FALL 2005 #2

Let H be a normal subgroup of a finite group G. Let $S \subseteq G$ be a conjugacy class of elements in G, and assume that $S \subseteq H$. Prove that S is a union of n conjugacy classes in H, all having the same cardinality, where n equals the index $[G : H.C_G(x)]$ of the group generated by H and the centralizer in G of any element $x \in S$.

Solution: Since S is a conjugacy class of elements in G, the group G acts transitively on S by conjugation. Thus, by the orbit equation,

$$|S| = \frac{|G|}{|C_G(x)|} \text{ for any } x \in S.$$

Since H is normal in G, and since $S \subseteq H$, it follows that H also acts on S by conjugation. Let $\overline{x}^H$ denote the orbit of an element $x \in S$ under this action of H. (That is, $\overline{x}^H$ is a conjugacy class in H.) The set S is finite. So we write S as a disjoint union of n conjugacy classes in H:

$$S = \overline{x_1}^H \cup \overline{x_2}^H \cup \cdots \cup \overline{x_n}^H.$$

We will prove that every conjugacy class in H has the same cardinality, but first let us assume this temporarily.

Then $|S| = n|\overline{x}^H|$, where x is any element of S. By the orbit equation, it follows that $|\overline{x}^H| = \frac{|H|}{|H \cap C_G(x)|}$. Therefore:

$$|S| = \frac{|G|}{|C_G(x)|} = n|\overline{x}^H| = \frac{n|H|}{|H \cap C_G(x)|},$$

and hence:

$$n = \frac{|G||H \cap C_G(x)|}{|C_G(x)||H|} = \frac{|G|}{|HC_G(x)|}.$$

(Note that since H is normal in G, the subgroup $HC_G(x)$ exists and is equal to $H.C_G(x)$.) Thus:

$$n = [G : H.C_G(x)].$$

It remains to be shown that $|\overline{x}^H| = |\overline{y}^H|$ for all $x, y \in S$. Note that $x = gyg^{-1}$ for some $g \in G$ (since S is a conjugacy class in G). So for any $h, \tilde{h} \in H$, we have:

$$hxh^{-1} = \tilde{h}x\tilde{h}^{-1}\ (\in \overline{x}^H) \text{ if and only if } g^{-1}hgyg^{-1}h^{-1}g = g^{-1}\tilde{h}gyg^{-1}\tilde{h}^{-1}g\ (\in \overline{y}^H).$$

(Note that $g^{-1}hg \in H$ since H is normal in G.) Hence, the map $\overline{x}^H \to \overline{y}^H$ given by $hxh^{-1} \mapsto g^{-1}hgyg^{-1}h^{-1}g$ is a well-defined injection and therefore $|\overline{x}^H| \leq |\overline{y}^H|$.

By interchanging the roles of x and y in the above argument, it follows that $|\overline{y}^H| \leq |\overline{x}^H|$. So $|\overline{x}^H| = |\overline{y}^H|$.

□