The Schwarzschild solution and an introduction to black holes

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"If only it weren't so damnably difficult to find exact solutions!"

- Albert Einstein (in a letter to Max Born, circa 1936), quoted in [1], page 142

1 The one-body problem

In Newtonian gravity, the planets orbit the sun in ellipses. Well, not exactly. Newton understood that the gravitational influence of the planets themselves lead to perturbations away from perfect ellipses. For example, if not for the Earth, the moon would orbit the sun in a nearly circular orbit. Because of the Earth, the moon orbits the sun in something that resembles a Ptolemaic epicycle! The (extremely hard) problem of calculating the motion of a collection of many bodies under their collective gravitational influences is known as the *many-body problem*.

When we first learn how to calculate the planetary orbits in Newtonian theory, we make the simplifying assumption that the sun is not affected by the bodies that are in orbit around it. This allows us to avoid the subtleties associated with the many-body problem, and we still get accurate predictions. When the sun is regarded as the only source of gravity and the planets, comets, etc., take on the role of 'test particles,' it may be said that we are doing the *one-body problem* rather than the many-body problem.

Today I'm going to talk about the one-body problem in General Relativity. Actually, I shouldn't say 'the' one-body problem. In General Relativity, the gravitational field (or spacetime) of a one-body system depends on whether this body is rotating, has electric charge, etc. So more honestly, I should say that today I am going to discuss the one-body problem for a non-rotating, non-charged, point-like mass. I will derive, from Einstein's field equation, a spacetime metric corresponding to such a system. This metric is called the *Schwarzschild solution*. It was discovered by Karl Schwarzschild a few months after Einstein first published his field equation ([2], p. 607).

2 A derivation of Schwarzschild's solution

We seek a spherically symmetric solution to the Einstein field equation for the vacuum, with the cosmological constant set to zero. Since we seek a spherically symmetric solution, let's use spherical coordinates (r, θ, φ, t). We place the 'mass' m at the origin $r = 0$. At every worldpoint with r-coordinate $r > 0$, we have nothing but vacuum. By spherical symmetry, we look for a metric of the form:

$$ds^2 = f(r)dr^2 + g(r)r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right) - h(r)dt^2. \tag{1}$$

where $f(r), g(r)$, and $h(r)$ are unknown functions of r. By a suitable scaling of r, we can have $g(r) = 1$. It turns out to be helpful to write the coefficients $f(r)$ and $h(r)$ as exponentials. (The first time that I worked through this derivation, I did not use this convenience and I found that the calculations got very messy). So, instead of using Equation (1) as our ansatz, it is more convenient to use:

$$ds^2 = e^{A(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right) - e^{B(r)}dt^2, \tag{2}$$

where $A(r)$ and $B(r)$ are unknown functions (possibly imaginary) of r alone.

The metric tensor coefficients g_{ij} corresponding to Equation (2) are, with respect the coordinate basis induced by our spherical coordinates,

$$\begin{bmatrix} g_{rr} & g_{r\theta} & g_{r\varphi} & g_{rt} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\varphi} & g_{\theta t} \\ g_{\varphi r} & g_{\varphi\theta} & g_{\varphi\varphi} & g_{\varphi t} \\ g_{tr} & g_{t\theta} & g_{t\varphi} & g_{tt} \end{bmatrix} = \begin{bmatrix} e^{A(r)} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & 0 & -e^{B(r)} \end{bmatrix}. \tag{3}$$

So the inverse metric tensor is given by:

$$\begin{bmatrix} g^{rr} & g^{r\theta} & g^{r\varphi} & g^{rt} \\ g^{\theta r} & g^{\theta\theta} & g^{\theta\varphi} & g^{\theta t} \\ g^{\varphi r} & g^{\varphi\theta} & g^{\varphi\varphi} & g^{\varphi t} \\ g^{tr} & g^{t\theta} & g^{t\varphi} & g^{tt} \end{bmatrix} = \begin{bmatrix} e^{-A(r)} & 0 & 0 & 0 \\ 0 & r^{-2} & 0 & 0 \\ 0 & 0 & r^{-2}\sin^{-2}\theta & 0 \\ 0 & 0 & 0 & -e^{-B(r)} \end{bmatrix}. \tag{4}$$

From Lecture 3, we know that:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(g_{il,j} + g_{jl,i} - g_{ij,l} \right). \tag{5}$$

Whence, from (3), (4), and (5), the Christoffel symbols corresponding to our ansatz can be calculated. However, it is often faster and easier to calculate the Christoffel symbols by variational techniques, as I will do here.

