Spring 2007 Math 510 HW11 Solutions

Section 11.8

18. Let γ be a trail joining vertices x and y in a general graph. Prove that the edges of γ can be partitioned so that one part of the partition determines and chain joining x and y and the other parts determine cycles.

Proof. We use induction on the length of the trail γ. If the length of the trail γ is 1, then γ is already a chain joining x and y. Assume that the length of γ is n and for any trail of length less than n, its edges can be partitioned into a chain and and cycles. If all vertices of γ are distinct except possibly $x = y$. Then γ is already a chain joining x and y. Otherwise, let us write the trail γ

$$\gamma : \; x = y_0 \text{ ------ } y_1 \text{ ------ } y_3 \text{ ------ } \cdots \text{ ------ } y_n = y$$

and there is a $0 \leq j < k \leq n$ such that $y_j = y_k$. Then we consider the trails

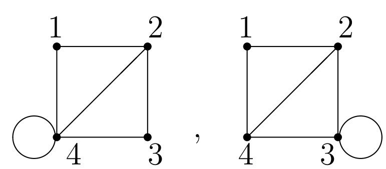
$$\gamma_1 : \; y_0 \text{ ------ } \cdots \text{ ------ } y_j \text{ ------ } y_{k+1} \text{ ------ } \cdots \text{ ------ } y_n \; ,$$

$$\gamma_2 : \; y_j \text{ ------ } y_{j+1} \text{ ------ } \cdots \text{ ------ } y_k \; .$$

Both γ_1 and γ_2 have length less than n. By induction hypothesis, the edges of γ_1 can be partitioned into a chain joining x and y and cycles. for the same reason, the edges of γ_2 can be partitioned into a chain joining y_j and y_k (thus a cycle) and other cycles. Note that γ_1 and γ_2 don't have any edges in common and each edge of γ is in one of γ_1 and γ_2. The partitions of the edges of γ_1 and γ_2 determines a partition of edges of γ. One of the part is a chain joining x and y and all other parts are cycles.

26. Determine the adjacency matrix of the first and second graphs in Fig. 11.39.

Solution. We first label the vertices of the graph as follows:



The adjacency matrix of a general graph is the matrix $A = (a_{ij})$ such that a_{ij} is the number of edges joining the vertices i and j. Note that matrix A is symmetric, i.e., $a_{ij} = a_{ji}$. The adjacency matrices for the two graphs are:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

30. Which complete graphs K_n have closed Eulerian trails? open Eulerian trails?

Solution. . A connected general graph has a closed Eulerian trail if and only if all vertices have even degree. Since each vertex in a complete graph K_n has degree $n-1$. Thus a complete graph K_n has a closed Eulerian trail if and only n is odd.

A connected general graph has an open Eulerian trail if and only if exactly two vertices has odd degree while all other vertices have even degree. Since all vertices of K_n has degree $n-1$. The only situation with exactly two vertices of odd degree happens when $n=2$.

41. Let $n \geq 3$ be an integer. Let G_n be the graph whose vertices are the $n!$ permutations of $\{1, 2, \ldots, n\}$, wherein two permutations are joined by an edge if and only if one can be obtained from the other by interchanging of two numbers (an arbitrary transposition). Deduce from from the results section 4.1 that G_n has a Hamilton cycle.

Solution. . Students need to read section 4.1 dealing how to list all permutations such that each permutation is included exactly once. The argument in 4.1 gives a list such that in the list, and each permutation is obtained from its predecessor by exchanging the two numbers. This can be done inductively for n.

For $n=3$, the list here gives a Hamilton cycle:

$$(1,2,3),(1,3,2),(3,1,2),(3,2,1),(2,3,1),(2,1,3),(1,2,3).$$

For $n>3$, suppose that a Hamilton cycle has been found for the graph G_{n-1} has been found, say,

$$y_1, y_2, \ldots, y_{(n-1)!}, y_1.$$

For each permutation $y_i = (i_1, i_2, \ldots, i_{n-1})$ of $\{1, 2, \ldots, n-1\}$, inserting n to one of the n spaces (between two numbers or at the beginning/end of the sequences) will produce a permutation of the set $\{1, 2, 3, \ldots, n-1, n\}$. They will be all distinct and with $n \cdot (n-1)!$ of them produces.

For $y_1 = (1, 2, 3, \ldots, n-1)$, we produce a chain in G_n as follows:

$$(1,2,\ldots,n-1,n),(1,2,\ldots,n,n-1),\ldots,(1,2,n,\ldots,n-1),(1,n,2,\ldots,n-1),(n,1,2,\ldots,n-1)=(n,y_1).$$

For Since y_2 is obtainable from y_1, then (n, y_2) can be obtained from (n, y_1) which will yield the following by moving n from left the right one at a time till (y_2, n).

In general for i odd, (y_i, n) can obtained from (y_{i-1}, n), from (y_i, n) moving n to the left one place at a time to produce a chain ending with (n, y_i) which passes to (n, y_{i+1}) and then moving to the right one place at time to get a chain ending at (y_{i+1}, n). Since $n \geq 3$, $(n-1)!$ even. The last chain so obtained is from $(n, y_{(n-1)!})$ to $(y_{(n-1)}, n)$. Since y_1 can be obtained from $y_{(n-1)!}$ be exchanging two numbers, the permutation (y_1, n) can be obtained from $(y_{(n-1)!}, n)$ be exchanging two numbers. Thus we get sequence of $n!$ permutations of $\{1, 2, \ldots, n\}$ each can be obtained from it predecessor by exchanging two numbers, i.e., there is an edge joining to permutations with one follows the other.

51(b). Find a knight's tour on the board of 6-by-6.

Solution.

1	18	3	30	35	20
4	29	36	19	6	31
17	2	5	32	21	34
28	11	26	23	14	7
25	16	9	12	33	22
10	27	24	15	8	13

For a 5-by-5 board:

1	14	9	20	3
24	19	2	15	10
13	8	25	4	21
18	23	6	11	16
7	12	17	22	5

62. Prove that if a tree has a vertex of degree p, then it has at least p pendent vertices.

Solution. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of a tree of order n. Since a tree is connected, we have $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$. Let k be the number of pendent vertices. Then $d_i = 1$ for $i = n-k+1, n-k+2, \ldots n-k+k$ and $d_i \geq 2$ for all $i = 1, \ldots, n-k$. Then we have

$$\begin{aligned} 2(n-1) &= d_1 + d_2 + \cdots + d_{n-k} + d_{n-k+1} + \cdots + d_n = d_1 + (d_2 + \cdots + d_{n-k}) + k \\ &\geq d_1 + 2(n-k-1) + k = d_1 + 2(n-1) - k. \end{aligned}$$

From $2(n-1) \geq d_1 + 2(n-1) - k$ we have $0 \geq d_1 - k$ or $k \geq d_1$. Now for any vertex v with $p = \deg(v)$ we have $k \geq d_1 \geq p$, i.e., there are at least p pendent vertices.