Representations of double affine Hecke algebras.

Notations

Irreducible root system

α_i, ω_i (simple roots, fundamental weights)

$X, Y, X^\vee, Y^\vee$ (weight lattice, root lattice, duals)

W, $\hat{W} = W \ltimes Y$ (Weyl group, Affine Weyl gp)

Δ, $\Delta_\pm$, $\Delta^\vee$, $\Delta^\vee_\pm$ (roots, positive roots, etc.)

$\theta \in \Delta_+$, $\Delta \times \mathbb{Z}$, $\Delta^\vee \times \mathbb{Z}$ (max root, affine roots)

$\alpha_0 = (-\theta, 1) \in \Delta \times \mathbb{Z}$

$T = k^\times \otimes_{\mathbb{Z}} Y^\vee$, $T^\vee = k^\times \otimes_{\mathbb{Z}} Y$ (k field)

$$k\hat{W}^{(ex)}_{(x)} = \underset{\substack{\Psi \\ s_\alpha}}{kW} \ltimes \underset{\substack{\Psi \\ x_\lambda}}{k[T]}, \quad k\hat{W}^{(ex)}_{(y)} = kW \ltimes \underset{\substack{\Psi \\ y_{\lambda^\vee}}}{k[T^\vee]}$$

$\underset{\substack{\Psi \\ t_w}}{H_W}$, $H_{\hat{W}^{(ex)}}$ Iwahori-Hecke algebras of W, $\hat{W}^{(ex)}_{(y)}$

t_w, $t_i = t_{s_i}$, $s_i = s_{\alpha_i}$, $s_0 = x_\theta s_\theta$

Def $\tau, \zeta \in k^\times$

$$H^{(ex)}_{\tau,\zeta} = k[T^\vee] \otimes \underbrace{H_W \otimes k[T]}_{H_{\hat{W}(ex)}}$$

relations: $\begin{cases} (t_i - \zeta)(t_i+1) = 0 \\ t_i p - {}^{s_i}p \cdot t_i = (\zeta - 1)\,(p - {}^{s_i}p)/(1 - y_{-\alpha_i}) \\ \text{etc} \end{cases}$

$p \in k[T^\vee]$, $y_{\alpha_0^\vee} = \tau\, y_{-\theta^\vee}$

$\widehat{W}^{ex}_{(x)} \curvearrowright k[T^\vee]$ by ${}^{x_\mu w} y_{\lambda^\vee} = y_{w\lambda^\vee}\, \tau^{-\mu \cdot w\lambda^\vee}$

where $\mu \cdot \lambda^\vee \in \frac{1}{m}\mathbb{Z}$ pairing $X/X^\vee$

Simple Modules ($k = \mathbb{C}$)

$G^\vee$ simple group, connected, simply connected, $\mathfrak{g}^\vee = \mathrm{Lie}\, G^\vee$

$T^\vee \subset G^\vee$ max torus.

$F = \mathbb{C}((\varpi))$

$\mathbb{C}^\times \curvearrowright G^\vee(F)$ st. $z \cdot g(\varpi) = g(z\varpi)$

$\mathcal{B} = \{ \text{Iwahori subalgebra} \subset \mathfrak{g}^\vee \otimes F \}$

$\mathcal{N} = \{ x \in \mathfrak{g}^\vee \otimes F \;;\; (\mathrm{ad}\, x)^\infty = 0 \}$

$\widetilde{\mathcal{N}} = \{ (b, x) \in \mathcal{B} \times (\mathfrak{g}^\vee \otimes F) \;;\; x \in b \cap \mathcal{N} \}$

$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ 2nd projection

$\check{T} \times \mathbb{C}^\times \circlearrowright \mathcal{B}$, $\check{\mathfrak{g}} \otimes F$, $\tilde{\mathcal{P}}$, $\mathcal{P}$, π equivariant
$\begin{array}{cc} \Psi & \Psi \\ s & \tau \end{array}$

Assume τ <u>not</u> root of 1.

$\Rightarrow \mathcal{B}^{(s,\tau)}$, $\tilde{\mathcal{P}}^{(s,\tau)}$ smooth schemes locally of f.t.

$\begin{array}{ccccc} A = \check{T} \times & \mathbb{C}^\times \times & \mathbb{C}^\times & \circlearrowright \tilde{\mathcal{P}} \\ \Psi & \Psi & \Psi & \Psi \\ a & s & \tau & \zeta \end{array}$, $\begin{cases} \tilde{\mathcal{P}}^a \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \\ \mathcal{P}^a \text{ affine cone of finite type} \end{cases}$

$\tilde{\mathcal{P}}^a \xrightarrow{\pi} \mathcal{P}^a$ proper on each connected component of $\tilde{\mathcal{P}}^a$.