As stated in the previous lecture, the Lagrangian for 'energy' is given by:

$$\begin{aligned} \mathcal{L}(r,\theta,\varphi,t;\dot{r},\dot{\theta},\dot{\varphi},\dot{t}) &= \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j \\ &= \frac{1}{2}\left(e^{A(r)}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\ \dot{\varphi}^2\right) - e^{B(r)}\dot{t}^2 \right). \end{aligned}$$

An affinely-parameterized worldline is a geodesic if and only if it is stationary with respect to the energy action. Consider then a test particle with a time-like worldline that is parameterized by proper time τ (an affine parameter). In coordinates, the worldline of this particle is given by a parametric curve $(r(\tau), \theta(\tau), \varphi(\tau), t(\tau))$, and we write $\dot{r}, \dot{\theta}$, $\dot{\varphi}$ and $\dot{t}$ to denote differentiation with respect to τ. As I said, this worldline is a geodesic if and only if it is stationary with respect to the energy action. So in order for this worldline to be a geodesic, we need the Euler-Lagrange equations to be satisfied:

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^k} \qquad k = 1,2,3,4, \tag{6}$$

where 'x^1, x^2, x^3,' and 'x^4' are understood to be 'r, θ, φ,' and 't' respectively. We find that:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{1}{2}\left(A'\dot{r}^2 e^A + 2r\left(\dot{\theta}^2 + \sin^2\theta\ \dot{\varphi}^2\right) - B'\dot{t}^2 e^B \right) \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2} r^2 \dot{\varphi}^2 \sin(2\theta) \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0, \tag{10}$$

where we write $A = A(r)$, $B = B(r)$, and $A' = \frac{d}{dr}A(r)$ and $B' = \frac{d}{dr}B(r)$. We also get that:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{r}} = e^{A}\left(A'\dot{r}^2 + \ddot{r}\right) \tag{11}$$

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r\left(2\dot{r}\dot{\theta} + r\,\ddot{\theta}\right) \tag{12}$$

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 2r\dot{r}\dot{\varphi}\sin^2\theta + r^2\dot{\theta}\dot{\varphi}\sin(2\theta) + r^2\sin^2\theta\,\ddot{\varphi} \tag{13}$$

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{t}} = -e^{B}\left(B'\dot{r}\dot{t} + \ddot{t}\right). \tag{14}$$

The double dot indicates the second derivative with respect to τ. Substituting these calculation into the Euler-Lagrange equations (6) and simplifying leads to:

$$\ddot{r} + \frac{1}{2}A'\dot{r}^2 - re^{-A}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) + \frac{1}{2}B'e^{B-A}\dot{t}^2 = 0 \tag{15}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \frac{1}{2}\sin(2\theta)\dot{\varphi}^2 = 0 \tag{16}$$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi} = 0 \tag{17}$$

$$\ddot{t} + B'\dot{r}\dot{t} = 0. \tag{18}$$

Equations (15) - (18) must be satisfied along a geodesic. Whence, we can 'read off' the Christoffel symbols by comparing (15) - (18) with the geodesic equation. It follows that the only non-vanishing Christoffel symbols are:

$$\Gamma^r_{rr} = \frac{1}{2}A', \quad \Gamma^r_{\theta\theta} = -re^{-A}, \quad \Gamma^r_{\varphi\varphi} = -re^{-A}\sin^2\theta, \quad \Gamma^r_{tt} = \frac{1}{2}B'e^{B-A}, \quad (19)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\theta_{\varphi\varphi} = -\frac{1}{2}\sin(2\theta), \qquad (20)$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \qquad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta, \qquad (21)$$

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{1}{2}B'. \qquad (22)$$

As pointed out in the previous lecture, Einstein's field equation for the vacuum can be written, with respect to a coordinate basis, as:

$$R_{ij} = 0 \qquad (23)$$

(if the cosmological constant is set equal to zero). Using the coordinate basis induced by our spherical coordinate system, we will denote the components of the Ricci tensor by R_{rr}, $R_{r\theta}$, ..., etc. Arranging these components into a matrix, we can write Equation (23) as:

$$\begin{bmatrix} R_{rr} & R_{r\theta} & R_{r\varphi} & R_{rt} \\ R_{\theta r} & R_{\theta\theta} & R_{\theta\varphi} & R_{\theta t} \\ R_{\varphi r} & R_{\varphi\theta} & R_{\varphi\varphi} & R_{\varphi t} \\ R_{tr} & R_{t\theta} & R_{t\varphi} & R_{tt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (24)$$

Equation (24) holds at every worldpoint in our spacetime with r-coordinate $r > 0$, because for $r > 0$, we have nothing but vacuum.

