thus the sl_2-submodule $T(n)$ generated by N_n is a Verma module which contains all quasi-primary vectors in N. Thus N cannot be generated by the quasi-primary vectors in N as sl_2-module, and we must have $M_n \neq 0$ for all $n \in \frac{1}{2}\mathbb{Z}$ with $n < 0$. If $L(1)M_1 \neq 0$, there exists an indecomposable sl_2-component N such that $L(1)N_1 \neq 0$. By what we just proved above, 0 is the smallest weight of N. Now by Proposition 3.3(iv) N is isomorphic to either the projective module $P(1)$ or the dual of the Verma module $M(0)$. In both cases, N is not generated by the quasi-primary vectors.

Conversely, if $M_n = 0$ for all $n \in \frac{1}{2}\mathbb{Z}$ with $n < 0$, then for any sl_2-indecomposable component N of M, case (ii) of Proposition 3.2 does not occur. The condition $L(1)M_1 = 0$ also rules out case (iv) of Proposition 3.2. Thus any component N has to correspond to one of the cases (i), (iii) or (v). In each of these cases, N is generated by the quasi-primary vectors. Therefore M is QP-generated. □

Corollary. *A vertex operator algebra V is QP-generated if and only if V has* *no negative weights and* $L(1)V_1 = 0$*.*

PROOF:— Note that a vertex operator algebra has only integer weights. □

3.4. PROPOSITION. *Let* V *be a vertex operator algebra and* M *a twisted* V*-module. Assume that* $n \in \mathbb{C}$ *satisfies one of the following:*

(i) $n \notin \frac{1}{2}\mathbb{Z}$,

(ii) $n < 1$,

(iii) $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$ *and* $L(1)$ *in injective on* M_{-n+1}.

Then there is a decomposition:

$$M_n = \ker(L(1)|M_n) \oplus L(-1)M_{n-1}.$$

PROOF:— We decompose M into a direct sum of indecomposable sl_2-modules. It is enough to show that the result holds for each indecomposable component. Let N be such an indecomposable component of M. If $N = M(\lambda)$ with $\lambda \notin \frac{1}{2}\mathbb{Z}$ or $\lambda \in \frac{1}{2}\mathbb{Z}$ and $\lambda > 0$, then both $L(-1)$ and $L(1)$ are isomorphisms between $N_{\lambda+i}$ and $N_{\lambda+i+1}$ for all $i \in \mathbb{Z}$ and $i \geq 0$. The decomposition of N_λ is also clear. If $N = P(k+1)$ for any $k \in \frac{1}{2}\mathbb{Z}$ and $k \geq 0$, then one only needs to take care of the decompositions of N_{-k} and N_{k+1}. The decomposition of N_{-k} is clear. But the decomposition at $k+1 = n$ fails since $\ker(L(1)|N_{k+1}) \cap L(-1)(N_k) \neq 0$. However, $n = k+1$ is excluded by assumption (iii) since $N_{-k} \subseteq \ker(L(1)|M_{-k})$. If $N = M(0)$ or $M(0)^*$, then similar arguments give the decomposition of N_n for $n = 0$ and $n > 1$. However, the decomposition of N_1 fails in these two cases. If $N = \mathbb{C}$, the result is trivial. Now the result follows from Corollary 3.2. □

4. Change of Virasoro elements

4.1. WEIGHT ONE SUBSPACE AND VIRASORO ELEMENTS. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. Our goal in this section is to change the Virasoro element ω to ω' such that $(V, Y, \mathbf{1}, \omega')$ is again a vertex operator algebra. Of course, the rank and the gradation will change accordingly. The reader can refer to [Lian] for a discussion of how to construct a Virasoro element for a pre-vertex operator algebra. Let $u \in V_1$ be a *lowest weight vector* for the Virasoro algebra, that is, $L(n)u = 0$ for all $n > 0$. We also assume that

$$u_n u = 0 \text{ if } n > 1 \text{ or } n = 0, \quad u_1 u = \lambda \mathbf{1} \tag{9}$$

for some $\lambda \in \mathbb{C}^*$. Then from the Jacobi identity (2), the component operators u_n of $Y(u, z)$ satisfy relations

$$[L(m), u_n] = -n u_{n+m} \tag{10}$$

$$[u_m, u_n] = m\lambda\delta_{m+n,0}. \tag{11}$$

Set

$$\omega' = \omega + u_{-2}\mathbf{1} \in V_2, \quad L'(z) = Y(\omega', z) = \sum_{m\in\mathbb{Z}} L'(m)z^{-m-2}.$$