$\mathcal{Z}^a = \tilde{\mathcal{P}}^a \times_{\pi^a} \tilde{\mathcal{P}}^a$ — b, b' in relative position $\leq w$

$K^A(\mathcal{Z}^a) = \varinjlim_{w \in \hat{W}} \underbrace{K^A(\mathcal{Z}^a_{\leq w})}_{\text{pro-object}}$ module / $R^A = K^A(\bullet)$

<u>Theorem</u> (V) (τ, ζ not roots of 1) $\exists$ ring homomorphism $H_{\tau\zeta} \to K^A(\mathcal{Z}^a) \otimes_{R^A} \mathbb{C}$ with dense image.

<u>Def</u> A $H_{\tau\zeta}$-module is integrable if R^A acts locally finitely.

Corollary 1 : classification of integrable simple modules.

Sketch : IM involution $H_{\tau,\zeta} \xrightarrow{\sim} H_{\tau^{-1},\zeta}$

$\Rightarrow$ assume that $\zeta^m \neq \tau^k \quad \forall\, m,k \in \mathbb{Z}_{>0}$

$\Rightarrow$ $\exists$ finite number of $\underbrace{\widetilde{G^\vee(F)}^{(s,\tau)}}_{\text{connected}}$ -orbits in $\mathcal{N}^a$

$\Rightarrow$ simple modules are classified by equivariant local systems.

Corollary 2 : Geometric approach to finite dimensional modules.

$\forall x \in \mathcal{N}^a, \quad H_{\tau,\zeta} \circlearrowright H_*(\mathcal{B}_x^{(s,\tau)}, \mathbb{C})$

$a = (s,\tau,\zeta)$

↑ affine Springer fiber

$x \in \mathcal{N}' \subset \mathfrak{g} \otimes F$ semisimple regular $\Rightarrow$ $\dim \mathcal{B}_x < \infty$

$x \in \mathcal{N}$ " " " + anisotropic

$\left(\text{i.e. } \nexists \text{ morphism } Z_{G^\vee(F)}(x) \to F^\times \text{ which is } \neq 1 \right)$

$\Rightarrow$ $\mathcal{B}_x$ finite dim. variety.

$\exists (a,x)$ ty $x \in \mathfrak{g}^a$, x semisimple regular anisotropic

$\Leftrightarrow$ x semisimple regular anisotropic homogeneous

$\left(\text{ie. } G^\vee(\bar{F})\text{-conjugated to } \mathfrak{g}^\vee \otimes \mathcal{O}^r,\ r \in \mathbb{Q} \right)$

Those x are classified

$\Rightarrow$ large familly of f.d. representations.

<u>Exemple</u> $k, m \in \mathbb{Z}_{>0}$, $(m,k) = 1$, $\zeta^m = \tau^k$

$$a = (\tau^{1/2h} \otimes 2\rho, \tau, \zeta)$$

h = Coxeter number

$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$

$$x = \sum_{(\alpha,\ell) \in \Pi_k} e_\alpha \otimes \mathcal{O}^\ell$$

$$\Pi_k = \{ (\alpha, \ell) \in \Delta \times \mathbb{Z} \ ;\ (ht\,\alpha, \ell) = (b,a) \text{ or } (b-h, 1+a) \}$$

$k = ah + b$, $a, b \in \mathbb{Z}_{\geq 0}$, $b < h$

$\Rightarrow$ yields a simple module of dimension k^r

$r = \operatorname{rank} G^\vee$

<u>Relation with the Cyclic quiver</u>

type A_{n-1}

Systems local are trivial

Assume: $k, m \in \mathbb{Z}_{>0}$, $(m,k) = 1$, $\zeta^m = \tau^k$

$\operatorname{Spec} s \subseteq \tau^{\frac{1}{m}\mathbb{Z}}$

$x \in \mathcal{N}^a \subset \mathrm{End}(F^{\oplus n})$

$$F^{\oplus n} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} V_i \,, \quad V_i = \{v \,;\, s v \in \zeta^i \tau^{\mathbb{Z}} v\}$$

F-subspace

$V_{i+m} = V_i$

$x(V_i) \subset V_{i+1}$

$$\mathbb{C}^{\oplus n} = F^{\oplus n} / (\varpi - 1) F^{\oplus n} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} V_i / (\varpi - 1) V_i$$

$\rightsquigarrow$ simple integrable modules classified by nilpotent representations of cyclic quiver + multiplicity formulas Standard/Simple given by IC of orbits.

