Representations of Algebraic Groups, Quantum Groups, and Lie Algebras

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Exceptional groups, Jordan algebras, and higher composition laws

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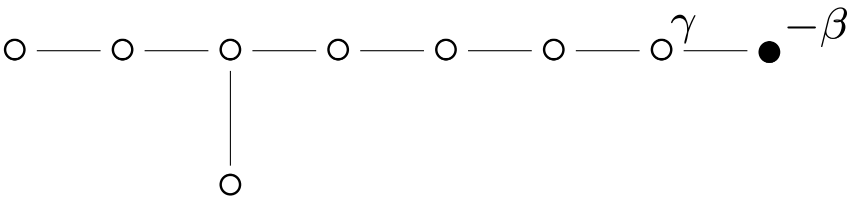
I. Algebraic groups and the Heisenberg parabolic

Let $\mathfrak{g}$ be a simple split Lie algebra over a field F.
$\mathfrak{g} \neq A_n, C_n$

Let Δ be its system of roots, Π be its system of simple roots.

Example

E_8 :



Let β be the highest root of Δ^+, and γ be the (unique) simple root connected to $-\beta$.

Let $P = L + U$ be the maximal parabolic subalgebra associated with the root γ
(L is the Levi factor, and U is nilpotent).

Let G be the semisimple algebraic group associated to L.

The Dynkin diagram of G is obtained by removing the vertex γ from the D.d. of $\mathfrak{g}$.

Let $V = U/[U,U]$.

We obtained a semisimple group G
acting in the vector space V

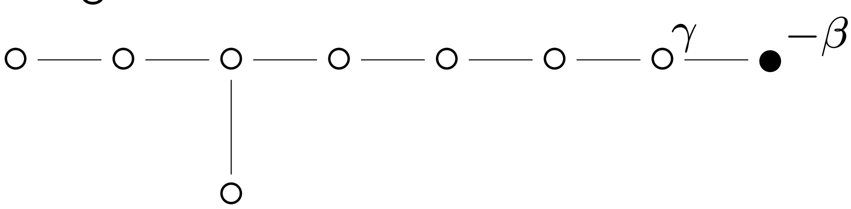
Summary:

$$\boxed{\mathfrak{g} = U^- \oplus \ L \oplus U}$$

$$G = \text{“Ad } L\text{”}, \qquad V = U/[U,U], \ \dim[U,U]{=}1$$

Examples

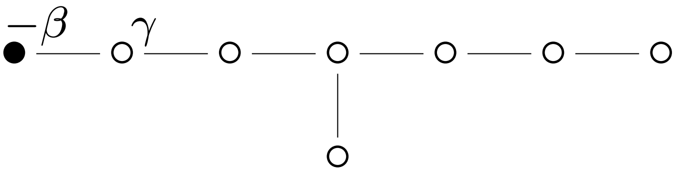
1. $\mathfrak{g} = E_8$



$G = E_7$

$\dim \mathfrak{g} = 248 = 57 + \quad (133+1) + (56+1)$
$\Rightarrow \dim V = 56.$

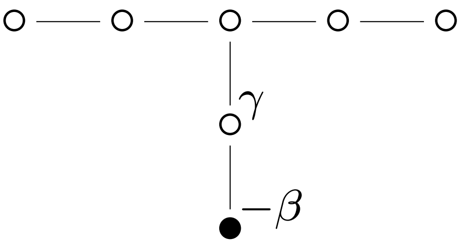
2. $\mathfrak{g} = E_7$



$G = D_6$

$$\boxed{(G, V) = (D_6, \text{half-spin})}$$

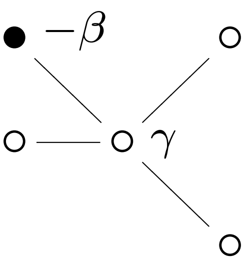
3. $\mathfrak{g} = E_6$



$G = A_5 \quad (= \mathsf{SL}_6)$

$$\boxed{(G, V) = (\mathsf{SL}_6, \wedge^3(F^6))}$$

4. $\mathfrak{g} = D_4$



$G = A_1 \times A_1 \times A_1$

$$\boxed{(G, V) = (\mathsf{SL}_2 \times \mathsf{SL}_2 \times \mathsf{SL}_2, F^2 \otimes F^2 \otimes F^2)}$$

II. The Freudenthal Construction

Given a vector space J (over a field F) with an admissible cubic form N

$$\mathcal{F} : J \mapsto (G, V),$$

where $V = J \oplus J \oplus F \oplus F$, $\quad \dim V = 2 \dim J + 2$
and G is a semisimple algebraic group acting in V
(G is the group of automorphisms of
a quartic form and a symplectic form on V)

$$q(x) = (\alpha\beta - (A, B))^2 + 4\alpha N(A) + 4\beta N(B) - 4(A^{\#}, B^{\#})$$

$$\{x, y\} = \alpha\delta - \beta\gamma + (A, D) - (B, C)$$

for $\quad x = \begin{bmatrix} \alpha & A \\ B & \beta \end{bmatrix}, \quad y = \begin{bmatrix} \gamma & C \\ D & \delta \end{bmatrix}$ in V

