A super duality and Kazhdan-Lusztig polynomials

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A brief history on Lie superalgebras $\mathfrak{gl}(m|n)$:

- Kac (1977): classification of finite-dimensional simple Lie superalgebras over $\mathbb{C}$.
- Kac (1978): characters for the (f.-d.) Kac modules; irreducible characters of the **typical** modules.

  (typical := $\exists$ unique object in the corresponding block; ∞-many simple objects in any *atypical* block.)
- Numerous partial results (Bernstein-Leites, Sergeev, Berele-Regev, Penkov-Serganova, van der Jeugt-Hughes-King-Thierry-Mieg, Cheng, W, Zhang, etc.)
- Serganova (1996): first solution to the problem of f.-d. irreducible characters of $\mathfrak{gl}(m|n)$ via Kazhdan-Lusztig-Vogan theory.

A new approach of Brundan (JAMS, 2003):

- Brundan defined on the Fock space

$$\mathcal{E}^{m|n} := \Lambda^m \mathbb{V} \otimes \Lambda^n \mathbb{V}^*$$

  (or rather its topological completion $\widehat{\mathcal{E}}^{m|n}$) the monomial, canonical, and dual canonical bases $\{K_\lambda\}$, $\{U_\lambda\}$ and $\{L_\lambda\}$, and the associated KL polynomials.

  Here $\mathbb{V}$ is the natural $U_q(\mathfrak{sl}(\infty))$-module and $\mathbb{V}^*$ its dual.

- The category $\mathcal{O}^+_{m|n}$ of f.-d. modules of $\mathfrak{gl}(m|n)$ is governed by the $q = 1$ specialization of KL polynomials:

  Kac $K(\lambda) \leftrightarrow K_\lambda$, tilting $U(\lambda) \leftrightarrow U_\lambda$, irreducible $L(\lambda) \leftrightarrow L_\lambda$.

- Explicit formulas for KL polynomials, etc.

- **Question:**

  any connections between representation theories of super $\mathfrak{gl}(m|n)$ and reductive $\mathfrak{gl}(m+n)$???

- 'Expected' answer from 'Super experts': no, for example,

○ any (atypical) block of $\mathfrak{gl}(m|n)$ contains ∞-many simples

○ a block of $\mathfrak{gl}(m+n)$ contains finitely many simples, as governed by the Weyl group S_{m+n}

- **Goal:** Super duality — a direct connection between representation theories of Lie algebras and Lie superalgebras

  Expected: this is a general 'desuperlizing' principle

- **Consequence:**

  Brundan-Serganova KL polynomials = usual parabolic KL;

  the usual parabolic KL polynomials compute the characters of finite-dimensional irreducible $\mathfrak{gl}(m|n)$-modules

- **Key:** pass to $n = \infty$!

Lie Superalgebra ABC:

- The general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$
- The odd part $:= \mathsf{g}_{-1} \oplus \mathfrak{g}_{+1}$

  The even part $\mathfrak{g}_0 := \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$
- (standard) Cartan subalgebra $\mathfrak{h}$, Borel subalgebra $\mathfrak{b}$, parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 + \mathfrak{b}$.
- Index set $I(m|n) = \{-m, \cdots, -1, 1, \cdots, n\}$.

Lie Superalgebra ABC (cont'd):

- $\delta_i :=$ the i-th fundamental weight.
- Symmetric bilinear form on $\mathfrak{h}^*$:

$$(\delta_i|\delta_j) = -\mathrm{sgn}(i)\delta_{ij}, \qquad i,j \in I(m|n).$$

- The sets of (dominant $=+$ or $++$ below) integral weights:

$$\begin{aligned} X_{m|n} &:= \{\lambda = \sum_{i\in I(m|n)} \lambda_i\delta_i \mid \lambda_i \in \mathbb{Z}\} \\ X^+_{m|n} &:= \{\lambda \in X_{m|n} \mid \lambda_{-m} \geq \cdots \geq \lambda_{-1}, \lambda_1 \geq \cdots \geq \lambda_n\} \\ X^{++}_{m|n} &:= \{\lambda \in X^+_{m|n} \mid \lambda_n \geq 0\} \end{aligned}$$

Also, $X^+_{m|\infty} :=$ set of dominant integral λ with $\lambda_i = 0$, $i \gg 0$.