Similar to quantum affine Schur algebras.

<u>Equivalence of Categories</u> (type A_{m-1}, $k = \mathbb{C}$)

$\mathcal{O}$ = f.g. integrable $H_{\tau,\zeta}$ - modules, i.e. $k[T^\vee]$ locally finite

$\hat{W} \circlearrowright T^\vee$, $\quad x_\alpha w \cdot (z \otimes \beta) = (z \otimes w\beta)(\tau \otimes \alpha) \in \mathbb{C}^\times \otimes Y = T^\vee$

$\forall \ell \in T^\vee$, ${}^\ell\mathcal{O} \subset \mathcal{O}$ full subcategory of modules supported set theoretically on $\hat{W}\ell$.

$$\mathcal{O} = \bigoplus_{\ell \in T^\vee / \hat{W}} {}^\ell\mathcal{O}$$

Step 1 : degeneration

$\hbar \in \mathbb{C}$, $\mathcal{H}'_\hbar$ = degenerate DAHA

$$= \mathbb{C}[\mathrm{Lie}\, T^\vee] \otimes \mathbb{C}\widehat{W}$$

$$s_i p - ({}^{s_i}p)\, s_i = \hbar \; \frac{p - {}^{s_i}p}{\alpha_i^\vee}$$

$$p \in \mathbb{C}[\mathrm{Lie}\, T^\vee] = \mathrm{Sym}(X^\vee \otimes \mathbb{C})$$

$\widehat{W} \curvearrowright \mathbb{C}[\mathrm{Lie}\, T^\vee]$ s.t. ${}^{x_\alpha w}\lambda^\vee = w\lambda^\vee - \underbrace{\alpha \cdot w\lambda^\vee}_{\text{pairing}}$

$\rightsquigarrow$ Categories $\mathcal{O}'$, ${}^\lambda\mathcal{O}'$ st. $\mathcal{O}' = \bigoplus_{\lambda \in \mathrm{Lie}T^\vee/\widehat{W}} {}^\lambda\mathcal{O}'$.

Lemma $\ell = e^{u\lambda}$, $\zeta = e^{u\hbar}$, $\tau = e^{u}$, $u \in \mathbb{C} \setminus$ countable set

$\widehat{W}_\ell$ generated by reflexions $\Longrightarrow$ ${}^\ell\mathcal{O} \simeq {}^\lambda\mathcal{O}'$.

Step 2 : monodromy of KZ

S_ζ := Quantized affine Schur algebra = $\mathrm{End}_{H_{\widehat{W}}} Q$

$Q = \bigoplus_{\substack{J \subseteq \{\alpha_i\} \\ (i \neq 0)}} H_{\widehat{W}} \otimes_{H_J} \mathbb{C}[T^\vee]$ where $H_J = \langle t_j, \mathbb{C}[T^\vee] \rangle_{\alpha_j \in J}$

Theorem (Varagnolo + V)

$\hbar \notin \frac{1}{2}\mathbb{Z} \Longrightarrow \mathcal{O}'_\hbar \simeq$ finite dimensional S_ζ - modules

$\zeta = e^{\hbar}$

Sketch of proof :

$T_0 = T \setminus$ root hyperplanes.

Recall $\quad H'_h = \underbrace{\mathbb{C}[\mathrm{Lie}\, T^\vee] \otimes \mathbb{C}W}_{\substack{\text{degenerate affine} \\ \text{Hecke algebra}}} \otimes \mathbb{C}[T]$ $\quad \mathbb{C}W \otimes \mathbb{C}[T] = \mathbb{C}\widehat{W}$

$\Rightarrow$ acts on $\quad \mathbb{C}[T] \simeq H'_h \underset{\mathbb{C}[\mathrm{Lie}T^\vee]\otimes\mathbb{C}W}{\otimes} \mathbb{C}$

$\Rightarrow$ Faithful representation on $\mathbb{C}[T]$ s.t. $\check{\omega}_i \in \mathbb{C}[\mathrm{Lie}T^\vee]$ acts via Dunkl operator

$$D_i = \partial_{\omega_i^\vee} - h \sum_{\alpha \in \Delta_+} \frac{\alpha \cdot \check{\omega}_i}{1 - x_{-\alpha}} (1 - s_\alpha) + h\, \mathrm{ht}(\check{\omega}_i)$$

Derivation $x_\lambda \mapsto (\lambda \cdot \check{\omega}_i)\, x_\lambda$

$H'_{h,0} = \mathbb{C}[T_0] \underset{\mathbb{C}[T]}{\otimes} H'_h$ localized algebra.