Remark Lie(G) is the TKK-construction of J

Examples

1. $J =$Mat$_{33}(F)$, $N =$ det

$G =$ SL$_6$ $(= A_5)$, $V = \wedge^3(F^6)$

2. $J = F \oplus F \oplus F$, $N(\alpha, \beta, \gamma) = \alpha\beta\gamma$

$G =$ SL$_2 \times$ SL$_2 \times$ SL$_2$, $V = F^2 \otimes F^2 \otimes F^2$

3. $J =$ Skew$_6(F)$, $N =$ Pfaff $= \sqrt{\text{det}}$

$G = D_6$, $V =$half-spin

4. $J = 27$-dim exceptional Jordan algebra, N

$G = E_7$,
V is the 56-dim module

Note
Examples $1, 3, 4$ have uniform description as 3×3 Hermitian matrices over split composition algebras of

dim = 2	binarions	$\mathbb{B}$	$(\cong F \times F)$
dim = 4	quaternions	$\mathbb{Q}$	
dim = 8	octonions	$\mathbb{O}$	

Theorem (S.K.)

Let $G_{\mathbb{Z}}, V_{\mathbb{Z}}$ be as in examples $1, 3, 4$
(i.e., $G_{\mathbb{Z}} = \mathsf{SL}_6(\mathbb{Z}), D_6, E_7$)

- The group $G_{\mathbb{Z}}$ acts transitively on the set of projective elements in $V_{\mathbb{Z}}$ of discriminant D

- If D is a *fundamental discriminant*, then every element of discriminant D is projective, and hence $G_{\mathbb{Z}}$ acts transitively on the set of elements of discriminant D

An integer D is called a fundamental discriminant if D is the discriminant of the full ring of integers in a quadratic field

- $D \equiv 1 \pmod 4$ and D is squarefree

- $D = 4k$ and k is a squarefree integer
  that is $\equiv 2$ or $3 \pmod 4$.

III. Number theoretical background

Gauss's composition law ($\sim$ 1801)

$$\boxed{\mathsf{Sym}^2\mathbb{Z}^2/\mathsf{SL}_2(\mathbb{Z}) \quad \overset{1-1}{\longleftrightarrow} \quad \Big(S, \mathsf{Cl}(S)\Big)}$$

where S is a quadratic order (a subring in a quadratic field extension of $\mathbb{Q}$),

Cl(S) is the set of (narrow) ideal classes of S

Higher composition laws
(Manjul Bhargava, $\sim$ 2000)

More examples of pairs $(G_{\mathbb{Z}}, V_{\mathbb{Z}})$ such that

$$V_{\mathbb{Z}}/G_{\mathbb{Z}}$$

are described in terms of ideal classes in number rings

M.B.:

- 5 HCL assoc. to quadratic rings
- 6 HCL assoc. to higher degree rings

S.K.:

For each HCL assoc. to quadratic rings, there is a Jordan algebra J such that the pair $(G_{\mathbb{Z}}, V_{\mathbb{Z}})$ is obtained from J via the Freudenthal construction

Summary
The Freudenthal construction and higher composition laws

$\mathfrak{g}$	J	Group G	Rep. V	$G_{\mathbb{Z}}$-orbits in $V_{\mathbb{Z}}$ parametrized by
B_3	$F \oplus F$	$(\mathrm{SL}_2)^2$	$V_2 \otimes \mathrm{Sym}^2 V_2$	
D_4	$F \oplus F \oplus F$	$(\mathrm{SL}_2)^3$	$V_2 \otimes V_2 \otimes V_2$	Ideal classes
D_5	$F \oplus Q_4$	$\mathrm{SL}_2 \times \mathrm{SO}_6$	$V_2 \otimes V(\omega_1)$	in quadratic rings
G_2	F	SL_2	$\mathrm{Sym}^3 V_2$	(M. Bhargava)
F_4	$\mathcal{H}_3(F)$	C_3	$V(\omega_3)$	
E_6	$\mathcal{H}_3(\mathbb{B})$	A_5	$V(\omega_3)$	Quadratic rings (M.B, S.K.)
E_7	$\mathcal{H}_3(\mathbb{Q})$	D_6	half-spin	Quadratic rings (S.K.)
E_8	$\mathcal{H}_3(\mathbb{O})$	E_7	minuscule	Quadratic rings (S.K.)
$\mathfrak{so}_{n+6}$	$F \oplus Q_n$	$\mathrm{SL}_2 \times \mathrm{SO}_{n+2}$	$V_2 \otimes V(\omega_1)$	?