From Lecture 4, we know that:

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}. \qquad (25)$$

Thus, we can calculate the components of the Ricci tensor from the Christoffel symbols found in Equations (19) - (22). By this procedure, we find that the only components of the Ricci tensor that are not identically equal to zero are the diagonal components:

$$R_{rr} = -\frac{1}{2}B'' - \frac{1}{4}(B')^2 + \frac{1}{4}A'B' + A'r^{-1} \tag{26}$$

$$R_{\theta\theta} = -e^{-A} + \frac{1}{2}rA'e^{-A} - \frac{1}{2}rB'e^{-A} + 1 \tag{27}$$

$$R_{\varphi\varphi} = \sin^2\theta \left(-e^{-A} + \frac{1}{2}rA'e^{-A} - \frac{1}{2}rB'e^{-A} + 1\right) \tag{28}$$

$$R_{tt} = e^{B-A}\left(\frac{1}{2}B'' + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' + B'r^{-1}\right). \tag{29}$$

Aside: One might wonder if the Ricci tensor is diagonal with respect to a coordinate basis whenever the metric tensor is diagonal. This is true in two dimensions, but not in three dimensions or higher. For example, in $\mathbb{R}^3$, consider the well-behaved diagonal metric whose only non-vanishing components (with respect to the usual (x, y, z) rectangular coordinates) are given by $g_{xx} = 1$, $g_{yy} = 1$, and $g_{zz} = e^x + e^y$. One can show that $R_{xy} = \frac{1}{4}e^{x+y}(e^x + e^y)^{-2}$. Whence, the Ricci tensor is not 'diagonal.'

Plugging Equations (26) - (29) into (24), we obtain a system of differential equations:

$$-\frac{1}{2}B'' - \frac{1}{4}(B')^2 + \frac{1}{4}A'B' + A'r^{-1} = 0 \tag{30}$$

$$-e^{-A} + \frac{1}{2}rA'e^{-A} - \frac{1}{2}rB'e^{-A} + 1 = 0 \tag{31}$$

$$\sin^2\theta \left(-e^{-A} + \frac{1}{2}rA'e^{-A} - \frac{1}{2}rB'e^{-A} + 1\right) = 0 \tag{32}$$

$$e^{B-A}\left(\frac{1}{2}B'' + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' + B'r^{-1}\right) = 0. \tag{33}$$

Equation (33) yields:

$$\frac{1}{2}B'' + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' + B'r^{-1} = 0. \tag{34}$$

Adding Equations (30) and (34), one gets that $A' + B' = 0$. Whence, $A + B = K =$ constant. At this point we make a physical argument. Since 'the gravitational field should vanish at infinity,' we ask that our ansatz, Equation (2), go over to the Minkowksi metric in spherical coordinates as $r \to \infty$. Thus, we need to have $\lim_{r\to\infty} e^{A(r)} = 1$ and $\lim_{r\to\infty} e^{B(r)} = 1$. That is, $\lim_{r\to\infty} A(r) = 0$ and $\lim_{r\to\infty} B(r) = 0$. Whence, the constant K is zero and we have that $A + B = 0$, or $B = -A$.

Substituting $B = -A$ into Equation (31), we get that:

$$-e^B - \frac{1}{2}rB'e^B - \frac{1}{2}rB'e^B + 1 = 0. \tag{35}$$

Whence,

$$1 = e^B + rB'e^B. \tag{36}$$

Note that:

$$\left[re^B\right]' = e^B + rB'e^B. \tag{37}$$

Hence,

$$1 = \left[re^B\right]', \tag{38}$$

or

$$re^B = \int 1dr = r + C, \tag{39}$$

where C is a constant of integration. Thus,

$$e^B = 1 + \frac{C}{r} \tag{40}$$

and

$$e^A = e^{-B} = \left(1 + \frac{C}{r}\right)^{-1}. \tag{41}$$

Substituting Equations (40) and (41) into Equation (2) we get:

$$ds^2 = \left(1 + \frac{C}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) - \left(1 + \frac{C}{r}\right) dt^2. \tag{42}$$

The value of the constant C remains to be determined. You would probably guess that C must have something to do with the mass m that is 'generating' the spacetime. Indeed it turns out that $C = -2m$. To get this, we use another physical argument. Consider a test particle that is 'forced' to stay at the spatial point with spherical coordinates $(r, \theta, \varphi) = (r_0, 0, 0)$ for all time (in the metric given by (42)). The worldline of such a particle, parameterized by proper time τ, can be expressed $\gamma(\tau) = (r_0, 0, 0, \tau/\sqrt{1 + C/r_0})$. The *four-velocity* (normalized tangent vector) of $\gamma(\tau)$ is $(v^r, v^\theta, v^\varphi, v^t) = (0, 0, 0, 1/\sqrt{1 + C/r_0})$, and the *four-acceleration* $(a^r, a^\theta, a^\varphi, a^t)$ is given by the covariant derivative of the four-velocity along $\gamma(\tau)$. Thus:

$$\begin{aligned} a^r = \dot{v}^r + \Gamma^r_{ij} v^i v^j &= \Gamma^r_{tt} v^t v^t \\ &= \frac{1}{2} B' e^{B-A}\Big|_{r=r_0} \left(\frac{1}{1 + \frac{C}{r_0}}\right) \\ &= \frac{1}{2} B' e^{2B}\Big|_{r=r_0} \left(\frac{1}{1 + \frac{C}{r_0}}\right) \\ &= \frac{-C}{2r_0^2\left(1 + \frac{C}{r_0}\right)} \left(1 + \frac{C}{r_0}\right)^2 \left(\frac{1}{1 + \frac{C}{r_0}}\right) \\ &= \frac{-C}{2r_0^2} \end{aligned} \tag{43}$$

$$a^\theta = \dot{v}^\theta + \Gamma^\theta_{ij} v^i v^j = \Gamma^\theta_{tt} v^t v^t = 0 \tag{44}$$

$$a^\varphi = \dot{v}^\varphi + \Gamma^\varphi_{ij} v^i v^j = \Gamma^\varphi_{tt} v^t v^t = 0 \tag{45}$$

$$a^t = \dot{v}^t + \Gamma^t_{ij} v^i v^j = \Gamma^t_{tt} v^t v^t = 0 = 0. \tag{46}$$

So the only non-vanishing component of the four-acceleration is the radial component $a^r = -C/r_0^2$. Now, in the 'Newtonian limit,' the 'force' exerted by a particle that maintains a distance r_0 from a point-mass m is given by the mass of the particle times the 'acceleration' m/r_0^2. Whence, in order to recover Newtonian theory in the non-relativistic limit, we require that $C = -2m$.

Thus, in all it's glory, our metric looks like this:

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right) - \left(1 - \frac{2m}{r}\right) dt^2. \qquad (47)$$

This is the **Schwarzschild solution**. It is the unique spherically symmetric solution to Einstein's field equation for the vacuum with no cosmological constant. Any spherically symmetric solution to the Einstein vacuum equations (with $\Lambda = 0$) must be locally isometric to the Schwarzschild solution ([3], p. 149).

3 Interpretation and black holes

The Schwarzschild solution is a good approximation for the spacetime structure in the neighborhood of a star, such as the sun. Most experimental tests of General Relativity in the solar system have been based on the predictions given by this solution, and General Relativity has passed every experimental test that has been carried out so far.

In the context of the sun however, the solution only makes sense for r larger than the sun's radius. This is because the interior of the sun is not even close to being a vacuum (remember, the Schwarzschild solution assumes a vacuum). Note that the Schwarzshild solution, as presented in Equation (47), appears to be problematic at $r = 2m$. At this radius, called the **Schwarzschild radius**, the metric coefficient $g_{tt} = -(1-2m/r)$ becomes zero and $g_{rr} = (1 - 2m/r)^{-1}$ blows up! This is merely a result of a 'bad' choice of coordinates, as I will explain below. Another problem occurs at $r = 0$. Here, $g_{tt} = -(1 - 2m/r)$ blows up and $g_{rr} = (1 - 2m/r)^{-1}$ goes to zero. This is not merely a coordinate artifact and is a real singularity in the geometry. Before getting into that however, I would like to talk about what happens when a star dies.

When a star runs out of nuclear fuel and 'dies,' it transitions into one of three possible final states. If the mass of a dying star is less than the **Chandrasekhar limit** (about 1.4 solar masses), then the dead star will undergo gravitational collapse until the electron degeneracy pressure (a quantum mechanical effect) becomes great enough to halt the collapse. The result is a stable structure called a **white dwarf**. This will be the fate of our sun, billions of years from now. If a dying star has a mass higher than the Chandrasekhar limit, but less than the **Oppenheimer-Volkoff limit** (about 3 to 4 solar masses), then electron degeneracy pressure will not be able to stop the gravitational collapse. Instead, electrons and protons will combine to make neutrons and neutrinos. Neutrinos are almost massless and most of them would probably just fly away. What is left is an outrageously dense soup of neutrons - in some sense a giant atomic nucleus - and the degeneracy pressure of this 'nucleus' is great enough to withstand further gravitational collapse. Such an object is called a **neutron star**. If a dying star has a mass higher than the Oppenheimer-Volkoff limit, then no known physical process can hold back the gravitational collapse and it is believed that the result will be a completely gravitationally collapsed object, or **black hole** ([4], p. 235).

So the Schwarzshild solution, taken in 'the most literal sense possible,' describes the spacetime geometry of a spherically symmetric black hole in a vacuum. Now, let's deal with the fact that the metric, as represented by Equation (47), appears to 'go crazy' at $r = 2m$ and at $r = 0$.