Lie Superalgebra ABC (cont'd):

- Sets of 'ρ-shifted weights':

$$\begin{aligned}
\mathbb{Z}^{m|n} &:= \{f : I(m|n) \to \mathbb{Z}\},\\
\mathbb{Z}^{m|n}_{+} &:= \{f \in \mathbb{Z}^{m|n} \mid f(-m) > \cdots f(-1); f(1) < \cdots < f(n)\},\\
\mathbb{Z}^{m|n}_{++} &:= \{f \in \mathbb{Z}^{m|n}_{+} \mid f(n) \leq n\}.
\end{aligned}$$

- Bijections: $X^{+}_{m|n} \xrightarrow{\cong} \mathbb{Z}^{m|n}_{+}, \qquad X^{++}_{m|n} \xrightarrow{\cong} \mathbb{Z}^{m|n}_{++}$

sends $\lambda \mapsto f_\lambda$, where

$$f_\lambda(i) = (\lambda + \rho|\delta_i), \qquad i \in I(m|n),$$

and $\rho := m\delta_{-m} + \cdots + 2\delta_{-2} + \delta_{-1} - \delta_1 - 2\delta_2 - \cdots - n\delta_n$.

E.g. $f_\emptyset = (m, \cdots, 2, 1|1, 2, \cdots, n)$.

Categories:

- $\mathcal{O}^{+}_{m|n}$= Category of fin. generated integral-weight modules M of $\mathfrak{g}$ which are locally finite over $\mathfrak{p}$, semisimple over $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$;
- $\mathcal{O}^{++}_{m|n}$ = the full subcategory of $\mathcal{O}^{+}_{m|n}$ which consists of modules whose composition factors are of the form $L_n(\lambda), \lambda \in X^{++}_{m|n}$;
- Grothendieck group $G(\mathcal{O}^{+}_{m|n})$ and its completion $\widehat{G}(\mathcal{O}^{+}_{m|n})$

Objects in $\mathcal{O}^+_{m|n}$:

- For $\lambda \in X^+_{m|n}$, the *Kac module*

$$K(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L^{\mathfrak{g}_0}(\lambda)$$

- the irreducible module $L(\lambda)$
- the tilting module $U(\lambda)$ characterized by
  1. indecomposable;
  2. $U(\lambda)$ has a Kac flag with $K(\lambda)$ at the bottom;
  3. $\text{Ext}^1(K(\mu), U(\lambda)) = 0$, for all $\mu \in X^+_{m|n}$

  (existence of $U(\lambda)$ due to Brundan for a finite n and CWZ for $n = \infty$)

Super Bruhat Ordering:

- Let $\mathbf{1}_{i,j} \in \mathbb{Z}^{m|n}$ be the function which equals 1 at i and j, and 0 elsewhere, for given $i < 0 < j \in I(m|n)$.
- Write $f \downarrow g$ (for $f, g \in \mathbb{Z}^{m|n}_+$) if

  $g = (f - r\mathbf{1}_{i,j})^+$ for some $i < 0 < j$, $r > 0$, such that $f(i) = f(j)$.

  Here $f^+ :=$ the dominant conjugate of f by $S_m \times S_n$.
- *Super Bruhat ordering* on $\mathbb{Z}^{m|n}_+$ (and on $X^+_{m|n}$):

  $f \succ g \Leftrightarrow \exists h_1, \ldots, h_r \in \mathbb{Z}^{m|n}_+$ such that $f = h_1 \downarrow \cdots \downarrow h_r = g$.

Blocks and Central Characters:

- $f, g \in \mathbb{Z}_+^{m|n}$ have same central characters $\chi_f = \chi_g$

  $\Leftrightarrow$

  f, g are linked by the super Bruhat ordering,

  i.e. $\exists h \in \mathbb{Z}_+^{m|n}$ such that $f \succ h, g \succ h$.

- $\chi_f = \chi_g \Leftrightarrow \mathsf{wt}^\epsilon(f) = \mathsf{wt}^\epsilon(g)$

  where $\mathsf{wt}^\epsilon : \mathbb{Z}^{m|n} \to \sum_{a \in \mathbb{Z}} \mathbb{Z}\epsilon_a$:

$$\mathsf{wt}^\epsilon(f) := \sum_{i \in I(m|n)} -\mathsf{sgn}(i)\epsilon_{f(i)}$$

  and $\epsilon_a, a \in \mathbb{Z}$, denotes the fundamental weights for $\mathcal{U}(sl(\infty))$.

