π - points for finite group schemes

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We replace/re-interpret cohomlogical methods by elementary representation theory as well as extend/refine earlier results of many other

We obtain a uniform approach to the study of an arbitrary finite group scheme over an arbitrary field $\Bbbk$ of characteristic $p>0$.

Mention a few antecedents:

Finite groups – $k\pi$:

Quillen

Alperin, Evens, Dade

Carlson

Avrunin, Scott

Benson, Rickhard

Restricted Lie alg – $V(\mathfrak{g})$

F., Parshall, Andersen-Jantzen, Nakano

Infinitesimal group schemes

Suslin, F., Bendel, Pertsova

<u>Def'n</u> A finite group schem[e] G/k is a functor

$$G: \left\{ \begin{matrix} \text{fin gen} \\ \text{comm } k\text{-alg} \end{matrix} \right\} \longrightarrow \text{Grps}$$

represented by a finite dim'l comm. k-alg $k[G]$ — coordina[te] algebra of G.

<u>Observation</u>: the k-linear dual of $k[G]$, which we denote by kG and call the group algebra of G is a finite dimensional co-commutative Hopf algebra

Examples

$G = \pi$ finite group

$\pi : A \longmapsto \pi^{\times \pi_0(\mathrm{Spec} A)}$

$k\pi$

$k[\pi] = k^{\times |\pi|}$

$\mathfrak{g}$ finite dim'l p-restricted Lie alg

$u(\mathfrak{g})$, $V(\mathfrak{g})$, $k\mathfrak{g}$ – restricted enveloping alg

$G = \ker\{F^r : \mathcal{G} \to \mathcal{G}^{(r)}\}$, r^{th} Frob.

kernel of alg. group $\mathcal{G}/k$

$G : A \longmapsto \mathrm{Ker}\{\mathcal{G}(A) \xrightarrow{F^r} \mathcal{G}^{(r)}(A)\}$

$k[G] = k[\mathcal{G}]/m_e^{p^r}$

$G = GL_{N(r)}^{\times i} \rtimes \Sigma_i$

Our "π-points" point of view gives uniform description of $H^{ev}(G, k)$ modulo nilpotents

Surprising, since cohomological behavior of finite groups is seemingly very different from that of Frobenius kernels

e.g.
G infinitesimal, $H < G$, then
$$H^{ev}(G,k)/_{nilp} \twoheadrightarrow H^{ev}(H,k)_{nilp}$$
<u>not</u> <u>true</u> for G finite

e.g. G finite, $cx(G) = \dim H^{ev}(G,k)$ realized as $cx(H)$ for some $H < G$ abelian
<u>not</u> <u>true</u> for G infinitesimal

<u>Defn</u> A G-module M is a
module for the group alg. kG.

<u>Fundamental example</u>

$\pi = \mathbb{Z}/p$ — finite repr. type
with p iso classes of indec
unique (1-dim'l) irred k
unique (free, p-dim'l) indec. proj

<u>WARNING.</u>
$\pi = \mathbb{Z}/p^{\times 2}$ has wild repr. type
for $p > 2$

$$k\mathbb{Z}/p = k[X]/_{X^p - 1} = k[t]/_{t^p}$$

<u>Defn</u> A π-point of G/k is a left flat k-alg homom

$$\alpha_K : K[t]/t^p \to KG$$

which factors through some

$$KC_K \rightarrowtail KG_K = KG, \text{ where}$$

$C_K \hookrightarrow G_K$ is a commutative subgroup scheme

some fld extension K/k

<u>Notes</u>:

- subtle role of Hopf alg. structure of kG
- necessary to consider field extensions K/k

We replace study of cohomology by study of π-points

Defn α_K, β_L π-points of G

Then β_L specializes to α_K

$$\beta_L \searrow \alpha_K$$

provided that $\forall$ fin gen kG

module M

$$\alpha_K^*(K \otimes M) \text{ free} \Rightarrow \beta_L^*(L \otimes M) \text{ fr}$$