We should remember that when a metric does peculiar things, in coordinates, it is not necessarily anything to panic over. For example, the Euclidean metric on the plane, in polar coordinates (r, θ), is given by:

$$ds^2 = dr^2 + rd\theta^2.$$

This metric is singular at the origin, but there is nothing special about the origin as far as geometry is concerned. The origin of the polar coordinate system is therefore a mere **coordinate singularity**. The Schwarzschild solution, being based on spherical polar coordinates also has a coordinate singularity at the 'poles,' where $\theta = 0$ or π. Since the coordinate system can just be 'rotated around' to move the 'poles,' the poles are clearly coordinate singularities only. The apparent singularity in the Schwarzschild solution at $r = 2m$ is also a coordinate singularity, but the singularity at $r = 0$ is not. The issue of determining precisely what is and what is not a 'mere' coordinate singularity is very subtle, but it is useful to look for coordinate-independent measures of the curvature ([4], p. 204). Recall that the Riemann tensor, in terms of a coordinate basis, is given by:

$$R^l_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^h_{ik}\Gamma^l_{jh} - \Gamma^h_{ij}\Gamma^l_{kh}.$$

These components are coordinate-dependent, but we can form invariant scalar quantities, such as the Ricci scalar $R = g^{ij}R_{ji} = R^i{}_i$ (where $R_{ij} = R^k{}_{ikj}$), as well as 'higher-order' scalars like $R^{ij}R_{ij}$, $R^{ijkl}R_{ijkl}$, and so on. We can conclude that we have a legitimate singularity in the curvature at some point if any one of these scalars blows up at that point - provided that the point is not 'infinitely far away' ([4], pp. 204 - 205). It turns out that ([4], p. 205):

$$R^{ijkl}R_{ijkl} = \frac{48m^2}{r^6}. \tag{48}$$

As you can see, this function is completely well-behaved except at $r = 0$. The singularity at $r = 0$ is said to be a **curvature singularity** because there exists an invariant measure of curvature that blows up there. Note that nothing pathological happens in Equation (48) at the Schwarzschild radius $r = 2m$.

If the apparent singularity at the Schwarzschild radius in Equation (47) really is merely a coordinate singularity, then we should be able to find new coordinates where the metric is well-behaved there. A particularly beautiful coordinate system that does the trick is the **Eddington-Finkelstein coordinate system**. Eddington-Finkelstein coordinates are based on the behavior of test photons propagating radially towards or away from the black hole, and hence the coordinates come in two flavors, *ingoing* and *outgoing* ([2], pp. 828 - 831). Today, I'll only discuss the *ingoing* kind.

To express the Schwarzschild solution in Eddington-Finkelstein coordinates, first define the so-called **tortoise coordinate** r^*, where:

$$r^* = \int \left(1 - \frac{2m}{r}\right)^{-1} dr. \tag{49}$$

The ingoing Eddington-Finkelstein time coordinate v is defined by:

$$v = t + r^*. \tag{50}$$

Thus, $dr^* = \left(1 - \frac{2m}{r}\right)^{-1} dr$ and $dv = dt + dr^*$. Substitution into Equation (47) yields:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dvdr + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right), \tag{51}$$

which is the Schwarzschild solution expressed in terms of (ingoing) Eddington-Finkelstein coordinates (r, θ, φ, v). With respect to these coordinates, the metric tensor can be put into the form:

$$\begin{bmatrix} g_{rr} & g_{r\theta} & g_{r\varphi} & g_{rv} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\varphi} & g_{\theta v} \\ g_{\varphi r} & g_{\varphi\theta} & g_{\varphi\varphi} & g_{\varphi v} \\ g_{vr} & g_{v\theta} & g_{v\varphi} & g_{vv} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2\theta & 0 \\ 1 & 0 & 0 & -(1-2m/r) \end{bmatrix}. \tag{52}$$

Since the determinant of this matrix is:

$$g = -r^4 \sin^2\theta, \tag{53}$$

the metric (51) is completely well-behaved everywhere, except at $\theta = 0$ or π (but these are merely coordinate singularities at the arbitrarily chosen 'poles'), and at $r = 0$, where a genuine curvature singularity lurks. (In contrast, the determinant of the analogous matrix corresponding to the metric given by Equation (47) is *undefined* at $r = 2m$.) In particular, the metric (51) is well-behaved at $r = 2m$. This proves that the apparent singularity at $r = 2m$ in Equation (47) is merely a coordinate singularity.