Fock space:

- The quantum group $\mathcal{U} = U_q(\mathfrak{sl}(\infty))$;

  Chevalley generators E_a, F_a, $K_{a,a+1}, a \in \mathbb{Z}$
- $\mathbb{V}$ is the natural $\mathcal{U}$-module, with basis $\{v_a\}_{a\in\mathbb{Z}}$;

  $\mathbb{V}^*$ is the dual module to $\mathbb{V}$, with dual basis $\{w_a\}_{a\in\mathbb{Z}}$
- Fock space (i.e. space of tensors of q-wedges)

$$\mathcal{E}^{m|n} := \Lambda^m\mathbb{V} \otimes \Lambda^n\mathbb{V}^*$$

- *Monomial basis* for $\mathcal{E}^{m|n}$:

$$K_f := v_{f(-m)} \wedge \cdots \wedge v_{f(-1)} \otimes v_{f(1)} \wedge \cdots \wedge v_{f(n)}, \quad f \in \mathbb{Z}_+^{m|n}$$

(Dual) Canonical bases:

Existence and uniqueness of a suitable bar involution on $\widehat{\mathcal{E}}^{m|n}$ implies

Theorem 0.1 (Brundan for finite n). *Let* $n \in \mathbb{N} \cup \infty$. $\exists$ *unique* canonical basis $\{U_f\}_{f\in\mathbb{Z}_+^{m|n}}$ *and* dual canonical $\{L_f\}$ *for* $\widehat{\mathcal{E}}^{m|n}$ *s.t.*

1. $\overline{U}_f = U_f$ *and* $\overline{L}_f = L_f$;
2. $U_f \in K_f + \widehat{\sum}_{g\in\mathbb{Z}_+^{m|n};g\prec f} q\mathbb{Z}[q]K_g$;
3. $L_f \in K_f + \widehat{\sum}_{g\in\mathbb{Z}_+^{m|n};g\prec f} q^{-1}\mathbb{Z}[q^{-1}]K_g$

Kazhdan-Lusztig polynomials:

- (Brundan) KL polynomials $u_{g,f}(q) \in q\mathbb{Z}[q]$ and $\ell_{g,f}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$ associated to $f, g \in \mathbb{Z}_+^{m|n}$:

$$\begin{aligned} U_f &= K_f + \sum_{\substack{g \in \mathbb{Z}_+^{m|n} \\ g \prec f}} u_{g,f}(q) K_g, \\ L_f &= K_f + \sum_{\substack{g \in \mathbb{Z}_+^{m|n} \\ g \prec f}} \ell_{g,f}(q) K_g. \end{aligned}$$

Jantzen Translation Functors:

- V := the natural module of $\mathfrak{gl}(m|n)$
- Translation functors $E_a, F_a : \mathcal{O}^+_{m|n} \longrightarrow \mathcal{O}^+_{m|n}$ $(a \in \mathbb{Z})$:

sends M in a block $\mathcal{O}^+(\gamma)$ $(\gamma \in \sum_{a\in\mathbb{Z}} \mathbb{Z}\epsilon_a)$ to

$$\begin{aligned} F_aM &:= \mathrm{pr}_{\gamma-(\epsilon_a-\epsilon_{a+1})}(M \otimes V), \\ E_aM &:= \mathrm{pr}_{\gamma+(\epsilon_a-\epsilon_{a+1})}(M \otimes V^*). \end{aligned}$$

Additional care is needed when $n = \infty$.

KL conjecture for Super $\mathfrak{gl}(m|n)$:

Theorem 0.2 (Brundan for finite n; CWZ for $n = \infty$).

1. *$\exists$ a unique tilting module $U_n(\lambda)$ for each $\lambda \in X^+_{m|n}$;*
2. *Sending Chevalley generators E_a, F_a to translation functors E_a, F_a defines a $\mathcal{U}_{q=1}$-module structure on $G(\mathcal{O}^+_{m|n})$;*
3. $j : \widehat{G}(\mathcal{O}^+_{m|n}) \to \widehat{\mathcal{E}}^{m|n}, \qquad [K(\lambda)] \mapsto K_{f_\lambda}(1)$
   is an isomorphism of $\mathcal{U}_{q=1}$-modules;
4. $j : [L(\lambda)] \mapsto L_{f_\lambda}(1)$,
   $j : [U(\lambda)] \mapsto U_{f_\lambda}(1)$.