We write $\beta_L \sim \alpha_K \Longleftrightarrow \beta_L \searrow \alpha_K, \alpha_K \searrow$

$[\alpha_K]$ - equiv. class of α_K

Examples

- If K/k is algebraic

and $\alpha_K \searrow \beta_L$, then $\alpha_K \sim \beta_L$

" $[\alpha_K]$ is special (i.e, closed)

- $G = \mathbb{Z}/p^{\times r}$, $r > 1$

$K = k(y_1, \ldots, y_{r-1})$

$\alpha_K : K[t]/t^p \longrightarrow K\mathbb{Z}/p^{\times r} =$

$K[x_1, \ldots, x_r]/x_i^p$

$t \longmapsto y_1 x_1 + \cdots + y_{r-1} x_{r-1} + x_r$

If $\beta_L \searrow \alpha_K$, then $\alpha_K \sim \beta_L$

"α_K is generic"

- If G is a finite group, then natural bijection

$$\left\{\begin{array}{l}\text{conj classes}\\ \text{of max elem}\\ \text{ab. } p\text{-grps}\end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{l}\text{equiv classes}\\ \text{of generic}\\ \pi\text{-points}\end{array}\right.$$

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sl_2 – p-restricted Lie alg

minimal field of defn K
of gen π-pt

$\mathbb{Q}\, sl_2$

$\alpha_K : K[t]/t^p \to K\, sl_2$

K = fld of frac (N)

$N = \operatorname{Spec} \mathbb{Q}[x_{11}, x_{12}, x_{21}] / x_{11}^2 + x_{21}x_{12}$

$t \mapsto x_{12}e + x_{21}f + x_{11}h$

[Alternative point of view]

Analyze "nice" algebra maps

$$k[t]/t^p \longrightarrow kG$$

Possible extensions:

replace G by quantum version

super version

replace $k[t]/t^p$ by ??

<u>Def'n</u> The space of π-pts

$\Pi(G)$ is the set of equiv. classes of π-pts of G equipped with the topology whose closed subsets are subsets of the form

$$\Pi(G)_M = \left\{ [\alpha_K] : \alpha_K^*(K \otimes M) \text{ is not free} \right\}$$

with M finite dim'l kG-mod

<u>Remarks</u>:

- Any subset $S \subset \Pi(G)$ is of form $\Pi(G)_M$, some kG-mod M
- def'n does not involve cohomology

Theorem

$$\Psi_G : \Pi(G) \to \mathrm{Proj}\, H^{ev}(G,k)$$

$$[\alpha_K] \mapsto \ker\{\alpha_K^* : H^{ev}(G,K) \to H^{ev}(\ldots K$$

$$\cap\, H^{ev}(G,k)$$

is a homeomorphism

sending

$$\Pi(G)_M \xrightarrow{\sim} \mathrm{Proj}\left(\frac{H^{ev}(G,k)}{\mathrm{ann}_{H^{ev}(G,k)} Ext_G^*(M,M\ldots}\right.$$

Remarks

- For G elem ab. p-grp, this is essentially "Carlson's conjecture [rank versus cohomological var.]
- For general finite group G, reformulates work of Avrunin-Scott

"Proof"

a.) Ψ_G is well defined and injective

Separate π-points using

Carlson modules

$$L_\zeta = \operatorname{Ker}\{\Omega^{2i}(k) \xrightarrow{\zeta} k\}$$

$$\operatorname{Var}(\operatorname{ann}_{H^{ev}(G,k)} \operatorname{Ext}^*_G(L_\zeta, L_\zeta)) = Z(\zeta)$$

b.) Ψ_G is ~~sur~~jective :

Detection modulo nilpotence of $H^{ev}(G,k)$ on quasi-elem ab subgrps by Quillen, S-F-B, Bendel, Suslin

$M \mapsto \Pi(G)_M$ is a finer invariant than the cohomological support variety for M infinite dimensional