$H'_{h,0} \simeq \mathrm{Diff}[T_0] \rtimes \mathbb{C}W$

Modules in $\mathcal{O}'$ are locally finite $/\mathbb{C}[\mathrm{Lie}T^\vee]$ and f.g.

$\Rightarrow$ yield vector bundles on T_0 with W-invariant integrable connections

$\overset{\text{Monodromy}}{\Longrightarrow}$ $\pi_1(T_0/W)$ - finite dimensional modules

8)

$\pi_1(T_0/W) =$ Braid group of $\hat{W}$

Action factorizes through $H_{\hat{W},e^h}$ - modules.

$\mathcal{M} : \mathcal{O}'_h \longrightarrow H_{\hat{W},e^h}$ - f.d. modules exact functor.

Recall: $\mathcal{A}, \mathcal{B}$ Abelian Categories, $P \in \mathrm{Ob}\,\mathcal{A}$ projective generator

$F : \mathcal{A} \to \mathcal{B}$ functor s.t. $\mathrm{End}_{\mathcal{A}} P = \mathrm{End}_{\mathcal{B}} FP$

$\Rightarrow \mathcal{A} \simeq (\mathrm{End}_{\mathcal{B}} FP)^{op}$ - mod.

$\rightsquigarrow$ ① compute $\mathcal{M}P$ for projective objects in $\mathcal{O}'_h$.

② find $P \in \mathrm{Proj}\,\mathcal{O}'_h$ s.t. $\mathcal{M}P = Q$.

$\Rightarrow$ $\exists$ quotient functor $\mathcal{O}'_h \to S_s$ - mod

same number of simple objects

$\Rightarrow$ Equivalence.

Recall that $\mathcal{O}'_h = \bigoplus_{\lambda \in \mathrm{Lie}\,T^\vee/\hat{W}} {}^\lambda\mathcal{O}'$.

Write ${}^\lambda\mathcal{O}' = \varinjlim_n {}^\lambda\mathcal{O}'_n$ (truncation)

Similarly $H_{\hat{W}}$ - mof $= \bigoplus_{\ell \in T^\vee/W} {}^\ell\underline{\mathcal{O}}$, ${}^\ell\underline{\mathcal{O}} = \varinjlim_n {}^\ell\underline{\mathcal{O}}_n$

Note that $\mathrm{Lie}\,T^\vee/\hat{W} = T^\vee/W$.

Truncations can be fixed such that

$$\mathcal{M} : {}^{\lambda}\mathcal{O}'_n \longrightarrow {}^{e}\underline{\mathcal{O}}_n$$

Pb1 Find projective $P \in {}^{\lambda}\mathcal{O}'_n$ such that

$$\mathcal{M}P = Q \otimes_{\mathbb{C}[T^\vee]} \mathbb{C}[T^\vee] / I(e)^n$$

Pb2 $\mathcal{M}$ fully faithful on projective modules if $h \notin \frac{1}{2}\mathbb{Z}$.

Pb1 (Summand $J = \emptyset$)

We want $\qquad (\mu \in \lambda, \lambda \in \mathrm{Lie}\, T^\vee / \hat{W})$

$$\mathcal{M}\Bigg(\underbrace{H' \otimes_{\mathbb{C}[\mathrm{Lie}T^\vee]} \mathbb{C}[\mathrm{Lie}T^\vee] / I(\mu)^n}_{P'(\mu)}\Bigg) = \underbrace{H_{\hat{W}} \otimes_{\mathbb{C}[T^\vee]} \mathbb{C}[T^\vee] / I(e^\mu)^n}_{P(e^\mu)}$$

$\forall w \in \hat{W}$, $\exists \quad \phi_w \in \mathrm{Hom}_{H'}(P'(\mu), P'(w\mu))$
(intertwiner)

$$P'(\mu) = \mathbb{C}[T] \otimes \underbrace{\mathbb{C}W \otimes \mathbb{C}[\mathrm{Lie}T^\vee] / I(\mu)^n}_{\underline{P}'(\mu) \text{ (a } H'_{\hat{W}}\text{-module)}} \quad \text{as } \mathbb{C}[T]\text{-module}$$

$\underline{P}'(\mu)$ = frame of $P'(\mu)$ (= vector bundle on T)

Apply invertible intertwiners

$\rightsquigarrow$ different frames $\underline{P}'(w\mu)$ of $P'(\mu)$.