Then $L'(m) = L(m) - (m+1)u_m$ for $m \in \mathbb{Z}$. In particular, $L'(-1) = L(-1)$ and $Y(L'(-1)v, z) = \frac{d}{dz}Y(v, z)$ for all $v \in V$. It is straightforward to verify from (10) and (11) that the operators $L'(m)$ satisfy the Virasoro algebra relation with central charge rank $V - 12\lambda$:

$$[L'(m), L'(n)] = (m-n)L'(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V - 12\lambda)$$

for $m, n \in \mathbb{Z}$.

In general, $(V, Y, \mathbf{1}, \omega')$ is not a vertex operator algebra. In fact, V may not be a direct sum of eigenspaces of $L'(0)$; and if it is, the finiteness condition and boundedness condition (1) may not hold. Thus we need an additional assumption on u : u_0 is *semi-simple* in the sense that V is a direct sum of eigenspaces of u_0. Since $L(0)$ and u_0 are commutative by (10), each V_n is a direct sum of u_0-eigenspaces. For convenience we set

$$V_{m,n} = \{v \in V_m | u_0 v = nv\}$$

for $m, n \in \mathbb{C}$. Then

$$V'_m = \{v \in V | L'(0)v = mv\} = \coprod_{s,t\in\mathbb{C},\, s-t=m} V_{s,t}.$$

We have

Proposition. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and let $v \in V_1$ be a lowest weight vector such that (9) holds. Suppose that u_0 is semi-simple with only integer eigenvalues on V, $\dim V'_m < \infty$ for all $m \in \mathbb{Z}$, and $V_n = 0$ if $n \in \mathbb{Z}$ is sufficiently small. Then $(V, Y, \mathbf{1}, \omega')$ is a vertex operator algebra of rank equal to rank $V - 12\lambda$.* □

4.2. Examples of simple vertex operator algebra with nonzero subspace of negative weights. It was originally believed in [D1] and [Li] that any simple vertex operator algebra has the properties: a) the subspace of negative weights is zero, b) the subspace V_0 is one-dimensional. In fact it is proved in [Li] that a simple vertex operator algebra V which is QP-generated in the sense of Section 3.3 has properties a) and b). However, one should notice from Theorem 3.3 that the condition that V is simple is not essential, although the QP condition is essential at least for the property a). Unfortunately, we found counter-examples when we were trying to prove these properties for simple vertex operator algebras in general. This section is devoted to these examples.

We are working in the setting of [FLM] and we assume that the reader is familiar with the vertex operator algebra V_L constructed from an even lattice L. For our purpose, we shall take L to be the root lattice of $sl(2,\mathbb{C})$. That is, $L=\mathbb{Z}\alpha$ where α is the unique positive root of $sl(2,\mathbb{C})$ with $\langle\alpha,\alpha\rangle=2$. Then

$$V_L=\mathbb{C}[\alpha(-1),\alpha(-2),...]\otimes\mathbb{C}[L]$$

where $\alpha(-n)$ for $n>0$ are commuting variables and $\mathbb{C}[L]$ is the group algebra with basis e^β for $\beta\in L$, $\mathbf{1}=1\otimes e^0$ and $\omega=\frac{1}{4}\alpha(-1)^2$. Moreover, $(V_L,Y,\mathbf{1},\omega)$ is a simple vertex operator algebra. For $\lambda\in\mathbb{C}$, set $u=\lambda\alpha(-1)\in(V_L)_1$. Then u has the property given in Proposition 4.1. Recall from [FLM] that $u_0=\lambda\alpha(0)$ and $\alpha(0)$ acts on V_L by

$$\alpha(0)(v\otimes e^\beta)=\langle\alpha,\beta\rangle v\otimes e^\beta$$

for $v\in\mathbb{C}[\alpha(-1),...]$ and $\beta\in L$. Also recall that if $v=\alpha(-1)^{i_1}\cdots\alpha(-k)^{i_k}$ for $i_j\geq 0$ and $k\geq 1$ then

$$L(0)(v\otimes e^\beta)=(i_1+2i_2+\cdots ki_k+\frac{1}{2}\langle\beta,\beta\rangle)v\otimes e^\beta.$$