Main tool for passing to $n = \infty$:

- Truncation maps $\mathfrak{Tr}_{n',n}$ $(n < n' \leq \infty)$ between Fock spaces:

$$\widehat{\mathcal{E}}^{m|\infty} \longrightarrow \cdots \longrightarrow \widehat{\mathcal{E}}^{m|n+1}_{+} \xrightarrow{\mathfrak{Tr}_{n+1,n}} \widehat{\mathcal{E}}^{m|n}_{+} \longrightarrow \cdots$$

  compatible with the monomial, (dual) canonical bases for varied n.

- Truncation functors $\mathfrak{tr}_{n',n}$ $(n < n' \leq \infty)$ of categories:

$$\mathcal{O}^{+}_{m+\infty} \longrightarrow \cdots \longrightarrow \mathcal{O}^{++}_{m|n+1} \xrightarrow{\mathfrak{tr}_{n+1,n}} \mathcal{O}^{++}_{m|n} \longrightarrow \cdots$$

  send tilting, Kac, irreducible modules to the corresponding ones (or to zero ‘when not well-defined’).

Reductive Lie Algebra $\mathfrak{gl}(m+n)$ ABC:

- $\mathfrak{gl}(m+n) \supset$ parabolic $\mathfrak{p}_c := \mathfrak{g}_0 + \mathfrak{b}_c \supset$ Borel $\mathfrak{b}_c \supset$ Cartan $\mathfrak{h}_c$, where $\mathfrak{g}_0 = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$
- $\mathcal{O}^+_{m+n}$: category of finitely generated integral-weight $\mathfrak{gl}(m+n)$-modules,

  locally finite over $\mathfrak{p}_c$, semisimple over $\mathfrak{g}_0$

  Objects: generalized Verma $\mathcal{K}(\lambda)$; the irreducible $\mathcal{L}(\lambda)$; tilting module $\mathcal{U}(\lambda)$, $\lambda \in X^+_{m+n}$
- $\delta'_i :=$ i-th fundamental weight, $i \in I(m|n)$
- $\rho_c := m\delta'_{-m} + \cdots + 2\delta'_{-2} + \delta'_{-1} + 0\delta'_1 - \delta'_2 - \cdots - (n-1)\delta'_n$
- $X^+_{m+n} \xrightarrow{\sim} \mathbb{Z}^{m+n}_+$ (ρ_c-shifted): counterparts of $X^+_{m|n} \xrightarrow{\sim} \mathbb{Z}^{m|n}_+$

Reformulation of KL for $\mathcal{O}^+_{m+n}$ (following Brundan):

- introduce the Fock space

$$\mathcal{E}^{m+n} := \Lambda^m \mathbb{V} \otimes \Lambda^n \mathbb{V}$$

  and its canonical basis $\{\mathcal{U}_f\}$, dual canonical basis $\{\mathcal{L}_f\}$

- define KL polynomials $\mathfrak{u}_{g,f}(q) \in \mathbb{Z}[q], \mathfrak{l}_{g,f}(q) \in \mathbb{Z}[q^{-1}]$:

$$\mathcal{U}_f = \sum_{g \in \mathbb{Z}_+^{m+n}} \mathfrak{u}_{g,f}(q) \mathcal{K}_g, \qquad \mathcal{L}_f = \sum_{g \in \mathbb{Z}_+^{m+n}} \mathfrak{l}_{g,f}(q) \mathcal{K}_g$$

- Category $\mathcal{O}^+_{m+n}$ is governed by KL polynomials at $q = 1$: g.Verma $\mathcal{K}(\lambda) \leftrightarrow \mathcal{K}_\lambda$, tilting $\mathcal{U}(\lambda) \leftrightarrow \mathcal{U}_\lambda$, irreducible $\mathcal{L}(\lambda) \leftrightarrow \mathcal{L}_\lambda$
- (CWZ) Explicit formulas for $\mathfrak{u}_{g,f}(q)$ and $\mathfrak{l}_{g,f}(q)$
- related works: Beilinson-Bernstein, Brylinski-Kashiwara, Collingwood-Irving, Enright-Shelton, Lascoux-Schützenberger, Leclerc-Miyachi, etc.