Because of spherical symmetry, we can begin to understand the Schwarzschild spacetime by considering null geodesics in the *radial* direction, using the metric given by Equation (51). Along such a wordline, we have $ds = d\theta = d\varphi = 0$. Hence, using an affine parameter λ, a radial null worldline in Eddington-Finkelstein coordinates satisfies:

$$0 = -\left(1 - \frac{2m}{r}\right)\dot{v}^2 + 2\dot{v}\dot{r}, \tag{54}$$

where the over-dot denotes differentiation with respect to the affine parameter λ. (I'll leave you with the exercise of showing that such a worldline actually satisfies the geodesic equation and so these radial null worldines actually are geodesics.) Thus, a radial null geodesic either satisfies:

$$\dot{v} = 0, \tag{55}$$

or

$$\dot{v} = 2\left(1 - \frac{2m}{r}\right)^{-1}\dot{r}. \tag{56}$$

Multiplying both sides of Equations (55) and (56) by $d\lambda/dr$, Equation (55) becomes:

$$\frac{dv}{dr} = 0, \tag{57}$$

and Equation (56) becomes:

$$\frac{dv}{dr} = 2\left(1 - \frac{2m}{r}\right)^{-1}. \tag{58}$$

By plotting out various solutions to Equations (57) and (58), we get results like those shown in Figure 1.

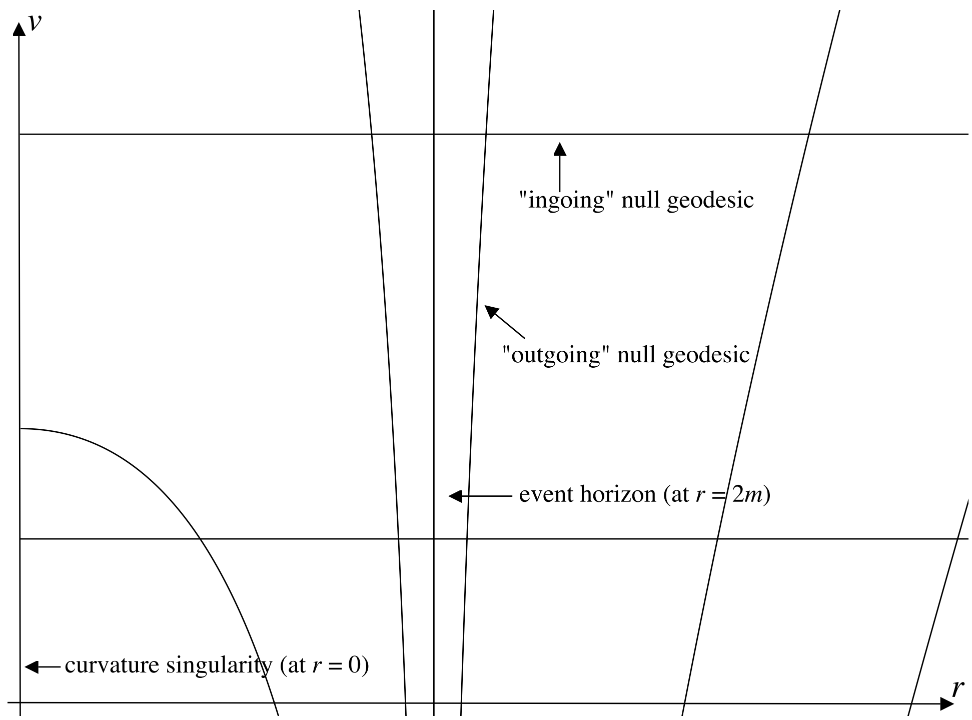

Figure 1: Plots of various radial null geodesics in (ingoing) Eddington-Finkelstein coordinates.

In Figure 1, 'ingoing' null geodesics plot as straight horizontal lines. 'Outgoing' null geodesics plot out differently. Notice that for $r < 2m$, even 'outgoing' null geodesics are pulled into the curvature singularity at $r = 0$. At the Schwarzschild radius, $r = 2m$, 'outgoing' null geodesics at $r = 2m$ plot out as vertical lines. (At $r = 2m$, Equation (54) reduces to $\dot{v}\dot{r} = 0$; whence r remains constant if $\dot{v} \neq 0$.) Thus, an 'outwardly' aimed photon fired at $r = 2m$ will simply 'stay put.' For $r > 2m$, outgoing photons will fly off safely towards infinity. Hence, an event that takes place at a radius $r \leq 2m$ cannot be observed at a radius $r > 2m$. The Schwarzschild radius $r = 2m$ is therefore called an **event horizon**, and this represents the 'boundary' of the black hole. The event horizon of the black hole is a therefore a point-of-no-return, or more accurately, a 'spherical-surface-of-no-return.' A starship that ventures past the event horizon of the black hole can never be seen or heard from again by the outside world. Any light signal that the starship emits will be pulled into the mencing curvature singularity at $r = 0$, never even making it to the event horizon. This is why black holes are called 'black holes.' No light can come out of them. Since light cannot escape a black hole, nothing else can either. Nothing can locally outrun light. Indeed, for the same reason, one cannot survive inside of the black hole indefinitely. An object inside the black hole must eventually encounter the singularity at $r = 0$. Stephen Hawking wrote ([5], p. 89): "One could well say of the event horizon what the poet Dante said of the entrance to Hell: 'All hope abandon, ye who enter here.'"