We find that

$$L'(0)(v\otimes e^\beta)=(i_1+2i_2+\cdots ki_k+\frac{1}{2}\langle\beta,\beta\rangle-\lambda\langle\alpha,\beta\rangle)v\otimes e^\beta. \tag{12}$$

Proposition. *The structure $(V_L,Y,\mathbf{1},\omega')$ is a vertex operator algebra if and only if $\lambda\in\frac{1}{2}\mathbb{Z}$.*

Proof:— Clearly, $\lambda\in\frac{1}{2}\mathbb{Z}$ if $(V,Y,\mathbf{1},\omega')$ is a vertex operator algebra . In fact, in (12) taking $\beta=\alpha$ gives the result. Now assume that $\lambda\in\frac{1}{2}\mathbb{Z}$. Again by (12) all the eigenvalues of $L'(0)$ are integers. Thus it is sufficient to prove that each eigenspace $(V_L)'_n$ of $L'(0)$ is finite-dimensional and $(V_L)'_m=0$ if m is sufficiently small. Set

$$\mathbb{C}[L]_m=\text{span}\,\{e^\beta|\beta\in L,L'(0)e^\beta=me^\beta\}=\{e^{n\alpha}|n^2-2n\lambda=m,n\in\mathbb{Z}\}$$

It is easy to see that $\mathbb{C}[L]_m$ is finite-dimensional for $m\in\mathbb{Z}$ and $\mathbb{C}[L]_n=0$ if n is sufficiently small. The result now follows immediately. $\square$

Finally we have

Corollary. *Let $1\leq\lambda\in\frac{1}{2}\mathbb{Z}$. Then $(V_L,Y,\mathbf{1},\omega')$ is a simple vertex operator algebra such that* $\dim(V_L)'_0$ *is greater than 1 and* $(V_L)'_{1-2\lambda}\neq 0$. $\square$

Bibliography

[BGG] I.N. Bernstein, I.M. Gelfand, S.I Gelfand, A category of $\mathfrak{g}$-modules, Functional Anal. Appl. **10** (1976), 1–8.

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA **83** (1986), 3068-3071.

[CI] D. Collingwood and R. Irving, A decomposition theorem for certain self-dual modules in the category $\mathcal{O}$, Duke Math. J., **58** (1989), 89–101.

[D1] C. Dong, Vertex algebras associated with even lattices, J. Algebra **161** (1993), 245–265.

[D2] C. Dong, Twisted modules for vertex algebras associated with even lattices, J. Algebra, to appear.

[FFR] A. J. Feingold, I. B. Frenkel and J. F. X. Ries, Spinor construction of vertex operator algebras, triality and $E_8^{(1)}$, Contemporary Math. **121,** 1991.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Memoirs American Math. Soc.* **494**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., Vol. 134, Academic Press, 1988.

[H1] J. E. Humphreys, Introduction to Lie algebras and representation theory, Springer-Verlag, 1972.

[H2] J. E. Humphreys, Finite dimensional and infinite dimensional modules for semisimple Lie algebras, p. 1–64. Lie theory and their applications, Queen's Paper in: Pure Appl. Math. No. 48, Kingston, Ont. 1978.

[I] R. Irving, Projective modules in the category $\mathcal{O}_S$: self-duality. Trans. Amer. Math. Soc., **291** (1985), 701–732.

[Li] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, J. Pure Appl. Algebra, to appear.

[Lian] B. H. Lian, On the classification of simple vertex operator algebras, *Comm. Math. Phys.,* to appear.

[MS] J. McKay and H. Strauss, The q-series of monstrous moonshine & the decomposition of the head characters, Comm. in Algebra **18** (1990), 253-278.

[X] X. Xu, Twisted modules of colored lattice vertex operator superalgebras, preprint.

Department of Mathematics
University of California
Santa Cruz, CA 95064