Claim: $\exists\, w \in \hat{W}$ st. monodromy of KZ has a cyclic vector in $\underline{P}'(w\mu)$

(use deformation, ie. $\mathbb{C} \to \mathbb{C}((\omega))$, and explicit solution of type A_1 in terms of Γ functions).

<u>Pb</u> 2
- faithfullness is a general property because KZ has regular singularities.
- For fully faithfullness we must check that localization

$$H'_{\mathfrak{a}}\text{-mod} \to H'_{\mathfrak{a}0}\text{-mod}$$

is surjective on Hom.

(explicit computation on residue of connection)

Hope: <u>p-adic groups</u>

F local field, $q = \#$ residue field of F, $q = p^n$

$\bar{k} = k$, $\ell = \operatorname{char} k$, $\ell \neq p$.

G reductive group

$\mathcal{U}_k$ = unipotent block $\subset$ smooth $kG(F)$-modules

$I \subset G(F)$ Iwahori.

$\underline{H} = \operatorname{End}_{kG(F)} k[I \backslash G(F)]$ Iwahori Hecke algebra

Bernstein:

$\ell = 0 \Rightarrow \mathcal{U}_k \xrightarrow{\sim} \underline{H}\text{-mod}, \; V \mapsto V^I$

If $\ell > 0$ then

(a) $k[I \backslash G(F)]$ not projective $kG(F)$-mod

$\Rightarrow V \mapsto V^I$ not exact

(b) $\left\{ \begin{array}{c} \text{simple objects} \\ \text{in } \mathcal{U}_k \end{array} \right\} \neq \left\{ \begin{array}{c} \text{simple quotient} \\ \text{of } k[I \backslash G(F)] \end{array} \right\}$

<u>Thm (Vigneras)</u> $G = GL_d$ ($\ell > 0$, hence q root of 1 in k)

$\exists$ Abelian subcategory $\mathcal{U}'_k \subset \mathcal{U}_k$ equivalent to modules over quantized affine Schur algebra.

Hope : $U_k \simeq$ a category of $H_{\tau\zeta}$-modules

($\tau \in k^\times$ large order, order of ζ in $k^\times/_{\tau^{\mathbb{Z}}}$ = order of q)

Remark: τ, ζ general position $\Rightarrow$ $H_{\tau\zeta}$ Morita equivalent to affine Hecke algebra, for all types

<u>Roots of unity - Modular quantization</u>

(In progress)

$$O = \{ g \in GL_n ; \ rk(g - id) \leq 1 \}$$

$$\tilde{O} \xrightarrow{\mu} O \quad \text{resolution}$$

$$T_s = \{ (g, g', o) \in GL_n^2 \times \tilde{O} ; \ [g, g'] = s\,\mu(o) \}$$

$(s \in \mathbb{C}^\times)$

$$H(s) = T_s^{\text{stable}} / GL_n \quad \text{quasi-projective}$$

deformation of $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$.

$$H(s) \xrightarrow{\pi} N(s) := \mathrm{Spec}\, \mathbb{C}[T_s]^{GL_n}, \text{ proper map}$$

$\tau^\ell = 1$ primitive, ℓ not too small

$\exists$ sheaf of Azumaya algebras $\mathcal{A}_{\tau\zeta} \in \mathrm{Coh}\, \tilde{\mathcal{N}}^{(\zeta^{2e})}$ such that

(a) $\mathrm{Spec}\; Z\,H_{\tau\zeta} = \mathcal{N}_{\zeta^{-2e}}$

(b) $\forall\, \xi \in \mathcal{N}_{\zeta^{-2e}}$, $D^b(H_{\tau\zeta}\text{-mod}_{\xi}) \simeq D^b(\mathcal{A}_{\tau\zeta}\text{-mod}_{\xi})$

modules supported (set theoretically) at ξ

Coherent sheaves of modules supported (set th.) on $\pi^{-1}(\xi)$

(c) $\mathcal{A}_{\tau\zeta}$ splits on fibers of π

Restrictions: (1) $\zeta \notin$ finite set

(2) Claim (c) for $\zeta^{2\ell} = 1$.

(3) $\ell \gg 0$