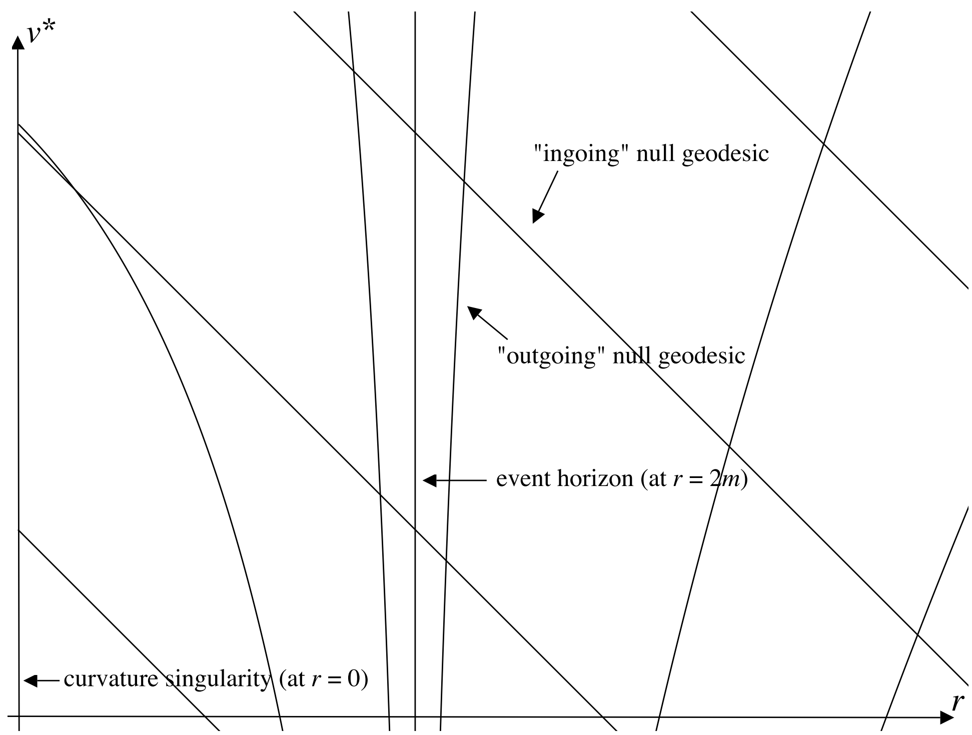

Figure 2: Plots of various radial null geodesics in the 'modified' Eddington-Finkelstein coordinate system (r, v^*).

Eddington-Finkelstein coordinates are very beautiful and enlightening.

However, it may feel a bit awkward to draw a null geodesic as a horizontal line. For large r-values (i.e., 'at infinity'), the Schwarzschild spacetime should look like Minkowski spacetime. In Minkowski spacetime, we are accustomed to having coordinates where null geodesics plot out as straight lines with a 45° slope. So let's try to get something that makes the causal structure of Schwarzshild spacetime look like this at large r-values. We can accomplish this by defining a new coordinate v^* such that:

$$v^* = v - r. \tag{59}$$

Equations (57) and (58) then become:

$$\frac{dv^*}{dr} = -1, \tag{60}$$

and

$$\frac{dv^*}{dr} = 2\left(1 - \frac{2m}{r}\right)^{-1} - 1. \tag{61}$$

Using (r, v^*)-coordinates, plots of various radial null geodesics are shown in Figure 2. Note that in this figure, all 'ingoing' null geodesics are situated at a 45° angle. The 'outgoing' null geodesics look curvy, with one exception: at $r = 2m$, outgoing geodesics plot as straight vertical lines ('$dv^*/dr = \pm\infty$'). Notice that the causal structure gets closer and closer to that of Minkowski spacetime as we move to larger radial coordinate values. (Indeed, as $r \to \infty$, Equation (61) becomes $dv^*/dr = 1$.) Again, we find that inside the event horizon, even 'outgoing' null geodesics are inevitably pulled into the curvature singularity at $r = 0$. Again, once inside the event horizon, one cannot get out, as is clear from the causal structure.

