Divisibility of the Steinberg Modules over a Levi Factor

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Theorem. *Let G be a reductive affine algebraic group defined and split over the prime field in a algebraically closed field k of characteristic p and L a split Levi subgroup of G. Let $St_r(G)$ and $St_r(L)$ be the r-th Steinberg modules for G and L respectively. Then $St_r(G)|_L \cong St_r(L) \otimes V$ for some L-module V.*

Choose a Borel subgroup B and a parabolic subgroup P such that P contains the Borel subgroup opposite to B and $P = U(P)L$. Here $U(P)$ is the unipotant radical of P. We will denote the r-th Frobenius kernel of a subgroup of G attaching the subscript r. By [Jantzen's book, p.225], we have

$$St_r(G)_{G_rB} \cong \mathrm{Ind}_B^{G_rB}((p^r-1)\rho);$$

$$St_r(L)_{L_rL^-} \cong \mathrm{Ind}_{L^-}^{L_rL^-}((p^r-1)\rho_L).$$

Here $L^- = L \cap B$ is a Borel subgroup of L and ρ_L is the half sum of all positive roots of L.

Note that we have the following subgroup schemes (they are the thickenings of certain subgroups in the language of CPS [Sect. 4, Math. Z. 182 (1983),447–471]).

$$L^- = P_rL \cap B, \;\; G_rB = P_rL^-B.$$

Applying Theorem 4.1 of CPS [loc.cit.], we have

$$\mathrm{Ind}_B^{G_rB}((p^r-1)\rho)|_{P_rL} \cong \mathrm{Ind}_{L^-}^{P_rL^-}((p^r-1)\rho) \cong \mathrm{Ind}_{L_rL^-}^{P_rL^-}(\mathrm{Ind}_{L^-}^{L_rL^-}((p^r-1)\rho)).$$

Note that $\mathrm{Ind}_{L^-}^{L_rL^-}((p^r-1)\rho) \cong_{L_rL^-} St_r(L) \otimes (p^r-1)(\rho-\rho_L)$ by tensor identity and the latter has a P-module structure. Thus we can apply the tensor identity to get

$$\mathrm{Ind}_B^{G_rB}((p^r-1)\rho)|_{P_rL^-} \cong St_r(L) \otimes ((p^r-1)(\rho-\rho_L) \otimes \mathrm{Ind}_{L_rL^-}^{P_rL^-}k$$

However $\mathrm{Ind}_{L_rL^-}^{P_rL^-}k \cong k[U(P)_r]$ as P_rL^--modules. The latter is a P-module via conjugation action since $U(P)_r$ is a normal subgroup scheme of P. Thus we have the L^--module isomorphism

$$St_r(G) \cong_{L^-} St_r(L) \otimes ((p^r-1)(\rho-\rho_L) \otimes k[U(P)_r].$$

Since both sides have L-module structures and $\mathrm{Ind}_{L^-}^{L}k = k$ (L/L^- connected and projective), we can apply the induction functor $\mathrm{Ind}_{L^-}^{L}$ and then tensor identity on both sides to get the isomorphism of L-modules

$$St_r(G) \cong_L St_r(L) \otimes ((p^r-1)(\rho-\rho_L) \otimes k[U(P)_r]$$

which gives the actual divisibility.

Remark: One can use the induction characterized in "Induced representations of Hopf algebras: Apllication to quantum groups at roots of 1" by Lin and the analogous Mackey decomposition theorem proved in "A Mackey decomposition theorem and cohomology for quantum groups at roots of 1" by Lin to prove the analogous divisibility theorem for quantum groups at roots of 1, where the coordinate algebra $k[U(P)_r]$ should be replaced by an induced module analogous to $\mathrm{Ind}_{L_rL^-}^{P_rL^-}k$.

References

[1] Donkin, Stephen, On tilting modules for algebraic groups, *Math. Z*

[2] Alperin, J. and Mason, G., Partial Steinberg modules for finite groups of Lie type. *Bull. London Math. Soc.*