(for finite groups, essentially same as considered by Benson-Carlson-Rickhard)

<u>Theorem</u> Let M_i be arbitrary G-mod.

$$\Pi(G)_{M_1 \oplus M_2} = \Pi(G)_{M_1} \cup \Pi(G)_{M_2}$$

$$\Pi(G)_{M_1 \otimes M_2} = \Pi(G)_{M_1} \cap \Pi(G)_{M_2}$$

$$\Pi(G)_M = \emptyset \Leftrightarrow M \text{ is projective}$$

"Proof" of projectivity test:

use earlier work of Pevtsova

base change so that

$$G \simeq G^0 \rtimes \pi_0(G)$$

$$\Pi(G)_M = \{[\alpha_K] : \alpha_K^*(K \otimes M) \text{ is not free}\}$$

Let stmod(G) be the stable module category of fin gen. G-mo

$$\mathrm{Hom}_{\mathrm{stmod}(G)}(M_1, M_2) = \frac{\mathrm{Hom}_G(M_1, M_2)}{\langle f: M_1 \to M_2 \text{ which factor thru proj}}$$

This is a $\otimes$-Δ-category

Recall: $M[-1] = \Omega M = \mathrm{Ker}\{P(M) \twoheadrightarrow M\}$

Rickard idempotents $E_{\mathcal{C}}, F_{\mathcal{C}}$ assoc. to thick, $\otimes$-ideal subcat. $\mathcal{C} \subset \mathrm{stmod}(G)$ $E_{\mathcal{C}} \to k \to F_{\mathcal{C}} \to E_{\mathcal{C}}[$

These are infinite dimensional: need invariant which behaves well with respect to $\otimes$, projectivity

<u>Theorem</u> G finite grp scheme/fld k

$$\mathcal{C} \subset \mathrm{stmod}(G) \longmapsto W_{\mathcal{C}} = \bigcup_{M \in \mathcal{C}} \Pi(G)_M \subset \Pi(G$$

$$\mathcal{C}_S = \{M : \Pi(G)_M \subset S\} \longleftarrow\!\!| \quad S \subset \Pi(G$$

Induce mutually inverse lattice
isomorphisms

$$\left\{\begin{array}{l}\text{thick, } \otimes\text{-ideal} \\ \mathcal{C} \subset \mathrm{stmod}(G)\end{array}\right\} \simeq \left\{\begin{array}{l}\text{subspaces } S \subset \Pi(G \\ \text{closed under spec}\end{array}\right.$$

(cf. Benson-Carlson-Rickhard for fi
groups)

<u>Prop^n</u> $\forall s \subset \Pi(G)$, $\exists M \ni$

$$\Pi(G)_M = S$$

Can choose M fin dim'l $\Longleftrightarrow S$ clos

Recover $\mathrm{Proj}\, H^{ev}(G,k)$ as a scheme in these representation theoretic terms.

<u>Theorem</u> Let $X = \mathrm{Proj}\, H^{ev}(G,k)$

Define sheaf of comm. k-alg on X, $\mathcal{E} = \mathrm{End}_{\mathrm{stmod}(G)/-}(k,k)$

$$\mathcal{E}(X-W) = \mathrm{End}_{\mathrm{stmod}(G)/\mathcal{C}_W}(k,k)$$

Then
$$\mathcal{O}_X \simeq \mathcal{E}$$

"Proof" (inspired by Balmer)

Use work of Carlson-Donovan-Wheele

Corollary $X = \mathrm{Proj}\, H^{ev}(G, k)$

$\exists$ functor

$$\mathrm{stmod}(G) \longrightarrow D^b(\mathrm{Coh}(X))$$

which induces isomorphism of latti[ces]
of thick, $\otimes$-ideal subcategories

$$\alpha_K : K[t]/t^p \longrightarrow KG$$

kG-modu[le]
M

? refined support varieties ?