Passing to $n = \infty$:

- Same techniques are applicable to the Fock spaces and categories on the reductive side

  E.g. truncations maps and truncation functors etc

- Explicit formulas for KL polynomials in $\widehat{\mathcal{E}}^{m+\infty}$

Isomorphism of Fock spaces:

$\Lambda^\infty \mathbb{V} =$ space of semi-infinite q-wedges of Kashiwara-Miwa-Stern;

$\Lambda^\infty \mathbb{V}^*$ is similarly defined

Proposition 0.3.

- *As* $\mathcal{U} = U_q(\mathfrak{sl}(\infty))$*-modules,*

$$\Lambda^\infty \mathbb{V} \cong \Lambda^\infty \mathbb{V}^*$$

  Indeed, both are isomorphic to the basic representation of $\mathcal{U}$;

- *This isomorphism identifies basis vectors* $v_{\lambda_1} \wedge v_{\lambda_2 - 1} \wedge v_{\lambda_3 - 2} \wedge \cdots$ *with*

$$w_{1-\lambda'_1} \wedge w_{2-\lambda'_2} \wedge w_{3-\lambda'_3} \wedge \cdots .$$

  Here λ' *is the conjugate partition of the partition* $\lambda = (\lambda_1, \lambda_2, \cdots)$.

Isomorphism of Fock spaces (cont'd):

Theorem 0.4. • *As* $\mathcal{U}$*-modules,* $\natural : \widehat{\mathcal{E}}^{m+\infty} \cong \widehat{\mathcal{E}}^{m|\infty}$;

- $\exists$ *bijections* $\mathbb{Z}_+^{m+\infty} \overset{\natural}{\leftrightarrow} \mathbb{Z}_+^{m|\infty}$ *and* $X^+_{m+\infty} \overset{\natural}{\leftrightarrow} X^+_{m|\infty}$;
- $\natural$ *is compatible with bar involutions, (super) Bruhat orderings, and thus canonical/dual canonical bases:*

  $\natural(\mathcal{K}_f) = K_{f^\natural}, \quad \natural(\mathcal{L}_f) = L_{f^\natural}, \quad \natural(\mathcal{U}_f) = U_{f^\natural}$

Super Duality:

Theorem 0.5.

- *$\exists$ isomorphism of the Grothendieck groups $\sharp : G(\mathcal{O}^+_{m+\infty}) \to G(\mathcal{O}^+_{m|\infty})$ which sends the tilting, generalized Verma, irreducible modules to the tilting, Kac, irreducible modules respectively;*
- *$\sharp$ is a ring isomorphism.*

Identifications of super and reductive KL polynomials:

- Given $\lambda, \mu \in X^{++}_{m|n}$, define $\lambda_\infty := (\lambda, 0, 0, \cdots) \in X^{+}_{m|\infty}$;
  write $\lambda^\natural_\infty = ((\lambda^\natural_\infty)^{<0}|(\lambda^\natural_\infty)^{>0}) \in X^{+}_{m+\infty}$
- Assuming lengths of partitions $(\mu^\natural_\infty)^{>0}$ and $(\lambda^\natural_\infty)^{>0}$ are $\leq N$, *truncate* $\lambda^\natural_\infty, \mu^\natural_\infty$ to $\lambda^{\natural,(N)}_\infty, \mu^{\natural,(N)}_\infty \in X^{++}_{m+N}$
- $\ell_{\mu,\lambda}(q) = \mathfrak{l}_{\mu^{\natural,(N)}_\infty, \lambda^{\natural,(N)}_\infty}(q)$; and,

$$
\begin{aligned}
u_{\mu,\lambda}(q) &= u_{\mu_\infty,\lambda_\infty}(q) && \text{by stability of Brundan KL,} \\
&= \mathfrak{u}_{\mu^\natural_\infty,\lambda^\natural_\infty}(q) && \text{by super duality,} \\
&= \mathfrak{u}_{\mu^{\natural,(N)}_\infty,\lambda^{\natural,(N)}_\infty}(q) && \text{by stability of KL;}
\end{aligned}
$$

- N depends on λ and μ and could be arbitrarily large for a given n

An Analogue:

- Compare with the involution ω on the ring of symmetric functions determined by

$$\omega(e_i) = h_i$$

- It is no longer an isomorphism when restricted to finitely many variables.