4 Putting in the cosmological constant: the Kottler solution

(This section may be skipped.) The cosmological constant has been making a comeback in physics over the previous decade. Once derided as "Einstein's blunder," the cosmological constant is now taken seriously as an explanation for the recently observed cosmic acceleration. The Schwarzschild solution assumes that the cosmological constant is zero. It is natural to ask what happens if this assumption is dropped. The analogue of the Schwarzschild solution in the presence of a nonzero cosmological constant Λ is called the **Kottler solution** and it looks like this ([6], section 5.2):

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right) - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2.$$

I will now derive the Kottler solution. Thanks to all the work that we've done so far, this will be a piece of cake!

From the previous lecture, we know that in the presence of a cosmological constant Λ, the vacuum Einstein equation can be written with respect to a coordinate basis as:

$$R_{ij} = \Lambda g_{ij}. \tag{62}$$

As in the Schwarzschild case, we seek a spherically symmetric metric of the form:

$$ds^2 = e^{A(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\ d\varphi^2\right) - e^{B(r)}dt^2, \tag{63}$$

where $A(r)$ and $B(r)$ are unknown functions of r, such that Equation (62) is satisfied. From our work on the Schwarzschild case, we already know the Christoffel symbols and the components of the Ricci tensor corresponding to Equation (63). The only difference is that, instead of getting Equations (30) - (33), we get, from Equation (62):

$$-\frac{1}{2}B'' - \frac{1}{4}(B')^2 + \frac{1}{4}A'B' + A'r^{-1} = \Lambda e^A \tag{64}$$

$$-e^{-A} + \frac{1}{2}rA'e^{-A} - \frac{1}{2}rB'e^{-A} + 1 = \Lambda r^2 \tag{65}$$

$$\sin^2\theta\left(-e^{-A} + \frac{1}{2}rA'e^{-A} - \frac{1}{2}rB'e^{-A} + 1\right) = \Lambda r^2 \sin^2\theta \tag{66}$$

$$e^{B-A}\left(\frac{1}{2}B'' + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' + B'r^{-1}\right) = -\Lambda e^B. \tag{67}$$

Note that Equation (67) yields:

$$\frac{1}{2}B'' + \frac{1}{4}(B')^2 - \frac{1}{4}A'B' + B'r^{-1} = -\Lambda e^A. \tag{68}$$

Adding Equations (64) and (68), we get that $A' + B' = 0$. So, just as in the Schwarzschild case, $A + B = K =$ constant. Now, since we are no longer assuming that the cosmological constant is zero, it is not clear (to me, at least) that we can repeat the argument used in the Schwarzschild case to get that $K = 0$. Instead, we just write $A = K - B$ and substitute this into Equation (65). This results in:

$$-e^{B-K} - rB'e^{B-K} + 1 = \Lambda r^2, \tag{69}$$

or:

$$\begin{aligned} 1 - \Lambda r^2 &= e^{B-K} + rB'e^{B-K} \\ &= \left[re^{B-K}\right]'. \end{aligned} \tag{70}$$

So

$$re^{B-K} = \int \left(1 - \Lambda r^2\right) dr = r - \frac{\Lambda r^3}{3} + C, \tag{71}$$

where C is another constant of integration. Thus,

$$e^B = \left(1 - \frac{\Lambda r^2}{3} + \frac{C}{r}\right) e^K. \tag{72}$$

In the case where $\Lambda = 0$, we should recover the Schwarzschild metric, which has $e^B = 1 - 2m/r$. Thus,

$$\left(1 + \frac{C}{r}\right) e^K = 1 - \frac{2m}{r}. \tag{73}$$

Taking the limit as $r \to \infty$ on both sides of Equation (73), we get that $e^K = 1$ (*if* $\Lambda = 0$, keep in mind that for all we know, K could be a function of Λ). Thus,

$$1 + \frac{C}{r} = 1 - \frac{2m}{r}, \tag{74}$$

and so $C = -2m$. Therefore,

$$e^B = \left(1 - \frac{\Lambda r^2}{3} - \frac{2m}{r}\right) e^K, \tag{75}$$

and

$$e^A = e^{K-B} = \left(1 - \frac{\Lambda r^2}{3} - \frac{2m}{r}\right)^{-1}. \tag{76}$$

Substituting Equations (75) and (76) into Equation (63), we get:

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right) - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) e^K dt^2.$$

By a suitable rescaling of the time coordinate t, we can get this into the form:

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \; d\varphi^2\right) - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2.$$

This is the Kottler solution as it is usually presented. It is the unique spherically symmetric solution to Einstein's field equation for the vacuum with cosmological constant ([6], Section 5.2).

5 What next?

Next time I'll discuss exact solutions corresponding to charged and rotating black holes. We'll use Penrose diagrams to understand their causal structure and see what wormholes and parallel universes have to do with black holes!

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[6] V. Perlick, "Gravitational Lensing from a Spacetime Perspective," Living Rev. Relativity 7, (2004), No. 9. URL (cited on November 5, 2008): http://www.livingreviews.org/lrr-2004-9